CHAPTER 4

RECURRANCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS OF GENERALIZED ORDER STATISTICS

1. Introduction

Some recurrence relations for moments of generalized order statistics have been obtained by Kamps (1995 b), Cramer and Kamps (2000), Kamps and Cramer (2001), Pawlas and Szynal (2001a, b), Athar and Islam (2004) among others. We have established recurrence relations for single and product moments of generalized order statistics (gos) for doubly truncated Weibull distribution and also for a general class of distributions \( F(x) = 1 - \exp\left[\frac{-1}{c}(h(x) - h(\alpha))\right], \ x \in (\alpha, \beta). \)

2. Recurrence relations for moments of the doubly truncated Weibull distribution

A random variable \( X \) is said to have Weibull distribution if the pdf of \( X \) is of the form

\[
f_1(x) = px^{p-1} e^{-x^p}, \ x > 0, \ p > 0
\]  

and the corresponding df is

\[
F_1(x) = 1 - e^{-x^p}, \ x > 0, \ p > 0
\]  

Now if for given \( P_1 \) and \( Q_1 \)

\[
\int_0^{Q_1} f_1(x)dx = Q \quad \text{and} \quad \int_0^{P_1} f_1(x)dx = P
\]

then the truncated pdf is given by

Part of the results of this chapter is contained in Khan et al. (2007 e) and Anwar et al. (2007).
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\[ f(x) = \frac{p x^{p-1} e^{-x^p}}{p - Q}, \quad -\log(1 - Q) \leq x^p \leq -\log(1 - P), \quad p > 0 \quad (2.3) \]

and the corresponding truncated df is

\[ \bar{F}(x) = -P_2 + \frac{1}{p} x^{1-p} f(x), \quad (2.4) \]

where

\[ Q_1^p = -\log(1 - Q), \quad P_1^p = -\log(1 - P), \quad Q_2 = \frac{1 - Q}{p - Q} \quad \text{and} \]

\[ P_2 = \frac{1 - P}{p - Q}. \]

2.1: Recurrence relations for single moments

**Case I:** \( m_i = m_j = m, \quad i, j = 1, 2, \ldots, n - 1. \)

Here we shall produce the following lemmas proved by Athar and Islam (2004), which will be used in sequel:

Let \( \xi(x) \) be a monotonic and continuous function of \( x \) then for \( 2 \leq r \leq n, \ n \geq 2 \) and \( k = 1, 2, \ldots \).

(i) \[ E[\xi \{ X_{(r,n,m,k)} \}] - E[\xi \{ X_{(r-1,n,m,k)} \}] = \frac{C_{r-2}}{(r-1)!} \sum_{i=1}^{m} \xi' \xi' \left( 1 - F(x) \right)^{1-r} g_{m-r-1}(F(x)) dx \quad (2.5) \]

For \( \xi(x) = x^j \), we have

\[ E[X^j (r,n,m,k)] - E[X^j (r-1,n,m,k)] = \frac{C_{r-2}}{(r-1)!} \sum_{i=1}^{m} x^{j-1} \left[ \bar{F}(x) \right]^{1-r} g_{m-r-1}(F(x)) dx \quad (2.6) \]
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(ii) \[ E[\xi \{ X (r-1,n,m,k) \}] - E[\xi \{ X (r-1,n-1,m,k) \}] = -\frac{(m+1)C_{r-2}^{(r-1)}}{\gamma_1 (r-2)!} \int_0^1 \xi' (x) \left( 1 - F(x) \right)^{\gamma_r} g_{r-1}^m (F(x)) \, dx \quad (2.7) \]

For \( \xi (x) = x^j \), we have

\[ E[X^j (r-1,n,m,k)] - E[X^j (r-1,n-1,m,k)] = \frac{(m+1)C_{r-2}^{(r-1)}}{\gamma_1 (r-2)!} \int_0^1 x^{j-1} \left( 1 - F(x) \right)^{\gamma_r} g_{r-1}^m (F(x)) \, dx \quad (2.8) \]

(iii) \[ E[\xi \{ X (r,n,m,k) \}] - E[\xi \{ X (r-1,n-1,m,k) \}] = \frac{C_{r-2}^{(r-1)}}{(r-1)!} \int_0^1 \xi' (x) \left( 1 - F(x) \right)^{\gamma_r} g_{r-1}^m (F(x)) \, dx \quad (2.9) \]

For \( \xi (x) = x^j \), we have

\[ E[X^j (r,n,m,k)] - E[X^j (r-1,n-1,m,k)] = \frac{C_{r-1}^{(r-1)}}{\gamma_1 (r-1)!} \int_0^1 x^{j-1} \left( 1 - F(x) \right)^{\gamma_r} g_{r-1}^m (F(x)) \, dx \quad (2.10) \]

**Theorem 2.1:** For the given Weibull distribution, truncated from both the sides, and \( n \in \mathbb{N}, m \in \mathbb{R} \), \( k = 1, 2, \ldots \) and \( 2 \leq r \leq n \).

