INTRODUCTION

The concept of generalization of continuous maps were studied by various topologists. Some of them are Dontchev and Przemski[41], Mashhour et al[104,105,107] and Arockiarani and Balachandran[10], Balachandran, Sundram and Maki[15], Sen and Bhattacharya[145], Sengul[146], T. Kesin and Noiri[83] introduced and studied g continuous, preclosed maps, weakly b continuous, almost b continuous maps respectively. In this chapter, we propose the concepts of various types of continuity based on bI sets in simple extension topology. We thus initiate the idea of generalized bI* continuity, approximately gbI* continuous functions, weakly bI* continuity, almost bI* maps and decomposition of almost bI* continuity in this chapter and study their characterizations.

5.1 Generalized bI* Continuity

This section defines a new form of continuity using the concept of generalized bI* closed set.

DEFINITION 5.1.1:
A function \( f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, J) \) is said to be gbI* continuous, if \( f^{-1}(V) \) is gbI* closed in X for every closed set V of Y.

DEFINITION 5.1.2:
A function \( f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, J) \) is said to be gbI* irresolute, if \( f^{-1}(V) \) is gbI* closed in X for every gbI* closed set V of Y.

DEFINITION 5.1.3:
A function \( f : (X, \tau^+) \rightarrow (Y, \sigma^+) \) is said to be gb* continuous, if \( f^{-1}(V) \) is gb* closed in X for every closed set V of Y.

DEFINITION 5.1.4:
A function \( f : (X, \tau^+) \rightarrow (Y, \sigma^+) \) is said to be gb* irresolute, if \( f^{-1}(V) \) is gb* closed in X for every gb* closed set V of Y.

THEOREM 5.1.5:
Let \( f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, J) \) be a map. Then the following are equivalent
(a) \( f \) is gbI\(^+\) continuous.

(b) the inverse image of every open set in \( Y \) is gbI\(^+\) open in \( X \).

**PROOF:**

Let \( f \) be gbI\(^+\) continuous and \( U \) be open in \( Y \). Then \( Y-U \) is closed in \( Y \). Since \( f \) is gbI\(^+\) continuous, \( f^{-1}(Y-U) \) is gbI\(^+\) closed. Therefore \( f^{-1}(U) \) is gbI\(^+\) open.

Let us now assume that the inverse image of every open set in \( Y \) is gbI\(^+\) open in \( X \).

Let \( V \) be closed in \( Y \). Then \( Y-V \) is open in \( X \). By (b) \( f^{-1}(Y-V) \) is gbI\(^+\) open. Hence \( X-f^{-1}(V) \) is gbI\(^+\) closed. Hence \( f \) is gbI\(^+\) continuous.

**THEOREM 5.1.6:**

Let \( f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, J) \) be a map. Then the following are equivalent

(a) \( f \) is gbI\(^+\) irresolute function.

(b) the inverse image of each gbI\(^+\) open set in \( Y \) is gbI\(^+\) open in \( X \).

**PROOF:**

Let \( f \) be gbI\(^+\) irresolute and \( U \) be gbI\(^+\) open in \( Y \). Then \( Y-U \) is gbI\(^+\) closed in \( Y \). Since \( f \) is gbI\(^+\) irresolute, \( f^{-1}(Y-U) \) is gbI\(^+\) closed. Therefore \( f^{-1}(U) \) is gbI\(^+\) open. Let us now assume that the inverse image of every gbI\(^+\) open set in \( Y \) is gbI\(^+\) open in \( X \). Let \( V \) be gbI\(^+\) closed in \( Y \). Then \( Y-V \) is gbI\(^+\) open in \( X \) which implies, \( f^{-1}(Y-V)=X-f^{-1}(V) \) is gbI\(^+\) open. Hence \( f \) is gbI\(^+\) irresolute.

**THEOREM 5.1.7:**

If \( f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, J) \) is biI\(^+\) continuous then \( f \) is gbI\(^+\) continuous.

**PROOF:**

Let \( V \) be any closed set in \( Y \). Since \( f \) is biI\(^+\) continuous, \( f^{-1}(V) \) is biI\(^+\) closed. But since every biI\(^+\) closed set is gbI\(^+\) closed we have \( f^{-1}(V) \) to be gbI\(^+\) closed. Hence \( f \) is gbI\(^+\) continuous.

**THEOREM 5.1.8:**

Let \( f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, J) \) be a function

(i) If \( f \) is gbI\(^+\) continuous, then for each point \( x \in X \) and for each open set \( V \) containing \( f(x) \) there exists a gbI\(^+\) open set \( U \) containing \( x \) such that \( f(U) \) is contained in \( V \).

(ii) If \( f \) is bijective, biI\(^+\) irresolute and open then \( f \) is gbI\(^+\) irresolute.
PROOF:

(i) Let \( x \in X \) and \( V \) be a open set in \( Y \) containing \( f(x) \). Since \( f \) is \( \text{gbI}^+ \) continuous, \( f^{-1}(V) \) to be \( \text{gbI}^+ \) open in \( X \). Let \( U = f^{-1}(V) \) then \( x \in U \) and \( f(U) \) is contained in \( V \).

(ii) Let \( V \) be \( \text{gbI}^+ \) closed set in \( Y \). Let \( U \) be any open set in \( X \) containing \( f^{-1}(V) \). i.e., \( f^{-1}(V) \subseteq U \Rightarrow V \subseteq f(U) \). Since \( f \) is open, \( f(U) \) is also open in \( Y \). Since \( V \) is \( \text{gbI}^+ \) closed in \( Y \), \( \text{bl}^+ \text{cl}(V) \subseteq f(U) \). Hence \( f^{-1}(\text{bl}^+ \text{cl}(V)) \subseteq f^{-1}(U) = U \). Since \( f \) is bijective \( \text{bl}^+ \text{cl}(V) \) is \( \text{bl}^+ \) closed in \( X \). Hence \( \text{bl}^+ \text{cl}(f^{-1}(V)) \subseteq \text{bl}^+ \text{cl}(f^{-1}(\text{bl}^+ \text{cl}(V))) = f^{-1}(\text{bl}^+ \text{cl}(V)) \subseteq U \). Thus \( f^{-1}(V) \) is \( \text{gbI}^+ \) closed. Hence \( f \) is \( \text{gbI}^+ \) irresolute.

DEFINITION 5.1.9:
A SEITS \((X, \tau^+, I)\) is called \( \text{bl}^+T_{1/2} \) space if every \( \text{gbI}^+ \) closed set is \( \text{bl}^+ \) closed.

THEOREM 5.1.10:
Let \( f : (X, \tau^+, I) \to (Y, \sigma^+, J) \) be a \( \text{gbI}^+ \) irresolute map. If \((X, \tau^+, I)\) is \( \text{bl}^+T_{1/2} \) space then \( f \) is a \( \text{bl}^+ \) irresolute map.

PROOF:
Let \( V \) be \( \text{gbI}^+ \) closed set in \( Y \). Since \( f \) is \( \text{gbI}^+ \) irresolute, \( f^{-1}(V) \) is \( \text{gbI}^+ \) closed. Since \( X \) is a \( \text{bl}^+T_{1/2} \) space, \( f^{-1}(V) \) is \( \text{bl}^+ \) closed. Hence \( f \) is a \( \text{bl}^+ \) irresolute map.

REMARK 5.1.11:
\( \text{gbI}^+ \) irresoluteness and \( \text{bl}^+ \) irresoluteness are independent as shown by the following example.

EXAMPLE 5.1.12:
(i) Let \( X \) be any arbitrary set and \( p \in X, \tau = \{X, \varnothing\} \) \( \tau^+ = \{X, \varnothing, \{b\}\} \); \( I = \{A \subseteq X / p \in A\}; \sigma = \varnothing(X) \)
let \( f : (X, \tau^+, I) \to (Y, \sigma^+, J) \) be the identity map. In \( Y \) every subset is \( \text{b}^+ \) closed and \( \text{gb}^+ \) closed.
In \( X \), every subset is \( \text{gbI}^+ \) closed. Therefore \( f \) is \( \text{gbI}^+ \) irresolute. Assume \( A \) to be a set of \( X \) not containing \( p \), then \( f^{-1}(A) = A \) is not \( \text{bl}^+ \) closed. Therefore \( f \) is not \( \text{bl}^+ \) irresolute.
Hence \( f \) is \( \text{gbI}^+ \) irresolute but not \( \text{bl}^+ \) irresolute.

(ii) Let \( X = \{a, b, c\} \) \( \tau = \{X, \varnothing, \{a\}\}; B = \{a, b\}; \tau^+ = \{X, \varnothing, \{a\}, \{a, b\}\} I = \{\varnothing\} \) and \( Y = \{a, b, c\} \).
Here a set is $b^+$ open iff it is $bl^+$ open and $gbl^+$ open iff it is $gbI^+$ open.

Let $f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, J)$ be the identity map. Then the inverse image of $b^+$ open sets in $Y$ are $b^+$ open in $X$. Therefore $f$ is $b^+$ irresolute. But $\{c\}$ is $gbI^+$ open in $Y$ and $f^{-1}(\{c\}) = \{c\}$ is not $gbI^+$ open in $X$. Hence $f$ is not $gbI^+$ irresolute.

**THEOREM 5.1.13:**

Let $f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, I)$ and $g : (Y, \sigma^+, I) \rightarrow (Z, \mu^+, K)$ be any two functions then

(a) $(g \circ f)$ is $gbI^+$ continuous if $g$ is continuous and $f$ is $gbI^+$ continuous.

(b) $(g \circ f)$ is $gbI^+$ irresolute if $g$ is $gb^+$ irresolute and $f$ is $gbI^+$ irresolute.

(c) $(g \circ f)$ is $gbI^+$ continuous if $g$ is $gb^+$ continuous and $f$ is $gbI^+$ irresolute.

