CHAPTER III

A STUDY ON RANDOM ARRIVAL OF SHIPMENTS TO MAXIMIZE THE REVENUE

Inventory control is one of the most important aspect of today’s complex supply chain management environment. The traditional design of an inventory system is hierarchical with transportation flows from one stage to next, i.e., the manufacturers to wholesalers and from wholesalers to retailers. More flexible systems also allow lateral transshipments with a stage, which can allow them to lower inventory levels and costs whilst still achieving the required service levels.

A single period inventory model is used to identify the amount of inventory to purchase a perishable goods or single opportunity to purchase. One of the inventory model that have recently renewed attention is that of the Newsboy problem. The single period inventory problem with either instantaneous or constant demand with and without backorders, has also been studied in the context of the classical newsboy problem and its extensions. The amount of the single order is based on balancing the cost of over and under estimating demand. This is a very common problem in areas such as

- Overbooking of airline seats or hotel rooms
- Ordering of fashion items
- Any type of one-time order

In some real life situations there is a part of the demand which cannot be satisfied from the inventory, leaving the system stock-out. In Newsboy problem, there is a on time supply of the items per day and the demand is probabilistic. If the order quantity is larger than the realized demand, the items which are left over at the end of period are sold at a salvage value or disposed off. In the past years many researchers have given considerable attention to the situation where the demand rate is dependent on the level of the on-hand inventory.
In order to make assumptions that are relevant to practical situations, we consider a multi period inventory problem. Shipments arrive at random according to a Poisson process. From these shipments, orders are filled subject to the inventory on hand. The arrival time of the next shipment is unknown, and the demand during lead time is either constant or probabilistic. Consequently, the amount of inventory which will on hand when the next shipment arrives is unknown, it may in fact be negative, meaning that some stock is backordered. The inventory level becomes known upon arrival of shipments. Under these circumstances the following model has been developed to find the economic order quantity.

The single period inventory problem with either instantaneous or constant demand with and without backorders has been studied. The multi period inventory problem of stochastic demand and shortages during lead time has also been well studied by several authors. In order to make assumptions that are relevant to practical situations, Rahim [60] considered a multi period inventory problem for the case of shipments arrived at random, from these shipments, orders are filled subject to the inventory on hand. He developed the model to find the economic order quantity. In this chapter, we propose a variation of the above said model to determine the optimal economic order quantity.

MODEL FORMULATION

A typical inventory situation is considered. Shipments arrive according to a Poisson process with inter arrival time distributed as follows:

\[ f(\tau) = \lambda e^{-\lambda \tau}, \quad \tau \geq 0, \]

where \( \frac{1}{\lambda} \) is the expected inter arrival time between consecutive shipments.
Let $S$ be a predetermined maximum inventory or capacity level. Every time a shipment arrives, an order of size $Q$ is placed so that the stock on hand becomes $S$. The arriving shipments are assumed to be sufficient to fill the orders, therefore, immediate replenishment is necessary as soon as shipment arrives. Thus $Q$ is a random variable which depends on the demand between successive arrivals. The demand $(D)$ per unit time is assumed to be constant throughout the planning horizon. If the demand between arrivals of two consecutive shipments is less than $S$, there is a surplus of inventory at the end of the cycle.

The cost of procurement is $C_p$ per unit, and is assumed to be independent of $Q$. Let the items be sold at $mC_p$, so that the profit margin per unit sold be $(m-1)C_p$. The cost of ordering ($C_o$) is constant and is ignored because it is incurred every time a shipment arrives. A cost of $C_h$ per unit is incurred for every time an item is carried as inventory.

**Fig. 3.1 Inventory model**

![Diagram of inventory model]
In traditional inventory models where there are no lost sales, demand is completely satisfied, and hence the minimization of total inventory-related costs becomes a valid objective. However in the present case, both the procurement costs and revenue incurred out of demand depend on the order level $S$, rather than the total demand. The order level $S$, in turn, depends on the fraction of demand actually fulfilled. The objective in this situation is the maximization of net revenue, rather than the minimization of total costs. The problem is to find the order level $S$ that maximizes the total net revenue by satisfying a fraction of the demand between successive arrivals.

$C_1$ - Expected net revenue generated due to satisfied demand.

$C_2$ - Expected cost of holding inventory.

$C_3$ - Expected cost of backorders.
Thus the total expected net revenue per cycle is \( R = C_1 - C_2 - C_3 \), where \( S \geq D \tau \), procurement (in next period) is \( D \tau \), and \( S \leq D \tau \), procurement is \( S + \beta(D \tau - S) \).