\[ E[X^j (r,n,m,k)] - E[X^j (r-1,n-1,m,k)] = -P_2 K \{ E[X^j (r,n-1,m,k+m)] - E[X^j (r-1,n-1,m,k+m)] \}
+ \frac{j}{p\gamma_1} E[X^{j-p} (r,n,m,k)] \quad , \quad (2.11) \]

where

\[ K = \frac{C_{r-2}^{(r-1)}}{C_{r-2}^{(r-1),k+m}} = \prod_{i=1}^{r-1} \left( \frac{\gamma_i^{(n-1)}}{\gamma_i^{(n-1)} + m} \right), \quad \gamma_i^{(n-1)} = k + (n-1-i)(m+1). \]
Proof: From (2.4) and (2.10), we have

\[ E[X^j(r,n,m,k)] - E[X^j(r-1,n-1,m,k)] \]

\[ = \frac{C_{r-1}}{\gamma_1 (r-1)!} \int_0^1 x^{j-1} [F(x)]^{\gamma r-1} \left\{ -P_2 + \frac{1}{p} x^{1-p} f(x) \right\} g_m^{r-1}(F(x)) \, dx \]

\[ = -P_2 \frac{C_{r-1}}{\gamma_1 (r-1)!} \int_0^1 x^{j-1} [F(x)]^{\gamma r-1} g_m^{r-1}(F(x)) \, dx \]

\[ + \frac{j}{p} \frac{C_{r-1}}{\gamma_1 (r-1)!} \int_0^1 x^{j-1} [F(x)]^{\gamma r-1} f(x) g_m^{r-1}(F(x)) \, dx \]

\[ = -P_2 \frac{C_{r-2}}{\gamma_1 (r-1)!} \int_0^1 x^{j-1} [F(x)]^{\gamma^{n-1,k+m}} g_m^{r-1}(F(x)) \, dx \]

\[ + \frac{j}{p} \int_0^1 E[X^{j-p}(r,n,m,k)] \]

as \( \gamma_r - 1 = \gamma_r^{(n-1,k+m)} = (k + m) + (n-1-r)(m+1) \), \( C_{r-1} = \gamma_1 C_{r-2}^{(n-1)} \) and hence the required result.

If we put \( p = 1 \) in the above expression, we get corresponding result for the exponential distribution. For the non-truncated case one has to put \( P = 1, Q = 0 \).

Remark 2.1: Recurrence relation for single moments of order statistics \((m = 0, k = 1)\) is

\[ E(X^j_{r,n}) - E(X^j_{r-1,n-1}) = -P_2 \{ E(X^j_{r,n-1}) - E(X^j_{r-1,n-1}) \} + \frac{j}{np} E(X^{j-p}_{r,n}) \]

or \( E(X^j_{r,n}) = Q_2 E(X^j_{r-1,n-1}) - P_2 E(X^j_{r,n-1}) + \frac{j}{np} E(X^{j-p}_{r,n}) \).

For \( r = 1 \)
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\[ E(X_{i:n}^j) = Q_2 Q_1^j - P_2 E(X_{i:n-1}^j) + \frac{j}{np} E(X_{i:n}^{j-p}) , \]

For \( r = n \)

\[ E(X_{n:n}^j) = Q_2 E(X_{n-1:n-1}^j) - P_2 P_1^j + \frac{j}{np} E(X_{n:n}^{j-p}) , \]

where by convention we use \( X_{n:n-1} = P_1 \) and \( X_{0:n} = Q_1 \)

as obtained by Khan et al. (1983a).

**Remark 2.2:** For the record values \( (m = -1) \) recurrence relation for single moments reduces as

\[
E(X_r^{(k)})^j - E(X_{r-1}^{(k)})^j = - P_2 \left( \frac{k}{k-1} \right)^{r-1} \{ E(X_r^{(k-1)})^j - E(X_{r-1}^{(k-1)})^j \} + \frac{j}{pk} E(X_r^{(k)})^{j-p}
\]

as \( K = \frac{C_n^{(n-1)}}{C_{n-2}^{(n-1,k+m)}} \prod_{i=1}^{r-1} \left( \frac{k}{k-1} \right) \), \( \gamma_1 = k \), for \( m = -1 \).

where \( X_r^{(k)} = X(r,n-1,k) \), \( r = 1,2,... \) is \( k \)-th record.

Similarly, the recurrence relations for single moments of order statistics with non-integral sample size for \( m = 0 \), \( \gamma_r = \alpha - r + 1 \), \( \alpha \in \mathbb{R}_+ \) may also be obtained.