(d) $(g \circ f)$ is $bI^+$ continuous if $(X, \tau^+, I)$ is $bI^+T_{1/2}$ and $g$ is $gb^+$ continuous and $f$ is $gbI^+$ irresolute.

**PROOF:**

(a) Let $g : (Y, \sigma^+, J) \rightarrow (Z, \mu^+, K)$ is continuous and $f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, J)$ is $gbI^+$ continuous.

To prove : $(g \circ f) : (X, \tau^+, I) \rightarrow (Z, \mu^+, K)$ is $gbI^+$ continuous.

Let $V$ be a closed set in $Z$. $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$

Since $g$ is continuous, $g^{-1}(V)$ is closed. Since $f$ is $gbI^+$-continuous, $f^{-1}(g^{-1}(V))$ is $gbI^+$-closed. Therefore $f$ is $gbI^+$-continuous.

(b) To prove: $(g \circ f) : (X, \tau^+, I) \rightarrow (Z, \mu^+, K)$ is $gbI^+$ irresolute.

Let $V$ be a $gbK^+$-closed in $Z$. $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$

Since $g$ is $gbI^+$-irresolute $g^{-1}(V)$ is $gbI^+$-closed in $Y$. Since $f$ is $gbI^+$-irresolute $f^{-1}(g^{-1}(V))$ is $gbI^+$-closed in $X$.

(c) To prove: $(g \circ f) : (X, \tau^+, I) \rightarrow (Z, \mu^+, K)$ is $gbI^+$ continuous.

Let $V$ be a closed in $Z$. $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$

Since $g$ is $gbI^+$-continuous, $g^{-1}(V)$ is $gbI^+$-closed in $Y$. Since $f$ is $gbI^+$-irresolute $f^{-1}(g^{-1}(V))$ is $gbI^+$-closed in $X$. Hence $(g \circ f)$ is $gbI^+$ continuous.

(d) To prove: $(g \circ f) : (X, \tau^+, I) \rightarrow (Z, \mu^+, K)$ is $bI^+$ continuous.

Let $V$ be a closed in $Z$.

To prove: $(g \circ f)^{-1}(V)$ is $bI^+$-closed in $X$.
\((g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))\).

Since \(g\) is gbI\(^+\)-continuous, \(g^{-1}(V)\) is gbI\(^+\)-closed.

Since \(f\) is gbI\(^+\)-irresolute \(f^{-1}(g^{-1}(V))\) is gbI\(^+\)-closed in \(X\).

Since \((X, \tau^+, I)\) is bI\(^+\)-T\(_{1/2}\) space, \(f^{-1}(g^{-1}(V))\) is bI\(^+\)-closed.

Hence \((g \circ f)\) is bI\(^+\)-continuous.

### 5.2 Approximately gbI\(^+\) Continuous Functions

This section defines and discusses the characterizations of approximately gbI\(^+\)continuous function.

**DEFINITION 5.2.1:**
A map \(f : (X, \tau^+, I) \to (Y, \sigma^+, J)\) is said to be a approximately bI\(^+\) continuous (apbI\(^+\) continuous) if \(bI^+ \text{cl}(F) \subseteq f^{-1}(U)\) whenever \(U\) is an open subset of \(Y\) and \(F\) is a gbI\(^+\) closed subset of \(X\) such that \(F \subseteq f^{-1}(U)\).

**DEFINITION 5.2.2:**
A map \(f : (X, \tau^+, I) \to (Y, \sigma^+, J)\) is said to be a approximately bI\(^+\) closed (apbI\(^+\) closed) if \(f(F) \subseteq bI^+ \text{int}(V)\) whenever \(V\) is a gbI\(^+\) open subset of \(Y\) and \(F\) is a closed subset of \(X\), \(f(F) \subseteq (V)\).

**DEFINITION 5.2.3:**
A map \(f : (X, \tau^+, I) \to (Y, \sigma^+, J)\) is said to be a approximately bI\(^+\) open (apbI\(^+\) open) if \(bI^+ \text{cl}(F) \subseteq f(V)\) whenever \(U\) is an open subset of \(X\), \(F\) is a gbI\(^+\) closed subset of \(Y\) and \(F \subseteq f(U)\).

**DEFINITION 5.2.4:**
A function \(f : (X, \tau^+, I) \to (Y, \sigma^+, J)\) is said to be bI\(^+\) closed (resp. bI\(^+\) open) if for every bI\(^+\) closed (resp. bI\(^+\) open) subset \(A\) of \(X\), \(f(A)\) is bI\(^+\) closed in \(Y\).

**DEFINITION 5.2.5:**
A map \(f : (X, \tau^+, I) \to (Y, \sigma)\) is said to be contra bI\(^+\) closed (resp. contra bI\(^+\) open) if \(f(U)\) is bI\(^+\) open (resp. bI\(^+\) closed) in \(Y\) for each closed (resp. open) set \(U\) of \(X\).
THEOREM 5.2.6:

Let $f: (X, \tau^+, I) \to (Y, \sigma^+, J)$ be a function. Then

1) If $f$ is contra $b I^+$ continuous, then $f$ is ap $b I^+$ continuous.

2) If $f$ is contra $b I^+$ closed, then $f$ is ap $b I^+$ closed.

3) If $f$ is contra $b I^+$ open, then $f$ is ap $b I^+$ open.

PROOF:

(1) Let $F \subseteq f^{-1}(U)$, where $U$ is open in $Y$ and $F$ is a $g b I^+$ closed subset of $X$. Therefore $b I^+ \text{cl}(F) \subseteq b I^+ \text{cl}(f^{-1}(U))$. Since $f$ is contra $b I^+$ continuous we have $b I^+ \text{cl}(F) \subseteq b I^+ \text{cl}(f^{-1}(U)) = f^{-1}(U)$. Thus $b I^+ \text{cl}(F) \subseteq f^{-1}(U)$. Hence $f$ is ap $b I^+$ continuous.

(2) Let $f(F) \subseteq V$, where $F$ is a closed subset of $X$ and $V$ is a $g b I^+$ open subset of $Y$. Therefore $f(F) = b I^+ \text{int} f(F) \subseteq b I^+ \text{int}(V)$. Thus $f$ is ap $b I^+$ closed.

3) Similar to (1) & (2).

DEFINITION 5.2.7:

A function $f: (X, \tau^+, I) \to (Y, \sigma^+, J)$ is said to be pre $I^+$ closed (resp pre $I^+$ open) if for every pre $I^+$ closed (resp pre $I^+$ open) subset $A$ of $X$, $f(A)$ is pre $I^+$ closed (resp pre $I^+$ open) in $Y$.

REMARK 5.2.8:

i) Clearly every continuous map in $\tau$ is ap $b I^+$ continuous

ii) Every pre $I^+$ closed map is ap $I^+$ closed.

THEOREM 5.2.9:

Let $f: (X, \tau^+, I) \to (Y, \sigma^+, J)$ be a map.

(1) If the open and $b I^+$ closed sets of $(X, \tau^+, I)$ coincide, then $f$ is ap $b I^+$ continuous if and only if $f$ is contra $b I^+$ continuous.

(2) If the open and $b I^+$ closed sets of $(Y, \sigma^+, J)$ coincide, then $f$ is ap $b I^+$ closed if and only if $f$ is contra $b I^+$ closed.

(3) If the open and $b I^+$ closed sets of $(Y, \sigma^+, J)$ coincide, then $f$ is ap $b I^+$ open if and only if $f$ is contra $b I^+$ open.
PROOF:

(1) Assume that $f$ is $\text{ap bi}^+$ continuous. Let $A$ be an arbitrary subset of $(X, \tau^+, I)$ such that $A \subseteq U$, where $U$ is open in $X$. Then by hypothesis $\text{bi}^+ \text{cl}(A) \subseteq \text{bi}^+ \text{cl}(U) = U$. Therefore all subsets of $(X, \tau^+, I)$ are $g \text{bi}^+$ closed (hence all are $g \text{bi}^+$ open). So for any open set $V$ of $(Y, \sigma^+, J)$ we have $f^{-1}(V)$ is $g \text{bi}^+$ closed in $(X, \tau^+, I)$. Since $f$ is $\text{ap bi}^+$ continuous, $\text{bi}^+ \text{cl}(f^{-1}(V)) \subseteq f^{-1}(V)$. Therefore $\text{bi}^+ \text{cl}(f^{-1}(V)) = f^{-1}(V)$. Thus $\text{bi}^+ \text{cl}(f^{-1}(V))$ is $\text{bi}^+$ closed in $(X, \tau^+, I)$ and $f$ is contra $\text{bi}^+$ continuous.

The converse is clearly true by theorem 5.2.6.

(2) Assume that $f$ is $\text{ap bi}^+$ closed. As in (1), we obtain that all subsets of $(Y, \sigma^+, J)$ are $g \text{bi}^+$ open. Therefore for any closed subset $F$ of $(X, \tau^+, I)$, $f(F)$ is $g \text{bi}^+$ open in $Y$. Since $f$ is $\text{ap bi}^+$ closed, we have $f(F) \subseteq \text{bi}^+ \text{int}(f(F))$. Therefore $f(F) = \text{bi}^+ \text{int}(f(F))$ and hence $f$ is contra $\text{bi}^+$ closed.

The converse is true by theorem 5.2.6

(3) Analogous to (1) and (2) making obvious changes.

LEMMA 5.2.10:
Let $A$ be a subset of a space $(X, \tau^+, I)$. Then

i) $\text{bi}^+ \text{cl}(A) = \text{sI}^+ \text{cl}(A) \cap \text{p I}^+ \text{cl}(A) = A \cup \{ (\text{int}(\text{cl}^+(A)) \cap \text{cl}^+(\text{int}(A)) \}$

ii) $\text{bi}^+ \text{int}(A) = \text{sI}^+ \text{int}(A) \cup \text{p I}^+ \text{int}(A) = A \cap \{ (\text{int}(\text{cl}^+(A)) \cup \text{cl}^+(\text{int}(A)) \}$

iii) $\text{bi}^+ \text{cl}(X \setminus A) = X \setminus \text{bi}^+ \text{int}(A)$

iv) $\text{bi}^+ \text{int}(X \setminus A) = X \setminus \text{bi}^+ \text{cl}(A)$

THEOREM 5.2.11:
If a map $f: (X, \tau^+, I) \rightarrow (Y, \sigma^+, J)$ is surjective, $\text{bi}^+$ irresolute and $\text{ap bi}^+$ closed, then the inverse image of each $g \text{bi}^+$ closed (resp. $g \text{bi}^+$ open) set in $Y$ is $g \text{bi}^+$ closed (resp. $g \text{bi}^+$ open) in $X$.