The maximum level of backorders is given by \( (D \tau - S)\beta \).

If we ignore the cost of ordering, the net revenue is considered to be the difference in the revenue obtained from items that are sold and the cost incurred on items that are procured. Thus the expected net revenue per inventory cycle depends upon the order level, the stock on hand at the end of the cycle, and the profit margin per unit.

Average revenue per cycle is,

\[
C_1 = (m-1)C_p \left[ \int_0^{S/D} D \tau e^{-\lambda \tau} d\tau + \int_{S/D}^\infty (S + (D \tau - S)\beta) \lambda e^{-\lambda \tau} d\tau \right]
\]

\[
= (m-1)C_p \left[ D\lambda \left( \int_0^{S/D} e^{-\lambda \tau} d\tau + \int_{S/D}^\infty \frac{e^{-\lambda \tau}}{\lambda} (D \tau - S) e^{-\lambda \tau} d\tau \right) \right]
\]

\[
= (m-1)C_p \left[ D\lambda \left( \frac{S}{-D\lambda} e^{-\lambda S/D} + \frac{1}{\lambda} \left( e^{-\lambda S/D} - 1 \right) \right) \right] + S \lambda e^{-\lambda S/D} \frac{1}{\lambda} + \beta D \left( e^{-\lambda S/D} \frac{1}{\lambda} \right)
\]

\[
= (m-1)C_p \left[ -Se^{-\lambda S/D} + De^{-\lambda S/D} - D + Se^{-\lambda S/D} + \beta D e^{-\lambda S/D} \right]
\]

i.e., \( C_1 = (m-1)C_p \frac{D}{\lambda} \left[ 1 - (1 - \beta)e^{-\lambda S/D} \right] \)

Inventory holding cost is incurred only when the quantity on hand is positive and is computed on the average inventory level.
The expected inventory holding cost per cycle is given by,

\[
C_2 = C_h \left[ \int_0^\frac{S}{D} (S\tau - \frac{D\tau^2}{2}) \lambda e^{-\lambda \tau} d\tau + \int_0^\frac{S}{D} \frac{S^2}{2D} \lambda e^{-\lambda \tau} d\tau \right]
\]

\[
= C_h \left[ \frac{S}{D} \lambda \int_0^\frac{S}{D} e^{-\lambda \tau} d\tau - \frac{D\lambda}{S} \int_0^\frac{S}{D} \tau e^{-\lambda \tau} d\tau \right] + C_h \left[ \frac{S^2 \lambda}{2D} \int_0^\frac{S}{D} \tau e^{-\lambda \tau} d\tau \right]
\]

\[
= C_h \left[ \frac{S^2 \lambda}{2D} \int_0^\frac{S}{D} e^{-\lambda \tau} d\tau \right]
\]

\[
= C_h \left[ \frac{D\lambda}{S} \int_0^\frac{S}{D} e^{-\lambda \tau} d\tau \right] + \frac{S^2 \lambda}{2D} \int_0^\frac{S}{D} \tau e^{-\lambda \tau} d\tau - C_h \frac{D\lambda}{S} \int_0^\frac{S}{D} \tau e^{-\lambda \tau} d\tau
\]

\[
= C_h \left[ \frac{D\lambda}{S} \int_0^\frac{S}{D} e^{-\lambda \tau} d\tau \right] - C_h \frac{D\lambda}{S} \int_0^\frac{S}{D} \tau e^{-\lambda \tau} d\tau + C_h \frac{S^2 \lambda}{2D} \left( \frac{1}{\lambda} e^{-\lambda \frac{S}{D}} \right)
\]

\[
= C_h \left[ -\frac{S^2}{D} e^{-\lambda \frac{S}{D}} - \frac{S}{\lambda} e^{-\lambda \frac{S}{D}} \right] + C_h \frac{D}{2D^2} \frac{S^2}{D} e^{-\lambda \frac{S}{D}}
\]

\[
+ C_h \frac{S}{\lambda} e^{-\lambda \frac{S}{D}} + C_h \left( \frac{1}{\lambda^2} \right) \left( e^{-\lambda \frac{S}{D}} - 1 \right)
\]

\[
= C_h \left[ -\frac{S^2}{D} e^{-\lambda \frac{S}{D}} - \frac{S}{\lambda} e^{-\lambda \frac{S}{D}} \right] + C_h \frac{D}{2D} \frac{S^2}{D} e^{-\lambda \frac{S}{D}} + C_h \frac{S^2 \lambda}{2D} \left( \frac{1}{\lambda} e^{-\lambda \frac{S}{D}} \right)
\]