**Theorem 2.2:** For the given Weibull distribution, truncated from both the sides, and \( n \in \mathbb{N}, m \in \mathbb{N}, k = 1,2,... \) and \( 2 \leq r \leq n \).

\[
E[X^j(r,n,m,k)] - E[X^j(r-1,n-1,m,k)] = \frac{(P-Q)}{p \gamma_1^*} K^* j E[\phi(X(r,n,m,k+1))]
\]  

(2.12)
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\[ \frac{j}{p \gamma_1} \{ -(1 - P) E[\phi(X(r,n,m,k)] + E[X^{j-p}(r,n,m,k)] \} \]  \hspace{1cm} (2.13)

where

\[ \phi(x) = x^{j-p} e^{x^p}, \quad K^* = \frac{C_{r-1}}{C_{r+1}} = \prod_{i=1}^{r} \left( \frac{\gamma_i}{\gamma_i + 1} \right). \]

**Proof:** In view of equation (2.3), (2.10) becomes

\[ E[X^j(r,n,m,k)] - E[X^j(r-1,n-1,m,k)] \]

\[ = \frac{C_{r-1}}{\gamma_1 (r-1)!} \int_0^h x^{j-1} [F(x)]^{\gamma_r} \left\{ \frac{(P-Q) f(x)}{px^{p-1} e^{-x^p}} \right\} g_m^{r-1}(F(x)) \, dx \]

\[ = \frac{(P-Q) C_{r-1}}{p \gamma_1 C_{r+1}} \int_0^h \phi(x) [\bar{F}(x)]^{\gamma_{(k+1)}-1} f(x) g_m^{r-1}(F(x)) \, dx \]

where \( \gamma_{(k+1)} = (k+1) + (n-r)(m+1) \) and hence the Theorem.

To prove (2.13), note that

\[ \frac{\bar{F}(x)}{f(x)} = -\frac{1}{p} \{ (1 - P) x^{1-p} e^{x^p} - x^{1-p} \} \]

and the result follows from (2.10).

**Theorem 2.3:** For the given Weibull distribution, truncated from both the sides, and \( n \in N, m \in \mathbb{R}, k = 1, 2, \ldots \) and \( 2 \leq r \leq n \).

\[ E[X^j(r,n,m,k)] - E[X^j(r-1,n-1,m,k)] \]

\[ = -P_2 K^* \{ E[X^j(r,n-1,m,k+m)] - E[X^j(r-1,n-1,m,k+m)] \} \]

\[ + \frac{j}{p \gamma_r} E[X_{j-p}(r,n,m,k)], \]  \hspace{1cm} (2.14)
where \( K^{**} = \frac{C_{r-2}}{C_{r-2}} = \prod_{i=1}^{r-1} \left( \frac{\gamma_i}{\gamma_i (n-1) + m} \right) = \prod_{i=1}^{r-1} \left( \frac{\gamma_i}{\gamma_i - 1} \right). \)

**Proof:** Proof follows on the lines of Theorem 2.1 using (2.4) and (2.6).

**Remark 2.3:** For the non-truncated exponential distribution *i.e.* at \( p = 1, \ P = 1 \) and \( Q = 0 \), we have

\[
E[X^j(r, n, m, k)] - E[X^j(r-1, n, m, k)] = \frac{j}{\gamma_r} E[X^{j-p}(r, n, m, k)],
\]

as given by Pawlas and Syznal (2001 a) and Cramer and Kamps (2000).

**Theorem 2.4:** For the given Weibull distribution, truncated from both the sides, and \( n \in N, \ m \in \mathbb{R}, \ k = 1, 2, ... \) and \( 2 \leq r \leq n \).

\[
E[X^j(r-1, n, m, k)] - E[X^j(r-1, n-1, m, k)]
= P_2 \frac{(m+1)(r-1)K^{**}}{\gamma_1}
\times \{ E[X^j(r, n-1, m, k + m)] - E[X^j(r-1, n-1, m, k + m)] \}
- \frac{(m+1)(r-1)}{p \gamma_r \gamma_1} j E[X^{j-p}(r, n, m, k)].
\tag{2.15}
\]

**Proof:** Proof follows from (2.4) and (2.8).

**Case II:** \( \gamma_i \neq \gamma_j, \ i, j = 1, 2, \cdots, n-1. \)

We have from Athar and Islam (2004)

(i) \[
E[\xi\{X(r, n, \tilde{m}, k)\}] - E[\xi\{X(r-1, n, \tilde{m}, k)\}]
= C_{r-2} \beta \int_0^\beta \int_0^\alpha \sum_{i=1}^r \alpha_i(r)[1 - F(x)]^{\gamma_i} dx
\tag{2.16}
\]
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(ii) \[ E[\xi\{X(r-1,n,\tilde{m},k)\}] - E[\xi\{X(r-1,n-1,\tilde{m}^*,k)\}] \]

\[
= \frac{\{r-1\} + \sum_{j=1}^{r-1} m_j}{\gamma_1} \beta \int_0^\beta \left[ \xi'(x) \sum_{i=1}^{r} \alpha_i(r)[1 - F(x)]^{\gamma_i} \right] dx 
\tag{2.17}
\]

(iii) \[ E[\xi\{X(r,n,\tilde{m},k)\}] - E[\xi\{X(r-1,n-1,\tilde{m}^*,k)\}] \]

\[
= \frac{\gamma_r}{\gamma_1} \beta \int_0^\beta \left[ \xi'(x) \sum_{i=1}^{r} \alpha_i(r)[1 - F(x)]^{\gamma_i} \right] dx 
\tag{2.18}
\]

where \( \tilde{m}^* = (m_2, m_3, \ldots, m_{n-1}) = \mathcal{N}^{n-2} \).