PROOF:
Let $A$ be a $g \text{bi}^+$ closed subset of $Y$. Suppose that $f^{-1}(A) \subseteq U$ where $U$ is an open subset of $X$. Taking complements we have, $X \setminus U \subseteq f^{-1}(Y \setminus A)$ (or) $f(X \setminus U) \subseteq (Y \setminus A)$. Since $f$ is $\text{ap bi}^+$ closed and by lemma 5.2.10, then $f(X \setminus U) \subseteq \text{bi}^+ \text{int}(Y \setminus A) = Y \setminus \text{bi}^+ \text{cl}(A)$. It follows that $X \setminus U \subseteq X \setminus (f^{-1}$
(bI^+ \text{cl } (A)) and hence \( f^{-1}(bI^+ \text{cl}(A)) \subseteq U.\) Since \( f \) is bI^+ irresolute, \( f^{-1} ( bI^+ \text{cl } (A)) \) is bI^+ closed. Thus we have bI^+ cl ( f^{-1} ( A)) \subseteq bI^+ cl ( f^{-1}bI^+ \text{cl } A) = f^{-1} ( bI^+ \text{cl } (A)) \subseteq U.\) This implies that \( f^{-1} ( A) \) is gbI^+ closed in X.

A similar argument shows that inverse images of gbI^+ open sets are gbI^+ open.

**THEOREM 5.2.12 :**

If a map \( f : ( X, \tau^+, I) \rightarrow ( Y, \sigma^+, J) \) is surjective bI^+ irresolute and ap bI^+ open, then the inverse image of each gbI^+ open set in Y is gbI^+ open in X.

**PROOF :**

Analogous to theorem 5.2.11 by including necessary changes.

**THEOREM 5.2.13 :**

If a map \( f : ( X, \tau^+, I) \rightarrow ( Y, \sigma^+, J) \) is apbI^+ continuous and bI^+ closed, then the image of each gbI^+ closed set in X is gbI^+ closed in Y.

**PROOF :**

Let F be a gbI^+ closed subset of X. Let \( f(F) \subseteq V \) where \( V \) is open in Y.

Then \( F \subseteq f^{-1} ( V) \) holds. Since \( f \) is apbI^+ continuous, we have \( bI^+ \text{cl } (F) \subseteq f^{-1} ( V) \)

Then \( f ( bI^+ \text{cl } (F)) \subseteq V. \) Therefore, we have \( bI^+ \text{cl } ( f (F)) \subseteq bI^+ \text{cl } ( f ( bI^+ \text{cl } (F))) = f ( bI^+ \text{cl } (F))) \subseteq V. \) Hence \( f(F) \) is gbI^+ closed in Y.

**THEOREM 5.2.14 :**

If \( f : ( X, \tau^+, I) \rightarrow ( Y, \sigma^+, J) \) is a continuous and bI^+ closed function, then \( f(A) \) is gbI^+ closed in Y for every gbI^+ closed set A of X.

**PROOF :**

Let A be a gbI^+ closed set in X. Let \( f(A) \subseteq V, \) where \( V \) be any open set in Y. Since \( f \) is continuous, \( f^{-1} ( V) \) is open in X and \( A \subseteq f^{-1} ( V). \) Then we have \( bI^+ \text{cl } (A) \subseteq f^{-1} ( V) \) and so \( f ( bI^+ \text{cl } (A)) \subseteq V. \) Since \( f \) is bI^+ closed, \( f ( bI^+ \text{cl } (A)) \) is bI^+ closed in Y and hence \( bI^+ \text{cl } ( f(A)) \subseteq bI^+ \text{cl } ( f ( bI^+ \text{cl } (A))) = f ( bI^+ \text{cl } (A))) \subseteq V. \) This shows that \( f(A) \) is gbI^+ closed in Y.
DEFINITION 5.2.15:
A map $f : (X, \tau^+, I) \to (Y, \sigma^+, J)$ is said to be contra bi+ irresolute if $f^{-1}(U)$ is bi+ closed in $X$ for each $U \in BI^+O(Y)$.

THEOREM 5.2.16:
Let $f : (X, \tau^+, I) \to (Y, \sigma^+, J)$ and $g : (Y, \sigma^+, J) \to (Z, \delta^+, k)$ be two maps such that $(g \circ f) : (X, \tau^+, I) \to (Z, \delta^+, k)$

1. If $g$ is bi+ continuous and $f$ is contra bi+ irresolute, then $(g \circ f)$ is contra bi+ continuous.
2. If $g$ is bi+ irresolute and $f$ is contra bi+ irresolute, then $(g \circ f)$ is contra bi+ irresolute.

PROOF: Obvious.

THEOREM 5.2.17:
Let $f : (X, \tau^+, I) \to (Y, \sigma^+, J)$ and $g : (Y, \sigma^+, J) \to (Z, \delta^+, K)$ be two maps such that $(g \circ f) : (X, \tau^+, I) \to (Z, \delta^+, K)$

1. If $f$ is preI+ closed and $g$ is ap bi+ closed, then $(g \circ f)$ is apbi+ closed.
2. If $f$ is apbi+ closed and $g$ is bi+ open, and $g^{-1}$ preserves gbI+ open sets, then $(g \circ f)$ is ap bi+ closed.
3. If $f$ is apbi+ continuous and $g$ is continuous, then $(g \circ f)$ is ap bi+ continuous.

PROOF:
(1) Suppose $B$ is an arbitrary closed subset in $X$ and $A$ is a gbI+ open subset of $Z$ for which $(g \circ f)(B) \subseteq A$. Then $f(B)$ is closed in $Y$ because $f$ is pre I+ closed. Since $g$ is an apbi+ closed set, $g(f(B)) \subseteq bi^+ int(A)$. This implies that $(g \circ f)$ is apbi+ closed.
(2) Suppose $B$ is an arbitrary closed subset of $X$ and $A$ is a gbI+ open subset of $Z$ for which $(g \circ f)(B) \subseteq A$. Hence $f(B) \subseteq g^{-1}(A)$.
Then $f(B) \subseteq bi^+ int(g^{-1}(A))$ because $g^{-1}(A)$ is gbI+ open and $f$ is apbi+ closed. Thus $(g \circ f)(B) = g(f(B)) \subseteq g ( bi^+ int(g^{-1}(A))) \subseteq bi^+ int(g g^{-1}(A)) \subseteq bi^+ int(A)$.
This implies that $(g \circ f)$ is apbi+ closed.
(3) Suppose $F$ is an arbitrary $g \beta I^+$ closed subset of $X$ and $U$ is a open set in $Z$ for which $F \subseteq (g \circ f)^{-1}(U)$. Then $g^{-1}(U) \in O(Y)$ because $g$ is continuous. Since $f$ is $ap\beta I^+$ continuous, then we have $bI^+ cl(F) \subseteq f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. This implies $(g \circ f)$ is $ap\beta I^+$ continuous.

5.3 Weakly $bI^+$ Continuous Functions
This section devotes itself to the notion of another type of continuity known as weakly $bI^+$ continuity and hence its properties are discussed.

**DEFINITION 5.3.1:**
A function $f: (X,\tau^+,I) \rightarrow (Y,\sigma^+,J)$ is said to be weakly $bI^+$ continuous functions ($w.bI^+.c$) if for each open set $V$ of $Y$ containing $f(x)$, there exists $U \in B^+O(X,x)$ such that $f(U) \subseteq \text{cl}^+(V)$.

**REMARK 5.3.2:**
(i) Every weakly continuous function is weakly $bI^+$ continuous.
(ii) Weakly $bI^+$ continuity implies weak $\beta I^+$ continuity.

The converse of these implications are not true in general.

**EXAMPLE 5.3.3:**
Let $X=Y=\{a,b,c\}$, $\tau = \{X,\varnothing\}; B=\{c\}; \tau^+ = \{X, \varnothing, \{c\}\}$; $\sigma = \{Y, \varnothing, \{b\},\{a,b\}\}; B = \{a\}; \sigma^+ = \{Y, \varnothing, \{b\},\{a\},\{a,b\}\}; I = \{a\}$; $f: (X,\tau^+,I) \rightarrow (Y,\sigma^+,J)$ identity function. Then $f$ is weakly $bI^+$ continuous but not weakly continuous and not $bI^+$ continuous.

**LEMMA 5.3.4:**
For a function $f: (X,\tau^+,I) \rightarrow (Y,\sigma^+,J)$ the following are equivalent
(a) $f$ is weakly $bI^+$ continuous at $x \in X$.
(b) $x \in \text{cl}^+(\text{int}(f^+(\text{cl}^+(V)))) \cup \text{int}(\text{cl}^+(f^+(\text{cl}^+(V))))$ for each neighbourhood $V$ of $f(x)$.
(c) $f^{-1}(V) \subseteq\beta I^+ \text{int}(f^+(\text{cl}^+(V)))$ for every open set $V$ of $Y$. 