\[
+ C_h \frac{S}{\lambda} e^{-\lambda \frac{S}{D}} + C_h \left( \frac{1}{\lambda^2} \right) \left( e^{-\lambda \frac{S}{D}} - 1 \right)
\]

\[
= C_h \left[ -\frac{S^2}{D} e^{-\lambda \frac{S}{D}} - \frac{S}{\lambda} e^{-\lambda \frac{S}{D}} \right] + C_h \frac{D}{2} \frac{S^2}{D} e^{-\lambda \frac{S}{D}} + C_h \frac{S^2 \lambda}{2D} \left( \frac{1}{\lambda} e^{-\lambda \frac{S}{D}} \right)
\]

\[
+ C_h \frac{S}{\lambda} e^{-\lambda \frac{S}{D}} + C_h \left( \frac{1}{\lambda^2} \right) \left( e^{-\lambda \frac{S}{D}} - 1 \right)
\]

ie., \( C_2 = C_h \frac{D}{\lambda^2} \left[ e^{-\lambda \frac{S}{D}} + \frac{\lambda S}{D} - 1 \right] \)
The expected backordering cost per cycle is computed based on the average level of backorders and the period of time the customers had to wait.

The expected backordering cost per cycle is,

\[
C_3 = C_b \left[ \int_{S/D}^{\infty} \frac{\beta(D \tau - S)^2}{2D} \lambda e^{-\lambda \tau} d\tau \right]
\]

\[
= C_b \frac{\beta \lambda}{2D} \left[ (D \tau - S)^2 e^{-\lambda \tau} \right]_{S/D}^{\infty} + \int_{S/D}^{\infty} \frac{\lambda}{2D} 2D(D \tau - S) d\tau
\]

\[
= C_b \frac{\beta}{1} \left[ (D \tau - S) e^{-\lambda \tau} \right]_{S/D}^{\infty} - C_b \frac{D \lambda}{\lambda^2} e^{-\lambda S/D}
\]

ie., \(C_3 = C_b \frac{\beta D}{\lambda^2} e^{-\lambda S/D}\)

Therefore the total expected net revenue per cycle is,

\[
R = C_1 - C_2 - C_3
\]

\[
R = (m-1) C_p \frac{D}{\lambda} \left[ 1 - (1 - \beta) e^{-\lambda S/D} \right] - C_h \frac{D}{\lambda^2} \left[ e^{-\lambda S/D} + \frac{\lambda S}{D} - 1 \right] - C_b \frac{D}{\lambda^2} e^{-\lambda S/D}
\]

\[
= \frac{D}{\lambda} \left[ \frac{(m-1)C_p}{\lambda} - \frac{C_h \frac{\lambda S}{D} - C_b}{\lambda} \right] - e^{-\lambda S/D} \left\{ (m-1)(1-\beta)C_p + C_h \frac{1}{\lambda} + C_b \frac{\beta}{\lambda} \right\}
\]

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\[
\frac{D}{\lambda} \left[ (m-1)C_p - C_h \frac{1}{\lambda} + \frac{\lambda S}{D} - C_b \frac{1}{\lambda} \right] - e^{-\frac{\lambda S}{D}} \left\{ (m-1)(1-\beta)C_p + C_h \frac{1}{\lambda} + C_b \beta \frac{1}{\lambda} \right\}
\]

The maximum total expected net revenue per cycle occurs for some \(S\) where the first partial derivative \(\frac{\partial R}{\partial S} = 0\).

Hence differentiating the above equation partially with respect to ‘\(S\)’ we get,

\[
\frac{\partial}{\partial S} \left\{ \frac{D}{\lambda} \left\{ -\frac{C_h}{D} + e^{\frac{\lambda S}{D}} \frac{\lambda}{D} \left[ (m-1)(1-\beta)C_p + C_h \frac{1}{\lambda} + C_b \beta \frac{1}{\lambda} \right] \right\} \right\} = 0
\]

ie.,

\[
e^{-\frac{\lambda S}{D}} \frac{\lambda}{D} \left[ (m-1)(1-\beta)C_p + C_h \frac{1}{\lambda} + C_b \beta \frac{1}{\lambda} \right] = \frac{C_h}{D}
\]

ie.,

\[
\frac{\lambda S^*}{D} = \log_e \left[ \lambda(m-1)(1-\beta)C_p / C_h + 1 + \frac{C_b}{C_h} \beta \right]
\]

ie.,

\[
S^* = \frac{D}{\lambda} \log_e \left[ \lambda(m-1)(1-\beta)C_p / C_h + 1 + \frac{C_b}{C_h} \beta \right] \quad (3.1)
\]

Further, to check the optimality, we need to show that \(\frac{\partial^2 R}{\partial S^2}\) is non-negative.