**Theorem 2.5:** For the given Weibull distribution, truncated from both the sides, and \( n \in \mathbb{N}, 2 \leq r \leq n \) and \( k = 1, 2, \ldots, \),

\[ E[X^J (r,n,\tilde{m},k)] - E[X^J (r-1,n-1,\tilde{m}^*,k)] \]

\[
= \frac{j}{p \gamma_1} \{-(1-P)E[\phi(X(r,n,\tilde{m},k))] + E[X^J-P (r,n,\tilde{m},k)]\}. \tag{2.19}
\]

\[
= \frac{(P-Q)}{p \gamma_1} K^* \int_0^1 x^{j-1} \sum_{i=1}^{r} \alpha_i(r)[\bar{F}(x)]^{\gamma_i} \frac{dx}{\overline{q}_1} \tag{2.20}
\]

**Proof:** From (2.18), we have

\[ E[X^J (r,n,\tilde{m},k)] - E[X^J (r-1,n-1,\tilde{m}^*,k)] \]

\[
= \frac{\gamma_r}{\gamma_1} \beta \int_0^\beta \left[ \xi'(x) \sum_{i=1}^{r} \alpha_i(r)[\bar{F}(x)]^{\gamma_i} \right] dx 
\tag{2.21}
\]

On using equation (2.4), **RHS** of (2.19) becomes

\[
= \frac{\gamma_r}{\gamma_1} \beta \int_0^\beta \left[ \xi'(x) \sum_{i=1}^{r} \alpha_i(r)[\bar{F}(x)]^{\gamma_i} \right] dx 
\]

and hence the required result.

Result (2.20) can be proved by using (2.3) in (2.21).
2.2: Recurrence relations for product moments

Case I: \( m_i = m_j = m \)

We have (Athar and Islam, 2004)
\[
E[\xi\{X(r, n, m, k), X(s, n, m, k)\}] - E[\xi\{X(r, n, m, k), X(s-1, n, m, k)\}]
\]
\[
= \frac{C_{s-2}}{(r-1)!(s-r-1)!} \int \int \frac{\partial}{\partial x} \xi(x, y)[1 - F(x)]^m f(x) g_m^{r-1}(F(x))
\]
\[
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1 - F(y)]^{\gamma_s} \, dy \, dx
\]
(2.22)

where \( \xi(x, y) = \xi_1(x) \xi_2(y) \)

**Theorem 2.6**: For the given Weibull distribution, truncated from both the sides, and \( 1 \leq r < s \leq n-1, \ m \in \mathbb{R}, \ n \geq 2 \) and \( k = 1, 2, \ldots. \)

\[
E[X^i(r, n, m, k) X^j(s, n, m, k)] - E[X^i(r, n, m, k) X^j(s-1, n, m, k)]
\]
\[
= -P_2 K_1 \left\{ E[X^i(r, n-1, m, k + m) X^j(s, n-1, m, k + m)]
\right.
\]
\[
- E[X^i(r, n-1, m, k + m) X^j(s-1, n-1, m, k + m)] \}
\]
\[
+ \frac{j}{p \gamma_s} E[X^i(r, n, m, k) X^{j-p}(s, n, m, k)],
\]
(2.23)

where
\[
K_1 = \frac{C_{s-2}}{C_{s-2}} \prod_{i=1}^{s-1} \frac{\gamma_i \gamma_i^{(n-1)}}{\gamma_i^{(n-1)} + m}
\]

**Proof**: Taking \( \xi(x, y) = x^i y^j \) in (2.22), we have
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\[ E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)] \]

\[ = \frac{C_{s-1}}{\gamma_s(r-1)!(s-r-1)!} \int x^i y^{j-1} [F(x)]^m f(x)g_m^{r-1}(F(x)) \]

\[ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s-1} dy \ dx \] (2.24)

Now using (2.4) in (2.24), we get

\[ E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)] \]

\[ = -P_2 \frac{C_{s-1}}{\gamma_s(r-1)!(s-r-1)!} \int x^i y^{j-1} [F(x)]^m f(x)g_m^{r-1}(F(x)) \]

\[ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s-1} dy \ dx \]

\[ + \frac{C_{s-1}}{p\gamma_s(r-1)!(s-r-1)!} \int x^i y^{j-p} [F(x)]^m g_m^{r-1}(F(x)) \]

\[ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} f(x)f(y) \ dy \ dx \]

\[ = -P_2 \frac{C_{s-2}}{(r-1)!(s-r-1)!} \int x^i y^{j-1} [F(x)]^m f(x)g_m^{r-1}(F(x)) \]

\[ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s^{(n-1,k+m)}} \ dy \ dx \]

\[ + \frac{j}{p\gamma_s} E[X^i(r,n,m,k)X^{j-p}(r,n,m,k)], \]

where \( \gamma_s - 1 = \gamma_s^{(n-1,k+m)}, C_{s-1} = \gamma_s C_{s-2} \) and hence the result.