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PROOF:

(a)⇒(b): Let V be any neighbourhood of f(x). Since f is (w.bI⁺) at x there exists U ∈ BI⁺O(X,x) such that f(U) ⊆ cl⁺(V). Then U ⊆ f⁻¹(cl⁺(V)). Since U is bI⁺ open,

\[ x ∈ U ⊆ \text{int}(cl⁺(U)) ∪ cl⁺(\text{int}(U)) \subseteq \text{int}(cl⁺(f⁻¹(cl⁺(V)))) ∪ cl⁺(\text{int}(f⁻¹(cl⁺(V)))) . \]

(b)⇒(c): Let x ∈ f⁻¹(V) so f(x) ∈ V. Then x ∈ f⁻¹(cl⁺(V)) and since

\[ x ∈ \text{int}(cl⁺(f⁻¹(cl⁺(V)))) ∪ cl⁺(\text{int}(f⁻¹(cl⁺(V)))) \]

Hence f⁻¹(V) ⊆ bI⁺ int(f⁻¹(cl⁺(V))).

(c)⇒(a): Let V be any neighbourhood of f(x) then x ∈ f⁻¹(V) ⊆ bI⁺ int(f⁻¹(cl⁺(V))).

Let U ⊆ bI⁺ int(f⁻¹(cl⁺(V))) then U ∈ BI⁺O(X,x) and f(U) ⊆ cl⁺(V). Hence f is weakly bI⁺ continuous at x ∈ X.

THEOREM 5.3.5:

For a function f: (X,τ⁺,I) → (Y,σ⁺,J) the following are equivalent

(a) f is weakly bI⁺ continuous at x ∈ X.

(b) bI⁺ cl(f⁻¹(int(cl⁺(B)))) ⊆ f⁻¹(cl⁺(B)) for every subset B of Y.

(c) bI⁺ cl f⁻¹(F) ⊆ f⁻¹(F) for every regular closed subset F of Y.

(d) bI⁺ cl(f⁻¹(V)) ⊆ f⁻¹(cl⁺(V)) for every open set V of Y.

(e) f⁻¹(V) ⊆ bI⁺ int(f⁻¹(cl⁺(V))) for every open set V of Y.

(f) f⁻¹(V) ⊆ cl⁺(int(f⁻¹(cl⁺(V)))) ∪ int(cl⁺(f⁻¹(cl⁺(V)))) for every open set V of Y.

PROOF:

(a)⇒(b): Let B be any subset of Y. Assume that x ∈ X \ f⁻¹(cl⁺(B)).

Then f(x) ∈ Y \ (cl⁺(B)) and there exists an open set V containing f(x) such that

\[ V \cap B = \emptyset, \text{hence } cl⁺(V) \cap int(cl⁺(B)) = \emptyset. \]

Since f is weakly bI⁺ continuous there is a U ∈ BI⁺ O(X,x) such that f(U) ⊆ cl⁺(V). Therefore, we have

\[ U \cap f⁻¹(int(cl⁺(B))) = \emptyset \text{ hence } x ∈ X \setminus bI⁺ cl⁺(f⁻¹ (int(cl⁺(B)))) . \]

Thus we obtain
\[ bI^* \text{cl}(f^{-1}(\text{int}(\text{cl}^+(B)))) \subseteq f^{-1}(\text{cl}^+(B)). \]

(b) \implies (c): Let \( F \) be any regular closed set of \( Y \).

Then we have \( bI^* \text{cl}(f^{-1}(\text{int}(F))) \subseteq bI^* \text{cl}(f^{-1}(\text{int}(F))) \subseteq f^{-1}(\text{int}(F)) \subseteq f^{-1}(F). \)

(c) \implies (d): For any open set \( V \) of \( Y \), \( \text{cl}^+(V) \) is regular closed in \( Y \) and we have

\[ bI^* \text{cl}(f^{-1}(V)) \subseteq bI^* \text{cl}(f^{-1}(\text{int}(\text{cl}^+(V)))) \subseteq f^{-1}(\text{cl}^+(V)). \]

(d) \implies (e): Let \( V \) be any open set of \( Y \), then \( Y \setminus \text{cl}^+(V) \) is open in \( Y \) and using lemma 5.2.10(iii), we have

\[ \text{cl}^+(Y \setminus \text{cl}^+(V)) \subseteq X \setminus f^{-1}(V). \]

Therefore we obtain \( f^{-1}(V) \subseteq bI^* \text{int}(f^{-1}(\text{cl}^+(V))). \)

(e) \implies (f): Let \( V \) be any open set of \( Y \). Using lemma 5.3.4 we have

\[ f^{-1}(V) \subseteq bI^* \text{int}(f^{-1}(\text{cl}^+(V))) \subseteq \text{cl}^+(\text{int}(f^{-1}(\text{cl}^+(V)))) \cup \text{int}(\text{cl}^+(f^{-1}(\text{cl}^+(V)))). \]

(f) \implies (a): Let \( x \) be any point of \( X \) and \( V \) be any open set of \( Y \) containing \( f(x) \). Then

\[ x \in f^{-1}(V) \subseteq \text{cl}^+(\text{int}(f^{-1}(\text{cl}^+(V)))) \cup \text{int}(\text{cl}^+(f^{-1}(\text{cl}^+(V)))). \]

It follows from lemma 5.3.4 that \( f \) is weakly \( bI^* \) continuous.

**THEOREM 5.3.6:**

The following are equivalent for a function \( f: (X, \tau^+, I) \rightarrow (Y, \sigma^+, J) \)

(a) \( f \) is weakly \( bI^* \) continuous at \( x \in X \).

(b) \( bI^* \text{cl}(f^{-1}(\text{int}(\text{cl}^+(V)))) \subseteq f^{-1}(\text{cl}^+(V)) \) for each \( V \in BI^*O(Y) \).

(c) \( bI^* \text{cl}(f^{-1}(V)) \subseteq f^{-1}(\text{cl}^+(V)) \) for each \( V \in PI^*O(Y) \).

(d) \( f^{-1}(V) \subseteq bI^* \text{int}(f^{-1}(\text{cl}^+(V))) \) for each \( V \in PI^*O(Y) \).

(e) \( bI^* \text{cl}(f^{-1}(\text{int}(\text{cl}^+(V)))) \subseteq f^{-1}(\text{cl}^+(V)) \) for each \( V \in SI^*O(Y) \).

**PROOF:**

(a) \implies (b): Let \( V \) be any \( bI^* \) open set of \( Y \). Hence that \( \text{cl}^+(V) \) is regular closed .Since \( f \) is weakly \( bI^* \) continuous from theorem 5.3.5 we have \( bI^* \text{cl}(f^{-1}(\text{int}(\text{cl}^+(V)))) \subseteq f^{-1}(\text{cl}^+(V)) \).

(b) \implies (c): Clear, since \( PI^* O(Y) \subseteq BI^* O(Y) \) and \( V \subseteq \text{int}(\text{cl}^+(V)) \).

(c) \implies (d): This is similar to the proof of the implication (d) \implies (e) in theorem 5.3.5.

(d) \implies (a): This follows from theorem 5.3.5 since every open set is \( \text{preI}^* \)-open.

(b) \implies (c): Clear, since \( SI^* O(Y) \subseteq BI^* O(Y) \).
(e) ⇒(a): Let \( F \) be any regular closed in \( Y \). Then \( F \) is semi \( I^+ \)-open in \( Y \) and
\[
\text{bl}^* \text{cl}(f^{-1}((\text{int}(\text{cl}(F)))) \subset f^{-1}(\text{cl}(F)) = f^{-1}(F)
\]
hence from theorem 5.3.5, \( f \) is weakly \( \text{bl}^+ \) continuous.

**THEOREM 5.3.7:**
The following are equivalent for a function \( f : (X, \tau^+, I) \to (Y, \sigma^+, J) \)
(a) \( f \) is weakly \( \text{bl}^+ \) continuous.
(b) \( f(\text{bl}^* \text{cl}(A)) \subset \text{cl}^{*}\_0(f(A)) \) for each subset \( A \) of \( X \), where \( \text{cl}^{*}\_0(A) = \{ x \in X / \text{cl}^{*}(U) \cap A \neq \varnothing \text{ where } U \in \tau^+ \text{ and } x \in U \} \).
(c) \( \text{bl}^* \text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}^{*}\_0(B)) \) for each subset \( B \) of \( Y \).
(d) \( \text{bl}^* \text{cl}^{*}(f^{-1}(\text{int}(\text{cl}^{*}\_0(B)))) \subseteq f^{-1}(\text{cl}^{*}\_0(B)) \) for each subset \( B \) of \( Y \).

**PROOF:**
(a) ⇒(b): Let \( x \in \text{bl}^* \text{cl}(A) \), \( V \) be any open set of \( Y \) containing \( f(x) \) then there exists \( U \in \text{BI}^* \text{O}(X, x) \) such that \( f(U) \subset \text{cl}^{*}(V) \). Then \( U \cap A \neq \varnothing \) and \( \varnothing \neq f(U) \cap f(A) \subset \text{cl}^{*}(V) \cap f(A) \) so that \( f(x) \in \text{cl}^{*}\_0(f(A)) \).

(b) ⇒(c): Let \( B \) be any subset of \( Y \). Let \( A = f^{-1}(B) \) in (b) then we have \( f(\text{bl}^* \text{cl}(f^{-1}(B))) \subset \text{cl}^{*}\_0(B) \) and \( \text{bl}^* \text{cl}(f^{-1}(B)) \subset f^{-1}(f(\text{bl}^* \text{cl}(f^{-1}(B)))) \subset f^{-1}(\text{cl}^{*}\_0(B)) \).

(c) ⇒(d): Let \( B \) be any subset of \( Y \). Since \( \text{cl}^{*}\_0(B) \) is closed in \( Y \), we have \( \text{bl}^* \text{cl}(f^{-1}(\text{int}(\text{cl}^{*}\_0(B)))) \subseteq f^{-1}(\text{cl}^{*}\_0(\text{int}(\text{cl}^{*}\_0(B)))) \subset f^{-1}(\text{cl}(\text{int}(\text{cl}^{*}\_0(B)))) \subset f^{-1}(\text{cl}(\text{cl}^{*}\_0(B))) \).