Since,

\[
\frac{\partial^2 R}{\partial S^2} = -e^{\frac{\lambda S}{D}} \frac{\lambda}{D} \left[ (m-1)(1-\beta)C_p + \frac{C_h}{\lambda} + \frac{C_b}{\lambda} \beta \right] \quad 0, \; S^* \text{ is maximum.}
\]

Therefore, the expected net revenue per cycle is,

\[
R_{\text{max}} = \frac{D}{\lambda} \left[ (m-1)C_p + \frac{C_h}{\lambda} - \frac{C_h S^* - C_b S^*}{D} - e^{-\frac{\lambda S^*}{D}} \left\{ (1-\beta)(m-1)C_p + (C_h + \beta C_b) / \lambda \right\} \right]
\]

\[
R_{\text{max}} = \frac{D}{\lambda} \left[ (m-1)C_p + \frac{C_h}{\lambda} - \frac{C_h S^*}{D} - C_b \frac{1}{\lambda} \right]
\]
Therefore, the maximum total expected net revenue over the time is given by

\[ T.R_{\text{max}} = \lambda R_{\text{max}} = (m-1)C_p D - SC_h \]  \hspace{1cm} (3.2)

**NUMERICAL ILLUSTRATION**

**Example.**

Let us take \( m=1.1 \).

Shipments arrive at an average rate of \( \lambda = 1 \) every month.

Demand for the item \( D = 10 \) units per month.

The fraction of demand backordered is \( \beta = 70\% \).

The procurement cost is \( C_p = \text{Rs.100} \) per unit.

The holding cost is \( C_h = \text{Rs.2} \) per unit per month.

The back ordering cost is \( C_b = \text{Rs.1.50} \) per month.

The profit margin per unit is 10\% of the procurement price.

Using these values in equation (3.1), we get the optimum order level \( S^* = 11.07 \).

Similarly, the maximum value of expected net revenue per cycle is

\[ T.R_{\text{max}} = \text{Rs.77.86} \text{ per month.} \]

**SENSITIVITY ANALYSIS**

Let us investigate the effect of optimum level \( S^* \) and expected net revenue \( T.R_{\text{max}} \) by changing the values of the parameters \( \lambda \), \( \beta \) and \( C_b \) in equations (3.1) and (3.2), we get,
## Effect of $\lambda$ on $S^*$ and $T.R_{\text{max}}$

Consider the sets 1, 2 and 3 from the above table, it should be noted that, when the arrival rate $\lambda$ increases, the order level $S$ decreases, the maximum net revenue $T.R_{\text{max}}$ increases. Hence the conclusion will be, if the shipments arrive more frequently, the required capacity or order level to meet the demand decreases and costs also decreases.

## Effect of $\beta$ on $S^*$ and $T.R_{\text{max}}$

The effect of changes in the percentage of backordered on the order level and maximum net revenue is shown in sets 1, 4 and 5 from the above table. It is observed that the percentage of stock that is backordered ($\beta$) has the same effect as that of the arrival rate.

ie., In set 1, when $\beta = 0.7$, $S^* = 11.069$ and $T.R_{\text{max}} = 77.861$. 

<table>
<thead>
<tr>
<th>Set</th>
<th>$\lambda$</th>
<th>$\beta$</th>
<th>$C_b$</th>
<th>$S^*$</th>
<th>$T.R_{\text{max}}$</th>
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<td>0.7</td>
<td>1.5</td>
<td>11.069</td>
<td>77.861</td>
</tr>
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<tr>
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In set 4, when $\beta$ increased to 0.90, $S^* = 7.770$ and $T.R_{\text{max}}$ becomes 84.459. The conclusion is, as $\beta$ increases, $S^*$ decreases and $T.R_{\text{max}}$ increases.

**Effect of $C_b$ on $S^*$ and $R_{\text{max}}$**

Consider the sets 1, 6 and 7 from the above table, we analyzed that when the cost of backordering $C_b$ increases, the order level $S^*$ increases and the maximum net revenue $T.R_{\text{max}}$ decreases.

**CONCLUSION**

In this Chapter, we have studied an inventory system with random arrival of shipments for multi period problem. The results obtained from the sensitivity analysis, we concluded that (i) When the backordering cost increases, the order level increases and net revenue decreases, (ii) When arrival rate increases, the order level decreases and net revenue increases. The case of limited capacity and the case of not ordering every time can be incorporated in the future study.