**Remark 2.4:** At \( p = 1, Q = 0, P = 1 \), Theorem 3.1 reduces to
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\[ E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)] \]

\[ = \frac{j}{\gamma_s} E[X^i(r,n,m,k)X^{j-1}(s,n,m,k)] \]

as obtained by Cramer and Kamps (2000) and for \( s = r + 1 \) and \( j = j + 1 \)

\[ E[X^i(r,n,m,k)X^{j+1}(r+1,n,m,k)] - E[X^i(r,n,m,k)X^{j+1}(r,n,m,k)] \]

\[ = \frac{j+1}{\gamma_{r+1}} E[X^i(r,n,m,k)X^{j}(r+1,n,m,k)] \]

or

\[ E[X^i(r,n,m,k)X^{j+1}(r+1,n,m,k)] \]

\[ = \frac{j+1}{\gamma_{r+1}} E[X^i(r,n,m,k)X^{j}(r+1,n,m,k)] + E[X^{i+j+1}(r,n,m,k)] \]

which is the result given by Pawlas and Syznal (2001 a) for the non-truncated exponential distribution.

**Remark 2.5:** Recurrence relations between product moments of order statistics \((m = 0, k = 1)\) is

\[ E(X_{r,s:n}^{(i,j)}) - E(X_{r,s-1:n}^{(i,j)}) = -P_2 \frac{n}{n-s+1} [E(X_{r,s:n}^{(i,j)}) - E(X_{r,s-1:n-1}^{(i,j)})] \]

\[ + \frac{j}{p(n-s+1)} E(X_{r,s:n}^{(i,j-p)}). \]

as \( K_1 = \frac{n}{n-s+1} \) and \( \gamma_s = n - s + 1 \) for \( m = 0, k = 1 \).

where \( X_{r,s:n}^{(i,j)} = X_{r:n}^{i} X_{s:n}^{j} \)

This is the relation obtained by Khan *et al.* (1983 b).
Case II: \( \gamma_i \neq \gamma_j \)

We have (Athar and Islam, 2004)

\[
E[\xi X(r, n, m, k), X(s, n, m, k)] - E[\xi X(r, n, m, k), X(s-1, n, m, k)]
\]

\[
= C_{s-2} \prod_{\alpha \leq x < y \leq \beta} \frac{\partial}{\partial y} \xi(x, y) \left[ \sum_{i=r+1}^{s} a_i^{(r)}(s) \left( \frac{1 - F(y)}{1 - F(x)} \right)^{\gamma_i} \right]
\]

\[
\times \left[ \sum_{i=1}^{r} a_i(r) \left( \frac{1 - F(y)}{1 - F(x)} \right)^{\gamma_i} \right] \frac{f(x)}{1 - F(x)} dy dx
\]

(2.25)

**Theorem 2.7:** For the given Weibull distribution, truncated from both the sides, and \( 1 \leq r < s \leq n - 1, \ n \geq 2 \) and \( k = 1, 2, \ldots \) and \( m = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1} \)

\[
E\{X^i(r, n, m, k)X^j(s, n, m, k)\} - E\{X^i(r, n, m, k)X^j(s-1, n, m, k)\}
\]

\[
= -\left( \frac{1 - P}{p} \right) \frac{j}{\gamma_s} E[\psi X(r, n, m, k)X(s, n, m, k)]
\]

\[
+ \frac{j}{p \gamma_s} E\{X^i(r, n, m, k)X^{j-p}(s, n, m, k)\}
\]

(2.26)

where \( \psi(x, y) = x^{i}y^{j} \).

**Proof:** Taking \( \xi(x, y) = x^{i}y^{j} \) in (2.25), we have

\[
E\{X^i(r, n, m, k)X^j(s, n, m, k)\} - E\{X^i(r, n, m, k)X^j(s-1, n, m, k)\}
\]

\[
= C_{s-2} \prod_{Q_{ls}^{R_{ls}}} \frac{\partial}{\partial y} x^{i}y^{j} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left( \frac{1 - F(y)}{1 - F(x)} \right)^{\gamma_i}
\]

\[
\times \left[ \sum_{i=1}^{r} a_i(r)[1 - F(x)]^{\gamma_i} \right] \frac{f(x)}{1 - F(x)} dy dx
\]

(2.27)

Multiplying and dividing (2.27) by \( 1 - F(y) \) and using (2.4) we get the required result.
3. Recurrence relations for the moments of the general class of distributions

Consider the general class of distributions

\[ F(x) = 1 - \exp\left(-\frac{1}{c} \{h(x) - h(\alpha)\}\right), \quad x \in (\alpha, \beta) \]  \hspace{1cm} (3.1)

and

\[ f(x) = \frac{h'(x)}{c} \exp\left(-\frac{1}{c} \{h(x) - h(\alpha)\}\right), \quad x \in (\alpha, \beta) \]

where \( c \) is a nonzero real constant and \( h(x) \) is a continuous, monotonic and differentiable function of \( x \) in the interval \((\alpha, \beta)\).