(d) ⇒(a): Let \( V \) be any open subset of \( Y \). Then \( V \subset \text{int}(\text{cl}^{*}(V)) = \text{int}(\text{cl}^{*}\_0(V)) \) and hence \( \text{bl}^* \text{cl}(f^{-1}(V)) \subseteq \text{bl}^* \text{cl}(f^{-1}(\text{int}(\text{cl}^{*}\_0(V)))) \subset f^{-1}(\text{cl}^{*}\_0(V)) \subset f^{-1}(\text{cl}^{*}(V)) \). It follows from 5.3.5 that \( f \) is weakly \( \text{bl}^+ \) continuous.

**COROLLARY 5.3.8:**
If \( f : (X, \tau^+, I) \to (Y, \sigma^+, J) \) is weakly \( \text{bl}^+ \) continuous if \( f^{-1}(\text{cl}^{*}\_0(B)) \) is \( \text{bl}^+ \) closed in \( X \) for every subset \( B \) of \( Y \).

**PROOF:**
Since \( f^{-1}(\text{cl}^{*}\_0(B)) \) is \( \text{bl}^+ \) closed in \( X \), we have \( \text{bl}^* \text{cl}(f^{-1}(B)) \subset \text{bl}^* \text{cl}(f^{-1}(\text{cl}^{*}\_0(B))) = f^{-1}(\text{cl}^{*}\_0B) \) and by theorem 5.3.7, \( f \) is weakly \( \text{bl}^+ \) continuous.
**THEOREM 5.3.9:**
A function $f: (X, τ^+, I) \rightarrow (Y, σ^+, J)$ is weakly $bI^+$ continuous if and only if the graph function $g: X \rightarrow X \times Y$ of $f$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is weakly $bI^+$ continuous.

**PROOF:**

**Necessity:** Suppose that $f$ is weakly $bI^+$ continuous. Let $x \in X$ and $W$ be an open set containing $g(x)$. There exists open sets $U_1 \subseteq X$ and $V \subseteq Y$ such that $g(x) = (x, f(x)) \in U_1 \times V$. Since $f$ is weakly $bI^+$ continuous, there exists $U_2 \in BI^+O(X, x)$ such that $f(U_2) \subseteq cl^{**}(V)$. Let $U = U_1 \cap U_2$, then $U \in BI^+O(X, x)$ [since the intersection of a $bI^+$ open set and an open set is $bI^+$ open] and $g(U) \subseteq cl^{**}(W)$.

**Sufficiency:** Suppose that $g$ is weakly $bI^+$ continuous. Let $x \in X$ and $V$ be an open set containing $f(x)$. Then $X \times V$ is an open set containing $g(x)$ and hence there exists $U \in BI^+O(X, x)$ such that $g(U) \subseteq cl^{**}(X \times V) = X \times cl^{**}(V)$. Therefore, we obtain $f(U) \subseteq cl^{**}(U)$ and hence $f$ is weakly $bI^+$ continuous.

**LEMMA 5.3.10:**
If $f: (X, τ^+, I) \rightarrow (Y, σ^+, J)$ is weakly $bI^+$ continuous and $g: (Y, σ^+, J) \rightarrow (Z, δ^+, K)$ is continuous, then the composition $g \circ f: (X, τ^+, I) \rightarrow (Z, δ^+, K)$ is weakly $bI^+$ continuous.

**PROOF:**
Let $x \in X$ and $W$ be an open set of $Z$ containing $g(f(x))$. Then $g^{-1}(W)$ is an open set of $Y$ containing $f(X)$ and there exists $U \in BI^+O(X, x)$ such that $f(U) \subseteq cl^{**}(g^{-1}(W))$. Since $g$ is continuous, we obtain $(g \circ f)(U) \subseteq g(cl^{**}(g^{-1}(W))) \subseteq g(g^{-1}(cl^{**}(W))) \subseteq cl^{**}(W)$, and hence $(g \circ f)$ is weakly $bI^+$ continuous.

**LEMMA 5.3.11:**
Let $f: (X, τ^+, I) \rightarrow (Y, σ^+, J)$ be an open continuous surjection and $g: (Y, σ^+, J) \rightarrow (Z, δ^+, K)$ is a function. If $g \circ f: (X, τ^+, I) \rightarrow (Z, δ^+, K)$ is weakly $bI^+$ continuous, then $g$ is weakly $bI^+$ continuous.

**PROOF:**
Let $W$ be an open set of $Z$. Since $g \circ f: X \rightarrow Z$ is weakly $bI^+$ continuous and $f$ is continuous we have $(g \circ f)^{-1}(W) \subseteq cl^{**} \left(int((g \circ f)^{-1}(cl^{**}(W)))\right) \cup int(cl^{**} \left(g \circ f)^{-1}(cl^{**}(W))\right) = cl^{**} \left(int(f^{-1}(g^{-1}(cl^{**}(W))))\right) \cup int(cl^{**} \left(f^{-1}(g^{-1}(cl^{**}(W))))\right)$. Since $f$ is open
continuous surjection we have \( g^{-1}(W) = f(f^{-1}(g^{-1}(\text{cl}^{*}(W)))) \) and
\[
g^{-1}(W) \subseteq f(\text{cl}^{*}((\text{int}(f^{-1}(g^{-1}(\text{cl}^{*}(W)))))) \cup f(\text{int}(\text{cl}^{*}(f^{-1}(g^{-1}(\text{cl}^{*}(W)))))))
\]
\[
\subseteq \text{cl}^{*}((\text{int}(f^{-1}(g^{-1}(\text{cl}^{*}(W)))))) \cup \text{int}(\text{cl}^{*}(f^{-1}(g^{-1}(\text{cl}^{*}(W))))))
\]
\[
\subseteq \text{cl}^{*}(\text{int}(g^{-1}(\text{cl}^{*}(W)))) \cup \text{int}(\text{cl}^{*}(g^{-1}(\text{cl}^{*}(W))))
\]
and by theorem 5.3.5 is weakly bI\(^{+}\) continuous.

Let \( \{X_{\alpha} : \alpha \in I\} \) and \( \{Y_{\alpha} : \alpha \in I\} \) be any two families of spaces with the same index set \( I \). Let \( f_{\alpha} : X_{\alpha} \rightarrow Y_{\alpha} \) be a function for each \( \alpha \in I \). The product space \( \prod \{X_{\alpha} : \alpha \in I\} \) will be denoted by \( \prod X_{\alpha} \) and \( f : \prod X_{\alpha} \rightarrow \prod Y_{\alpha} \) denote the product function defined by \( f((x_{\alpha})) \) for each \( \{x_{\alpha}\} \in \prod X_{\alpha} \). Moreover, let \( p_{\beta} : \prod X_{\alpha} \rightarrow X_{\beta} \) and \( q_{\beta} : \prod Y_{\alpha} \rightarrow Y_{\beta} \) be the natural projection then we have the following theorem.

**THEOREM 5.3.12:**
A function \( f : \prod X_{\alpha} \rightarrow \prod Y_{\alpha} \) is weakly bI\(^{+}\) continuous if and only if \( f_{\alpha} : X_{\alpha} \rightarrow Y_{\alpha} \) is weakly bI\(^{+}\) continuous for each \( \alpha \in I \).

**Proof:**

**Necessity:** Suppose that \( f \) is weakly bI\(^{+}\) continuous. Let \( \beta \in I \) and since \( q_{\beta} \) is continuous, by lemma 5.3.10 \( f_{\beta} \circ q_{\beta} = q_{\beta} \circ f_{\beta} \) is weakly bI\(^{+}\) continuous. Moreover \( p_{\beta} \) is open continuous surjection and by lemma 5.3.11, \( f_{\beta} \) is weakly bI\(^{+}\) continuous.

**Sufficiency:** Let \( x=(x_{\alpha}) \in \prod X_{\alpha} \). And \( W \) be any open set containing \( f(x) \) there exists a basic open set \( \prod V_{\alpha} \) such that \( f(x) \in \prod V_{\alpha} \subset W \) and \( \prod V_{\alpha} = \prod_{i=1}^{n} V_{\alpha_{i}} \times \prod_{i \neq i}^{n} Y_{\alpha} \) where \( V_{\alpha} \) is open in \( Y_{\alpha} \) for each \( \alpha = \alpha_{1}, \alpha_{2}, ..., \alpha_{n} \). Since \( f_{\alpha} \) is weakly bI\(^{+}\) continuous for each \( \alpha = \alpha_{1}, \alpha_{2}, ..., \alpha_{n} \) there exists a bI\(^{+}\) open set \( U_{\alpha} \) such that \( f_{\alpha}(U_{\alpha}) \subset \text{cl}^{*}(V) \). Now let us consider \( U=\prod_{i=1}^{n} U_{\alpha_{i}} \times \prod_{i \neq i}^{n} X_{\alpha_{i}} \) then \( U \) is bI\(^{+}\) open in \( \prod X_{\alpha} \) and \( f(U) \subset \prod_{i=1}^{n} f_{\alpha_{i}}(U_{\alpha_{i}}) \times \prod_{i \neq i}^{n} Y_{\alpha} \subset \prod_{i=1}^{n} \text{cl}^{*}(V_{\alpha_{i}}) \times \prod_{i \neq i}^{n} Y_{\alpha} \subset \text{cl}^{*}(W) \). Hence it follows that \( f \) is weakly bI\(^{+}\) continuous.

### 5.4 Almost bI\(^{+}\) Continuous Maps

This section proposes another type of continuity called almost bI\(^{+}\) continuity and discusses its properties.

**DEFINITION 5.4.1:**

Let \( (X, \tau^{+}, I) \) and \( (Y, \sigma^{+}, J) \) be extended ideal topological spaces then the function \( f : (X, \tau^{+}, I) \rightarrow (Y, \sigma^{+}, J) \) is almost bI\(^{+}\) continuous at \( x \in X \) if for each open set \( V \) of \( Y \) containing \( f(x) \), there exists \( U \in \text{BI}^{+}O(X, x) \) such that \( f(U) \subset \text{int} (\text{cl}^{+}(V)) \).
REMARK 5.4.2:
Almost $bI^+$ continuity implies almost $\beta I^+$ continuity and almost $b$ continuity.