Then

\[ 1 - F(x) = \frac{c}{h'(x)} f(x) \]  \hspace{1cm} (3.2)

3.1: Recurrence relations for single moments

Case 1: \( m_1 = m_2 = ... = m_{n-1} = m \)

**Theorem 3.1:** For distribution given in (3.1) and \( n \in \mathbb{N}, 2 \leq r \leq n, k = 1, 2, ... \) and \( m \in \mathbb{R} \).

\[ E[\xi\{X(r, n, m, k)\}] - E[\xi\{X(r - 1, n, m, k)\}] = \frac{c}{\gamma_r} E[\phi'\{X(r, n, m, k)\}] \]

where \( \phi'(x) = \frac{\xi'(x)}{h'(x)} \)  \hspace{1cm} (3.3)

**Proof:** From (2.5) and (3.2), we have

\[ E[\xi\{X(r, n, m, k)\}] - E[\xi\{X(r - 1, n, m, k)\}] = \frac{c}{(r-2)!} \int_{\alpha}^{\beta} \frac{\xi'(x)[1 - F(x)]^{r-1}}{h'(x)^2} \left(\frac{c}{h'(x)} f(x)\right) g_{\gamma_r}^{r-1}(F(x))dx \]
Recurrence relations for single and product moments of generalized order statistics

\[ E[\xi(X_{r,n})] = E[\xi(X_{r-1,n})] + \frac{c}{(n-r+1)} E[\phi'(X_{r,n})] \]

as obtained by Ali and Khan (1997).

**Remark 3.2:** The recurrence relation for single moments of \( k \)-th record values \((m = -1)\) will be

\[ E[\xi(X_r^{(k)})] = E[\xi(X_{r-1}^{(k)})] + \frac{c}{k} E[\phi'(X_r^{(k)})] \]

where \( X_r^{(k)} = X(r,n,-1,k), r = 1,2,... \) is \( k \)-th record.

Similarly, the recurrence relations for single moments of order statistics with non-integral sample size for \( m = 0, \gamma_r = \alpha + r + 1, \alpha \in \mathbb{R}_+ \) may also be obtained.

**Theorem 3.2:** For the distribution given in (3.1) and \( n \in \mathbb{N}, m \in \mathbb{R}, 2 \leq r \leq n \) and \( k = 1,2,... \)

(i) \[ [E[\xi(X_{r-1,n,m,k})]] - E[\xi(X_{r-1,n-1,m,k}))] = \frac{c(m+1)(r-1)}{\gamma_1 \gamma_r} E[\phi'(X(r,n,m,k))] \] (3.4)

(ii) \[ [E[\xi(X_{r,n,m,k})]] - E[\xi(X_{r-1,n-1,m,k}))] = \frac{c}{\gamma_1} E[\phi'(X(r,n,m,k))] \] (3.5)

**Proof:** Results can be established in view of (2.7), (2.9) and (3.2).
Case II: $\gamma_i \neq \gamma_j$

**Theorem 3.3:** For distribution given in (3.1) and 
$n \in \mathbb{N}, \tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in \mathbb{R}^{n-1}, \tilde{m}^* = (m_2, m_3, \cdots, m_{n-1}) \in \mathbb{R}^{n-2}$ 
$2 \leq r \leq n$ and and $k = 1, 2, \ldots$ 

(i) $E[\xi\{X(r, n, \tilde{m}, k)\}] - E[\xi\{X(r - 1, n, \tilde{m}, k)\}]$

\[= \frac{c}{\gamma_r} E[\phi'(X(r, n, \tilde{m}, k))] \quad (3.6)\]

(ii) $E[\xi\{X(r - 1, n, \tilde{m}, k)\}] - E[\xi\{X(r - 1, n - 1, \tilde{m}^*, k)\}]$

\[= -\frac{c}{\gamma_1 \gamma_r} \left\{ (r - 1) + \sum_{j=1}^{r-1} m_j \right\} E[\phi'(X(r, n, \tilde{m}, k))] \quad (3.7)\]

(iii) $E[\xi\{X(r, n, \tilde{m}, k)\}] - E[\xi\{X(r - 1, n - 1, \tilde{m}^*, k)\}]$

\[= \frac{c}{\gamma_1} E[\phi'(X(r, n, \tilde{m}, k))] \quad (3.8)\]

**Proof:** Results can be established in view of (2.16), (2.17), (2.18) and (3.2).

**Remark 3.3:** Theorems 3.1 and 3.2 can be deduced from Theorems 3.3 by replacing $\tilde{m}$ with $m$, $m \neq -1$.

**Examples**

1. **Burr distribution**

\[F(x) = \left( \frac{\beta}{\beta + x^\tau} \right)^\lambda, \quad 0 < x < \infty\]

Here we have 

\[h(x) = \ln(\beta + x^\tau), \quad c = \frac{1}{\lambda} > 0, \quad \xi(x) = x^{\lambda + \tau}\]
Then \( \phi'(x) = \frac{(j + \tau)}{\tau} (\beta x^j + x^{j+\tau}) \)

Therefore from (3.3), we have

\[
E[X^{j+\tau}(r,n,m,k)] = \frac{\beta(j + \tau)}{\lambda \gamma_r \tau - (j + \tau)} E[X^j (r,n,m,k)]
\]
\[
+ \frac{\lambda \gamma_r \tau}{\lambda \gamma_r \tau - (j + \tau)} E[X^{j+\tau} (r-1,n,m,k)]
\]

as obtained by Pawlas and Szynal (2001 a).

2. Power function distribution

\( F(x) = x^p, \quad 0 < x \leq 1 \)

We have \( h(x) = -\ln(1-x^p), \quad c = 1, \quad \xi(x) = x^{j+1} \)

then \( \phi'(x) = \frac{(j + 1)}{p} (x^{j+1-p} - x^{j+1}) \)

Therefore from (3.3), we get

\[
E[X^{j+1}(r,n,m,k)] = \frac{(j + 1)}{p \gamma_r + (j + 1)} E[X^{j+1-p}(r,n,m,k)]
\]
\[
+ \frac{p \gamma_r}{p \gamma_r + (j + 1)} E[X^{j+1}(r-1,n,m,k)]
\]

At \( p = 1 \), we have

\[
E[X^{j+1}(r,n,m,k)] = \frac{(j + 1)}{\gamma_r + (j + 1)} E[X^j (r,n,m,k)]
\]
\[
+ \frac{\gamma_r}{\gamma_r + (j + 1)} E[X^{j+1} (r-1,n,m,k)]
\]
which is recurrence relation for uniform distribution on (0,1) as obtained by Pawlas and Szynal (2001 a) and Cramer and Kamps (2000).

3. Pareto distribution

\[ F(x) = 1 - a^p x^{-p}, \quad a \leq x < \infty \]

We have

\[ h(x) = \ln x, \quad c = \frac{1}{p}, \quad \xi(x) = x^{j+1} \]

then \( \phi'(x) = (j + 1)x^{j+1} \)

Therefore from (3.3), we have

\[ E[X^{j+1}(r,n,m,k)] = \frac{p\gamma_r}{p\gamma_r - (j + 1)} E[X^{j+1}(r-1,n,m,k)] \]

as obtained by Cramer and Kamps (2000).

4. Weibull distribution

\[ F(x) = 1 - e^{-\theta x^p}, \quad 0 \leq x < \infty, \quad p, \theta > 0 \]

We have

\[ h(x) = x^p, \quad c = \frac{1}{\theta}, \quad \xi(x) = x^{j+1} \]

then \( \phi'(x) = \frac{(j + 1)}{p} x^{j+1-p} \)

Now in view of (3.3), we get

\[ E[X^{j+1}(r,n,m,k)] - E[X^{j+1}(r-1,n,m,k)] = \frac{(j + 1)}{p\theta \gamma_r} E[X^{j+1-p}(r,n,m,k)] \]

which is same as the result (2.14) obtained in Section 2.

Further at \( p = 1 \) and \( \theta = 1 \), this becomes

\[ E[X^{j+1}(r,n,m,k)] - E[X^{j+1}(r-1,n,m,k)] = \frac{(j + 1)}{\gamma_r} E[X^j(r,n,m,k)] \]
which is the recurrence relations for the moments of generalized order statistics from standard exponential distribution as given by Pawlas and Szynal (2001 a) and Cramer and Kamps (2000) as discussed in Section 2.

Similarly recurrence relations for moments of generalized order statistics for some other distributions may be obtained with proper choice of \( h(x) \) and \( c \).

### 3.2: Recurrence relations for product moments

**Case I:** \( m_1 = m_2 = ... = m_{n-1} = m \)

**Theorem 3.4:** For the distribution given in (3.1) and \( n \in \mathbb{N}, m \in \mathbb{N}, 1 \leq r < s \leq n - 1 \).

\[
E[\xi\{X(r, n, m, k), X(s, n, m, k)\}] - E[\xi\{X(r, n, m, k), X(s - 1, n, m, k)\}]
\]

\[
= \frac{c}{\gamma_s} E[\phi'\{X(r, n, m, k), X(s, n, m, k)\}]
\]

(3.9)

where \( \phi'(x, y) = \frac{\partial}{\partial y} \xi(x, y) \), \( \xi(x, y) = \xi_1(x) \xi_2(y) \)

**Proof:** (3.9) can be established in view of (2.22) and (3.2).

**Remark 3.4:** Under the assumption given in Theorem 3.4 with \( k = 1, m = 0 \), we get the recurrence relations for product moments of order statistics

\[
E[\xi(X_{r:n}, X_{s:n})] - E[\xi(X_{r:n}, X_{s-1:n})] = \frac{c}{n - s + 1} E[\phi(X_{r:n}, X_{s:n})]
\]

as obtained by Ali and Khan (1998).