LEMMA 5.4.3:
For a function $f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, J)$ the following are equivalent.
(a) $f$ is almost $bI^+$ continuous at $x \in X$.
(b) $x \in cl(\int (f^{-1}(scl^+*(V))) \cup intcl(f^{-1}(scl^+*(V)))$ for each neighbourhood $V$ of $f(x)$.
(c) $f^{-1}(V) \subset bI^+ int (f^{-1}(scl^+*(V)))$ for every open set $V$ of $Y$.

PROOF:
(a) $\Rightarrow$ (b).
Let $V$ be any neighbourhood of $f(x)$. Since $f$ is almost $bI^+$ continuous at $x$, there exists $U \in BI^+O(X, x)$ such that $f(U) \subset scl^+*(V)$. Then $U \subset f^{-1}(scl^+*(V))$.
Since $U$ is $bI^+$ open $x \in U \subset cl^+* \int(U) \cup int(cl^+*(U))$
$\subset cl^+* (\int(f^{-1}(scl^+*(V))) \cup int(cl^+*(f^{-1}(scl^+*(V))))$.
(b) $\Rightarrow$ (c): Let $x \in f^{-1}(V)$ and so $f(x) \in V$. Then $x \in f^{-1}(scl^+*(V))$
and since $x \in cl^+*(\int(f^{-1}(scl^+*(V))) \cup int(cl^+*(f^{-1}(scl^+*(V))))$.
We have
$x \in f^{-1}(scl^+*(V)) \cap [\int(cl(f^{-1}(scl^+*(V))) \cup int(cl^+*(f^{-1}(scl^+*(V))))]
= bI^+ \int[f^{-1}(scl^+*(V))].$Hence $f^{-1}(V) \subset bI^+ \int f^{-1}(scl^+*(V))$.
(c) $\Rightarrow$ (a): Let $V$ be an open neighbourhood of $f(x)$.
Then $x \in f^{-1}(V) \subset bI^+ \int f^{-1}(scl^+*(V)))$. Let $U = bI^+ \int f^{-1}(scl^+*(V)))$.
then $U \in BI^+O(X, x)$ and $f(U) \subset scl^+*(V)$. This proves that $f$ is almost $bI^+$ continuous at $x \in X$.

THEOREM 5.4.4:
For a function $f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, J)$ the following are equivalent.
(a) $f$ is almost $bI^+$ continuous.
(b) $f^{-1}(V) \in BI^+O(X)$ for every regular open set $V$ of $Y$.
(c) $f^{-1}(V) \in BI^+C(X)$ for every regular closed set $F$ of $Y$.

PROOF:
(a) $\Rightarrow$ (b) Obvious.
(b) $\Rightarrow$ (c) Let $F$ be a regular closed set of $Y$, then $Y/F$ is regular open.
Let $x \in f^{-1}(Y / F)$ then $f(x) \in Y / F$ and since $f$ is almost $\text{bl}^+$ continuous there exists $W_x \in \text{BI}^+O(X, x)$ such that $x \in W_x$ and $f(W_x) \in Y / F$.

Then $x \in W_x \subset f^{-1}(Y / F)$ so that $f^{-1}(Y / F) = \bigcup_{x \in f^{-1}(Y / F)} W_x \in \text{BI}^+O(X, x)$

Hence $f^{-1}(F) \in \text{BI}^+C(X)$.

(c)$\Rightarrow$(a) Obvious.

**THEOREM 5.4.5:**

The following are equivalent for a function $f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, J)$

(a) $f$ is almost $\text{bl}^+$ continuous.

(b) $\text{bl}^+\text{cl}(f^{-1} \text{cl}^+(\text{int}(\text{cl}^+(B)))) \subset f^{-1}(\text{cl}^+(B))$ for every subset $B$ of $Y$.

(c) $\text{bl}^+\text{cl}(f^{-1}(\text{cl}^+\text{int}(F))) \subset f^{-1}(F)$ for every closed $F$ of $Y$.

(d) $\text{bl}^+\text{cl}(f^{-1}\text{cl}^+(V)) \subset f^{-1}\text{cl}^+(V))$ for every open set $V$ of $Y$.

(e) $f^{-1}(V) \subset \text{cl}^+\text{int}(f^{-1}(\text{cl}^+(V)))$ for every open set $V$ of $Y$.

(f) $f^{-1}(V) \subset \text{cl}^+\text{int}(f^{-1}(\text{cl}^+(V))) \cup \text{int}(\text{cl}^+\text{f}^{-1}(\text{cl}^+(V)))$ for every open set $V$ of $Y$.

**PROOF:**

(a)$\Rightarrow$(b) : Let $B$ be any subset of $Y$. Assume that $x \in X \setminus f^{-1}\text{cl}^+(B)$. Then $f(x) \in Y \setminus \text{cl}^+(B)$ and there exists an open set $V$ containing $f(x)$ such that $V \cap B = \emptyset$.

Hence $\text{int}(\text{cl}^+(V)) \cap \text{cl}^+(\text{int}(\text{cl}^+(B))) = \emptyset$. Since $f$ is almost $\text{bl}^+$ continuous, there exists $U \in \text{BI}^+O(X, x)$, such that $f(U) \subset \text{int}(\text{cl}^+(V))$. Therefore we have,

\[ U \cap f^{-1}(\text{cl}^+(\text{int}(\text{cl}^+(B)))) = \emptyset. \]

Thus we obtain $\text{cl}^+\text{int}(\text{f}^{-1}(\text{cl}^+(B))) \subset f^{-1}(\text{cl}^+(B))$.

(b)$\Rightarrow$(c) : Let $F$ be any closed set of $Y$. Then we have $\text{cl}^+\text{int}(\text{f}^{-1}(\text{cl}^+(\text{int}(F)))) = \text{cl}^+\text{int}(\text{f}^{-1}(\text{cl}^+(F))) \subset f^{-1}(\text{cl}^+(F))$.

(c)$\Rightarrow$(d) : For any open set $V$ of $Y$, $\text{cl}^+(V)$ is regular closed in $Y$ and we have $\text{cl}^+\text{int}(\text{f}^{-1}(\text{cl}^+(V))) \subset \text{cl}^+\text{cl}(\text{f}^{-1}(\text{cl}^+(\text{int}(\text{cl}^+(V))))) \subset f^{-1}\text{cl}^+(V)$.

(d)$\Rightarrow$(e) : Let $V$ be any open set in $Y$. Then $Y \setminus \text{cl}^+(V)$ is open in $Y$ and hence we have $X \setminus \text{bi}^+\text{int}(\text{f}^{-1}\text{scl}^+(V)) = \text{bi}^+\text{cl}(\text{f}^{-1}(Y \setminus \text{cl}^+(V))) \subset f^{-1}(\text{cl}^+(Y \setminus \text{cl}^+(V))) \subset X \setminus f^{-1}(V)$.

Therefore we obtain $f^{-1}(V) \subset \text{bi}^+\text{cl}(\text{f}^{-1}\text{scl}^+(V))$.

(e)$\Rightarrow$(f) : Let $V$ be any open set of $Y$, then we have
f^{-1}(V) \subset bI^+ \text{ int } f^{-1}(scl^+(V))
\subset cl^+ \text{ int}(f^{-1}(scl^+(V))) \cup \text{ int } (cl^+(f^{-1}(scl^+(V)))

(f) \Rightarrow (a) : Let x be any point of X and V be any open set of Y containing f(x). Then \( x \in f^{-1}(V) \subset cl^+ \text{ int } (f^{-1}(scl^+(V))) \cup \text{ int } (cl^+(f^{-1}(scl^+(V)))
It follows from lemma 5.4.3 \( f \) is almost \( bI^+ \) continuous at any point of \( x \) of \( X \). Therefore \( f \) is almost \( bI^+ \) continuous.

**THEOREM 5.4.6:**
The following are equivalent for a function \( f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, J) \)
(a) \( f \) is almost \( bI^+ \) continuous.
(b) \( bI^+ \text{ cl}(f^{-1}(V)) \subset f^{-1}(cl^+(V)) \) for each \( V \in BI^+O(Y) \).
(c) \( bI^+ \text{ cl}(f^{-1}(V)) \subset f^{-1}(cl^+(V)) \) for each \( V \in PI^+O(Y) \).
(d) \( f^{-1}(V) \subset bI^+ \text{ int } (f^{-1}(\text{ int } (cl^+(V)))) \) for each \( V \in PI^+O(Y) \).
(e) \( bI^+ \text{ cl}(f^{-1}(V)) \subset f^{-1}(cl(V)) \) for each \( V \in SI^+O(Y) \).

**PROOF :**
(a) \Rightarrow (b). Let \( V \) be any \( bI^+ \) open set of \( Y \). It follows that \( cl^+(V) \) is regular closed. Since \( f \) is almost \( bI^+ \) continuous from theorem 5.4.4, \( f^{-1}(cl^+(V)) \) is \( bI^+ \) closed. Therefore we have \( bI^+ \text{ cl } (f^{-1}(V)) \subset f^{-1}(cl^+(V)) \)
(b) \Rightarrow (c). Obvious since \( PI^+O(Y) \subset BI^+O(Y) \).
(c) \Rightarrow (d). This is similar to the proof of the implication \( (d) \Rightarrow (e) \) in theorem 5.4.5
(d) \Rightarrow (a). Obvious from theorem 5.4.5, since every open set is pre \( I^+ \) open.
(b) \Rightarrow (e). Obvious, Since \( SI^+O(Y) \subset BI^+O(Y) \).
(e) \Rightarrow (a). Let \( F \) be any regular closed in \( Y \). Then \( F \) is semi \( I^+ \) open in \( Y \) and \( bI^+ \text{ cl } (f^{-1}(F)) \subset f^{-1}(cl^+(F)) = f^{-1}(F) \). Hence \( f^{-1}(F) \) is \( bI^+ \) closed. Therefore \( f \) is almost \( bI^+ \) continuous.

**THEOREM 5.4.7:**
Let \( f : (X, \tau^+, I) \rightarrow (Y, \sigma^+, J) \) be a function and \( g : X \rightarrow X \times Y \) be the graph defined by \( g(x) = (x, f(x)) \) for every \( x \in X \). Then \( g \) is almost \( bI^+ \) continuous if and only if \( f \) is almost \( bI^+ \) continuous.
PROOF:

Necessity: Let \( x \) be any point of \( X \) and \( V \) any regular open set of \( Y \) containing \( f(x) \). Then we have \( g(x) = (x, f(x)) \in X \times V \) is regular open in \( X \times Y \). Since \( g \) is almost \( bI^+ \) continuous there exists \( U \in BI^+O(X, x) \) such that \( g(U) \subseteq X \times V \). Therefore we obtain \( f(U) \subseteq V \) and hence \( f \) is almost \( bI^+ \) continuous.

Sufficiency: Let \( x \in X \) and \( W \) be a regular open set of \( X \times Y \) containing \( g(x) \). There exists a regular open set \( U_1 \) in \( X \) and a regular open set \( V \) in \( Y \) such \( U_1 \times V \subseteq W \). Since \( f \) is almost \( bI^+ \) continuous there exists \( U_2 \in BI^+O(X) \) such that \( x \in U_2 \) and \( f(U_2) \subseteq V \). Let \( U = U_1 \cap U_2 \) then we obtain \( x \in U \subseteq BI^+O(X) \). Hence we have \( g(U) \subseteq U_1 \times V \subseteq W \). Hence \( g \) is almost \( bI^+ \) continuous.

5.5 Decomposition Of Almost \( I^+ \) Continuous Functions

This section introduces \( \beta I^+ \) open and almost \( I^+ \) open sets and functions and further deals with the decomposition of almost \( I^+ \) continuity via the function under suitable conditions.

DEFINITION 5.5.1:
A subset \( A \) of a SEITS \( (X, \tau^+, I) \) is said to be
(i) almost-\( I^+ \)-open if \( A \subseteq Cl^+(Int(A^+)) \)
(ii) \( \beta I^+ \)-open if \( A \subseteq Cl^+(Int(Cl^+(A))) \)
(iii) \( +^* \) dense in itself if \( A \subseteq A^{+^*} \)

By \( \beta I^+O(X, \tau^+, I) \), we denote the family of all \( \beta I^+ \)-open sets of the space \( (X, \tau^+, I) \).

THEOREM 5.5.2:
(i) Every almost-\( I^+ \)-open set is \( \beta I^+ \)-open.
(ii) Every \( \beta I^+ \)-open set is \( \beta^+ \)-open.
(iii) Every open set is \( \beta I^+ \)-open.

PROOF:
(i) Let \( (X, \tau^+, I) \) be a SEITS and \( A \) be an almost-\( I^+ \)-open set of \( X \). Then \( A \subseteq Cl^+(Int(A^+)) \subseteq Cl^{+^*}(Int(Cl^{+^*}(A))) = Cl^{+^*}(Int(Cl^{+^*}(A))) \). Therefore, \( A \) is \( \beta I^+ \)-open.
(ii) & (iii) obvious.
The converses of the above theorem are not necessarily true as shown by the following example.

**EXAMPLE 5.5.3:**
Let \( X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \) and \( I = \{\emptyset, \{a\}\} \) let \( B = \{b, c\} \). Then \( A = \{a\} \) is a \( \beta^-I^+ \)-open set which is not almost-\( I^+ \)-open.

**THEOREM 5.5.4:**
A subset \( A \) of a space \( (X, \tau^+, I) \) is \( \beta^-I^+ \)-open if and only if \( \text{Cl}^{++}(A) = \text{Cl}^+\text{(Int}(\text{Cl}^{++}(A))) \).

**PROOF:**
Let \( A \) be a \( \beta^-I^+ \)-open set. Then we have \( A \subseteq \text{Cl}^+(\text{Int}(\text{Cl}^{++}(A))) \) and hence \( \text{Cl}^{++}(A) \subseteq \text{Cl}^+(\text{Int}(\text{Cl}^{++}(A))) \subseteq \text{Cl}^+(\text{Int}(\text{Cl}(A))) \subseteq \text{Cl}^{++}(A) \). Therefore, we have \( \text{Cl}^{++}(A) = \text{Cl}^+(\text{Int}(\text{Cl}^{++}(A))) \). The converse is obvious. Hence the proof.

**REMARK 5.5.5:**
The intersection of two \( \beta^-I^+ \)-open sets need not be \( \beta^-I^+ \)-open as shown by the following example

**EXAMPLE 5.5.6:**
Let \( X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}\} \) and \( I = \{\emptyset, \{c\}\} \) let \( B = \{c\} \). Then \( A = \{b\} \) is a not \( \beta^-I^+ \)-open whereas \( \{a, b\} \) and \( \{b, c\} \) are \( \beta^-I^+ \)-open.

**THEOREM 5.5.7:**
Let \( (X, \tau^+, I) \) be a SEITS and \( \{A_\alpha : \alpha \in \Delta\} \) be a family of subsets of \( X \) then,
(a) If \( \{A_\alpha : \alpha \in \Delta\} \subseteq \beta^-I^+O(X, \tau^+, I) \), then \( \cup\{A_\alpha : \alpha \in \Delta\} \in \beta^-I^+O(X, \tau^+, I) \).
(b) If \( A \in \beta^-I^+O(X, \tau^+, I) \) and \( U \in \tau^+ \), then \( A \cap U \in \beta^-I^+O(X, \tau^+, I) \).
PROOF:
(a) Since \( \{ A_\alpha : \alpha \in \Delta \} \subseteq \beta I^+ O(X, \tau^+, I) \), then \( A_\alpha \subseteq \text{Cl}^+ (\text{Int}(\text{Cl}^+ (A_\alpha))) \) for each \( \alpha \in \Delta \). Then we have
\[
\bigcup_{\alpha \in \Delta} A_\alpha \subseteq \bigcup_{\alpha \in \Delta} \text{Cl}^+ (\text{Int}(\text{Cl}^+ (A_\alpha))) \\
\subseteq \text{Cl}^+ (\text{Int}(\bigcup_{\alpha \in \Delta} \text{Cl}^+ (A_\alpha))) \\
\subseteq \text{Cl}^+ (\text{Int}(\text{Cl}^+ (\bigcup_{\alpha \in \Delta} A_\alpha))).
\]
This shows that \( \bigcup_{\alpha \in \Delta} A_\alpha \in \beta I^+ O(X, \tau^+, I) \).

(b) By the assumption, \( A \subseteq \text{Cl}^+ (\text{Int}(\text{Cl}^+ (A))) \) and \( U = \text{Int}(U^*) \). Thus we have
\[
A \cap U \subseteq \text{Cl}^+ (\text{Int}(\text{Cl}^+ (A))) \cap \text{Int}(U^*) \\
\subseteq \text{Cl}^+ (\text{Int}(\text{Cl}^+ (A)) \cap \text{Int}(U^*)) \\
= \text{Cl}^+ (\text{Int}(\text{Cl}^+ (A) \cap U)) \\
= \text{Cl}^+ (\text{Int}(\text{Int}(A^* \cup A) \cap U)) \\
= \text{Cl}^+ (\text{Int}((A^* \cap U) \cup (A \cap U))) \\
\subseteq \text{Cl}^+ (\text{Int}((A \cap U)^* \cup (A \cap U))) \\
= \text{Cl}^+ (\text{Int}(\text{Cl}^+ (A \cap U))).
\]
This shows that \( A \cap U \in \beta I^+ O(X, \tau^+, I) \).

DEFINITION 5.5.8:
A subset \( A \) of a space \((X, \tau^+, I)\) is said to be \( \beta-I^+ \)-closed if its complement is \( \beta-I^+ \)-open.

THEOREM 5.5.9:
A subset \( A \) of a space \((X, \tau^+, I)\) is \( \beta-I^+ \)-closed if and only if \( \text{Int}^+ (\text{Cl} (\text{Int}^+ (A))) \subseteq A \).

Proof:
Let \( A \) be a \( \beta-I^+ \)-closed set of \((X, \tau^+, I)\). Then \((X - A)\) is \( \beta-I^+ \)-open and hence \((X - A) \subseteq \text{Cl}^+ (\text{Int}(\text{Cl}^+ (X - A))) = X - \text{Int}^+ (\text{Cl}(\text{Int}^+ (A))) \).
Therefore, we have \( \text{Int}^+ (\text{Cl}(\text{Int}^+ (A))) \subseteq A \). Conversely, let \( \text{Int}^+ (\text{Cl}(\text{Int}^+ (A))) \subseteq A \).
Then \((X - A) \subseteq \text{Cl}^+ (\text{Int}(\text{Cl}^+ (X - A)))\) and hence \((X - A)\) is \( \beta-I^+ \)-open. Therefore, \( A \) is \( \beta-I^+ \)-closed. Hence the proof.

REMARK 5.5.10:
For a subset \( A \) of a space \((X, \tau^+, I)\), we have,
\[
X - \text{Int}(\text{Cl}^+ (\text{Int}(A))) \neq \text{Cl}(\text{Int}(\text{Cl}^+ (X - A)))
\]
as shown by the following example.
EXAMPLE 5.5.11:
Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $I = \{\emptyset, \{a\}\}$. Let $B = \{c\}$ then $\tau^+(B) = \{\emptyset, X, \{a\}, \{a, c\}\}$. Here when $A = \{a, c\}$ we have, $X - \text{Int}(\text{Cl}^{**}(\text{Int}(A))) = \{b, c\}$ and $\text{Cl}(\text{Int}(\text{Cl}^{**}(X - A))) = \emptyset$.