**Remark 3.5:** At \( m = -1 \); we have the recurrence relations for product moments of \( k\)-th record values
Recurrence relations for single and product moments of generalized order statistics

\[ E[\xi (X(r, n, m, k), X(s, n, m, k))] - E[\xi (X(r, n-1, k), (s-1, n-1, k))] \]

\[ = \frac{c}{k} E[\phi' (X(r, n-1, k), X(s, n-1, k))] \]

Case II: \( \gamma_i \neq \gamma_j \)

**Theorem 3.5:** For distribution given in (3.1) and for
\( k, n \in \mathbb{N}, \tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}, 1 \leq r < s \leq n - 1. \)

\[ E[\xi (X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k))] - E[\xi (X(r, n, \tilde{m}, k), X(s-1, n, \tilde{m}, k))] \]

\[ = \frac{c}{\gamma_s} E[\phi' (X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k))] \quad (3.10) \]

**Proof:** Proof follows from (2.25) and (3.2).

**Examples**

1. The Burr distribution

\[ F(x) = \left( \frac{\beta}{\beta + x^\tau} \right)^\lambda, \quad 0 < x < \infty \]

We have,

\[ h(x) = \ln(\beta + x^\tau), \quad c = \frac{1}{\lambda} > 0, \quad \xi_1 (x) = x^i, \quad \xi_2 (y) = y^{j+\tau}, \quad s = r + 1 \]

Thus \( \phi'(x, y) = \frac{\beta (j+\tau)}{\tau} x^i y^j + \frac{(j+\tau)}{\tau} x^i y^{j+\tau} \)

Therefore from (3.9), we have

\[ E[X^i (r, n, m, k) X^{j+\tau} (r+1, n, m, k)] \]

\[ = \frac{(j+\tau)\beta}{\lambda \gamma_{r+1} \tau - (j+\tau)} E[X^i (r, n, m, k) X^j (r+1, n, m, k)] \]

\[ + \frac{\lambda \gamma_{r+1} \tau}{\lambda \gamma_{r+1} \tau - (j+\tau)} E[X^{i+j+\tau} (r, n, m, k)] \]
as obtained by Pawlas and Szynal (2001 a).

2. Power function distribution

\[ F(x) = x^p, \quad 0 < x \leq 1 \]

We have \( h(x) = -\ln(1 - x^p), \quad c = 1, \quad \xi_1(x) = x^i \) and \( \xi_2(x) = y^{j+1} \),

then \( \phi'(x, y) = \frac{(j + 1)}{p} (x^i y^{j+1-p} - x^i y^{j+1}) \)

Therefore from (3.9), we have

\[ E[X^i (r, n, m, k) X^{j+1} (r + 1, n, m, k)] = \frac{(j + 1)}{p \gamma_{r+1} + (j + 1)} E[X^i (r, n, m, k) X^{j+1-p} (r + 1, n, m, k)] \]

\[ + \frac{p \gamma_{r+1}}{p \gamma_{r+1} + (j + 1)} E[X^{i+j+1} (r, n, m, k)] \]

3. Pareto distribution

\[ F(x) = 1 - a^p x^{-p}, \quad a \leq x < \infty \]

We have

\[ h(x) = \ln x, \quad c = \frac{1}{p}, \quad \xi_1(x) = x^i, \quad \xi_2(y) = y^j \]

Then \( \phi'(x, y) = j x^i y^j \)

Thus in view of (3.9)

\[ E[X^i (r, n, m, k) X^j (r + 1, n, m, k)] = \frac{p \gamma_{r+1}}{(p \gamma_{r+1} - j)} E[X^{i+j} (r, n, m, k)] \]

For \( p = 1 \)

\[ E[X^i (r, n, m, k) X^j (r + 1, n, m, k)] = \frac{\gamma_{r+1}}{\gamma_{r+1} - j} E[X^{i+j} (r, n, m, k)] \]
as obtained by Cramer and Kamps (2000).

4. Weibull distribution

\[ F(x) = 1 - e^{-\theta x^p}, \quad 0 \leq x < \infty, \quad p, \theta > 0 \]

We have

\[ h(x) = x^p, \quad c = \frac{1}{\theta}, \quad \xi_1(x) = x^j, \quad \xi_2(y) = y^{j+1} \]

and 

\[ \phi'(x, y) = \frac{(j + 1)}{p} x^j y^{j+1-p} \]

Therefore,

\[
E[X^i(r, n, m, k) X^{j+1}(r + 1, n, m, k)] \\
= \frac{(j + 1)}{p \theta \gamma_{r+1}} E[X^i(r, n, m, k) X^{j+1-p}(r + 1, n, m, k)]
\]

which is same as the result (2.26) obtained in Section 2.

For \( p = 1 \)

\[
E[X^i(r, n, m, k) X^{j+1}(r + 1, n, m, k)] \\
= \frac{(j + 1)}{\theta \gamma_{r+1}} E[X^i(r, n, m, k) X^{j}(r + 1, n, m, k)]
\]

as obtained by Cramer and Kamps (2000).

For \( i = 0 \), these results reduce to the results obtained for single moments.