REMARK 5.5.12:
If a subset $A$ of a space $(X, \tau^+, I)$ is $\beta$-$I^+$-closed, then $\text{Int}^{**}(\text{Cl}^{**}(\text{Int}(A))) \subseteq A$.

PROOF:
Let $A$ be any $\beta$-$I^+$-closed set of $(X, \tau^+, I)$. Since $\tau^+(I)$ is finer than $\tau$, we have $\text{Int}^{**}(\text{Cl}^{**}(\text{Int}(A))) \subseteq \text{Int}^{**}(\text{Cl}^{**}(\text{Int}^+(A))) \subseteq \text{Int}^{**}(\text{Cl}( \text{Int}^{**}(A)))$.

Therefore, by Theorem 5.5.9, we obtain $\text{Int}^{**}(\text{Cl}^{**}(\text{Int}(A))) \subseteq A$.

COROLLARY 5.5.13:
Let $A$ be a subset of a space $(X, \tau^+, I)$ such that $X - \text{Int}(\text{Cl}^{**}(\text{Int}(A))) = \text{Cl}^{**}(\text{Int}(\text{Cl}^{**}(X - A)))$.

Then $A$ is $\beta$-$I^+$-closed if and only if $\text{Int}^{**}(\text{Cl}^{**}(\text{Int}(A))) \subseteq A$.

PROOF:
This is an immediate consequence of Theorem 5.5.9.

Now we define $\beta$-$I^+$-continuous functions and its properties are studied.

DEFINITION 5.5.14:
A function $f: (X, \tau^+, I) \to (Y, \sigma^+)$ is said to be $\beta$-$I^+$-continuous if $f^{-1}(V)$ is $\beta$-$I^+$-open in $(X, \tau^+, I)$ for each open set $V$ of $(Y, \sigma^+)$. 

REMARK 5.5.15:
It is obvious from Theorem 5.5.3 that almost-$I^+$-continuity implies $\beta$-$I^+$-continuity and $\beta$-$I^+$-continuity implies $\beta^+$-continuity.

THEOREM 5.5.16:
For a function $f: (X, \tau^+, I) \to (Y, \sigma^+)$, the following conditions are equivalent:

(i) $f$ is $\beta$-$I^+$-continuous,

(ii) for each $x \in X$ and each $V \in Y$ containing $f(x)$, there exists $U \in \beta I^+O(X, \tau^+, I)$
containing \( x \) such that \( f(U) \subseteq V \),

(iii) The inverse image of each closed set in \( Y \) is \( \beta^{-1}\)-closed.

**PROOF:** Straightforward.

**DEFINITION 5.5.17:**

A function \( f : (X, \tau^+, I) \to (Y, \sigma^+, J) \) is said to be \( \beta^{-1}\)-irresolute if \( f^{-1}(V) \) is \( \beta^{-1}\)-open for every \( \beta^{-1}\)-open set \( V \) of \( (Y, \sigma^+, J) \).

**THEOREM 5.5.18:**

Let \( f : (X, \tau^+, I) \to (Y, \sigma^+, J) \) and \( g : (Y, \sigma^+, J) \to (Z, \eta) \) be two functions, where \( I \) and \( J \) are ideals on \( X \) and \( Y \) respectively. Then

(i) \( g \circ f \) is \( \beta^{-1}\)-continuous if \( f \) is \( \beta^{-1}\)-continuous and \( g \) is continuous.

(ii) \( g \circ f \) is \( \beta^{-1}\)-continuous if \( f \) is \( \beta^{-1}\)-irresolute and \( g \) is \( \beta^{-1}\)-continuous.

**PROOF:** Obvious.

**DEFINITION 5.5.19:**

Let \( (X, \tau^+, I) \) is a SEITS and \( A \) is subset of \( X \), we denote by \( \tau_A \) the relative topology on \( A \) and \( I_A = \{ A \cap II \in I \} \) is clearly an ideal on \( A \).

**THEOREM 5.5.20:**

Let \( (X, \tau^+, I) \) be a SEITS. If \( U \in \tau^+ \) and \( A \in \beta I^O(X, \tau^+, I) \), then \( U \cap A \in \beta IO^+(U, \tau_U, I_U) \).

**PROOF:**

Since \( U \in \tau^+ \) and \( A \in \beta I^O(X, \tau^+, I) \), by Theorem 5.5.7 we have

\[
A \cap U \subseteq Cl^+ \left( \text{Int} (Cl^+ (A \cap U)) \right) \quad \text{and hence}
\]

\[
A \cap U \subseteq U \cap Cl^+(\text{Int}(Cl^+ (A \cap U)))
\]

\[
\subseteq Cl^+(U \cap \text{Int}(Cl^+ (A \cap U)))
\]

\[
\subseteq Cl^+(\text{Int}(U \cap Cl^+(A \cap U)))
\]

\[
= Cl^+(\text{Int}_U (U \cap Cl^+(A \cap U))).
\]

Since \( U \in \tau^+ \), we obtain

\[
A \cap U \subseteq U \cap Cl^+(\text{Int}_U (Cl^+(A \cap U))) = Cl^+(\text{Int}_U (Cl^+(A \cap U))).
\]

This shows that \( A \cap U \in \beta I^O(U, \tau_U, I_U) \).
THEOREM 5.5.21:
Let \( f : (X, \tau^+, I) \rightarrow (Y, \sigma) \) be \( \beta-I^* \)-continuous function and \( U \in \tau^+ \). Then the restriction \( f |_U : (U, \tau^+_U, I|_U) \rightarrow (Y, \sigma) \) is \( \beta-I^* \)-continuous.

PROOF:
Let \( V \) be any open set of \( (Y, \sigma) \). Since \( f \) is \( \beta-I^* \)-continuous, we have \( f^{-1}(V) \subseteq \beta I^* O(X, \tau^+, I) \). Since \( U \in \tau^+ \), by theorem 5.5.20,
\[
U \cap f^{-1}(V) \subseteq \beta I^* O(U, \tau^+_U, I|_U)
\]
On the other hand, \( (f|_U)^{-1}(V) = U \cap f^{-1}(V) \) and \( (f|_U)^{-1}(V) \subseteq \beta I^* O(U, \tau^+_U, I|_U) \).
This shows that \( f|_U : (U, \tau^+_U, I|_U) \rightarrow (Y, \sigma) \) is \( \beta-I^* \)-continuous.

THEOREM 5.5.22:
A function \( f : (X, \tau^+, I) \rightarrow (Y, \sigma) \) be \( \beta-I^* \)-continuous if and only if the graph function \( g : X \rightarrow X \times Y \), defined by \( g(x) = (x, f(x)) \) for each \( x \in X \), is \( \beta-I^* \)-continuous.

PROOF:
Necessity: Suppose that \( f \) is \( \beta-I^* \)-continuous. Let \( x \in X \) and \( W \) be any open set of \( X \times Y \) containing \( g(x) \). Then there exists a basic open set \( U \times V \) such that \( g(x) = (x, f(x)) \subseteq U \times V \subseteq W \). Since \( f \) is \( \beta-I^* \)-continuous, there exists a \( \beta-I^* \)-open set \( U_0 \) of \( X \) containing \( x \) such that \( f(U_0) \subseteq V \). By theorem 5.5.7, \( U_0 \cap U \subseteq \beta I^* O(X, \tau^+, I) \) and \( g(U_0 \cap U) \subseteq U \times V \subseteq W \). This shows that \( g \) is \( \beta-I^* \)-continuous.

Sufficiency: Suppose that \( g \) is \( \beta-I^* \)-continuous. Let \( x \in X \) and \( V \) be any open set of \( Y \) containing \( f(x) \). Then \( X \times V \) is open in \( X \times Y \) and by \( \beta-I^* \)-continuity of \( g \), there exists \( U \subseteq \beta I^* O(X, \tau^+, I) \) containing \( x \) such that \( g(U) \subseteq X \times V \). Therefore, we obtain \( f(U) \subseteq V \). This shows that \( f \) is \( \beta-I^* \)-continuous.

Below are the theorems concerning the Decomposition Of Almost-I^*-Continuity

DEFINITION 5.5.23:
A function \( f : (X, \tau^+, I) \rightarrow (Y, \sigma) \) is said to be \( \gamma^* \)-I-continuous if the pre image of every open set in \( (Y, \sigma) \) is \( \gamma^* \)-dense-in-itself in \( (X, \tau^+, I) \).

THEOREM 5.5.24:
Let \( A \) be a subset of a SEITS \( (X, \tau^+, I) \), the following conditions are equivalent:

(a) \( A \) is almost-I^* -open.
(b) A is $\beta$-I$^+$-open and $^+\sigma$-dense-in-itself.

**PROOF:**

(a) $\Rightarrow$ (b). By theorem 5.5.2, every almost-I$^\ast$-open set is $\beta$-I$^+$-open. On the other hand, we have, $A \subseteq \text{Cl}^+ (\text{Int}(A^\ast)) \subseteq \text{Cl}^+ (A^\ast) = A^\ast$. This shows that A is $^\ast\sigma$-dense-in-itself.

(b) $\Rightarrow$ (a). By the assumption, $A \subseteq \text{Cl}^+ (\text{Int}(\text{Cl}^+ (A))) = \text{Cl}^+ (\text{Int}(A^\ast \cup A)) = \text{Cl}^+ (\text{Int}(A^\ast))$. This shows that A is almost-I$^\ast$-open.

Thus we have the following decomposition of almost-I$^\ast$-continuity.

**THEOREM 5.5.25:**

For a function $f : (X, \tau^+, I) \rightarrow (Y, \sigma^\ast)$, the following conditions are equivalent:

(a) $f$ is almost-I$^\ast$-continuous,

(b) $f$ is $\beta$-I$^+$-continuous and $^+\sigma$-I-continuous.

**PROOF:**

This is an immediate consequence of theorem 5.5.24.