CHAPTER II

STOCHASTIC MODEL FOR RESERVE INVENTORY BETWEEN MACHINES

2.1 RESERVE INVENTORY BETWEEN MACHINES IN SERIES

Operations Research has a number of branches like Mathematical programming, Game theory, Queuing problems, Reliability theory, Inventory control etc. All these disciplines depended more and more for their development and sophistication. The use of advanced probability theory for which stochastic process is a basic structure. Many of the real life problems, which are governed by chance mechanism are deeply involved with the concept of stochastic processes. So in conceptualizing the real life problems as mathematical models, stochastic process plays a predominant role.

Supply disruption can have drastic impacts on firms who fail to protect against them. Traditional inventory models focus on demand uncertainty and design the system to best mitigate that risk. However, the effects of supply disruptions can have very different implications for system design. In the past years, there has been an explosion of research on inventory and supply chain models with supply disruptions. In this chapter, we examine a single-stage inventory system with disruptions and introduce an effective approximation for systems with both disruptions and demand or yield uncertainty. We examine optimal base-stock inventory policies using finite-horizon, periodic review models for a single supplier whose single retailer is subject to stochastic disruptions.

Inventory control is the process of deciding what and how much of various items are to be kept in stock. The basic objective is to reduce the investment in inventories and ensuring that production process does not suffer at the same time.
The necessity for maintaining inventory arises in several situations in a production oriented inventory system, viz., hydro-electric system, thermal power station, etc. In the case of hydro-electric system, the input is the river-flow and the dam is the reserve to hydro-electric station. The optimal reserve in this case is not exactly the reserve in the dam but it is only the problem of optimal discharge during the different periods of a planning horizon. In the case of a thermal station, Coal or Lignite or Furnace oil is used as input in the thermal generator and the consumption is at the thermal power station. So the optimal reserve inventory is the reserve between the two systems, namely, the system that produces the output and the system that consumes the input obtained from the first system. One of the problems of interest in inventory is the determination of the optimal size of the buffer stock between the systems.

In Inventory management, the very objective is to determine the optimal stock level or the optimal order size by taking into account the related costs. A serial system in series as discussed in reliability theory is possible in many situations of inventory management.

If various manufacturing processes operate successively, then in the case of breakdown of one or any disturbance at some stage can affect the entire system. Hence stocking points of inventory are created between adjacent stages so as to achieve a certain degree of independence in operating the stages.

Sehik Uduman et. al., [71] have found a stochastic model for reserve inventory between two machines in series. In this section, we consider a modified model in which the system which is one in series is conceptualized and assumes the stochastic demand pattern. That is the output of the first machine $M_1$ is the input for the second machine $M_2$, the output of $M_2$ is the input for the third machine $M_3$ and so on, likewise the optimal reserve inventory between ‘n’ machines in series is obtained.
Here we have considered a model in which a separate stock is maintained in between the machines and assuming that the consumption rate is constant. Finally, we have obtained a generalized solution for each type. These results constitute a set of tools that will be useful for future study on inventory models.

Let $M_1, M_2$ be two machines.

- $h$ – cost of inventory holding per unit
- $d$ - idle time cost due to $M_2$ per unit time
- $r$ - constant rate of consumption
- $s$ - reserve inventory between $M_1$ and $M_2$
- $t$ - continuous random variable denoting the repair time of $M_1$ with $g(.)$ as p.d.f and $G(.)$ as c.d.f.

If $T$ is a random variable denoting idle time of $M_2$, it may be noted that

$$T = 0, \quad \text{if} \quad t \leq \frac{s}{r}$$

$$= \left( t - \frac{s}{r} \right) \quad \text{if} \quad t \gt \frac{s}{r}$$

The total expected total cost due to inventory holding and idle time of $M_2$ per unit time is,

$$E(C) = hs + \frac{d}{\mu} \int \left( t - \frac{s}{r} \right) g(t) \, dt,$$

Where $\frac{1}{\mu}$ denotes the number of breakdowns of $M_1$ per unit time.
To obtain the optimality ‘s’, we have to find \( \frac{dE(C)}{ds} = 0 \)

ie., \( \frac{d}{ds}\left( hs + \frac{d}{\mu} \int (t - \frac{s}{r}) g(t) dt \right) = 0, \)

After simplification, we get, \( G\left( \frac{s}{r} \right) = 1 - \frac{r\mu h}{d} \)

(2.1.1)

From this, we get the equation for optimal reserve inventory, it has a limitation that it should be less than unity, otherwise the solution is not feasible. Hence a new improved solution with no restriction was obtained as follows:

\[
E(C) = \left\{ \begin{array}{l}
ht \int_{\frac{s}{r}}^{\frac{s}{r}} (s - t) g(t) dt + \frac{d}{\mu} \int_{\frac{s}{r}}^{\infty} (t - \frac{s}{r}) g(t) dt \\
\end{array} \right.
\]

\( \frac{dE(C)}{ds} = 0, \) proves the optimality.

ie., \( \frac{d}{ds}\left\{ \begin{array}{l}
ht \int_{\frac{s}{r}}^{\frac{s}{r}} (s - t) g(t) dt + \frac{d}{\mu} \int_{\frac{s}{r}}^{\infty} (t - \frac{s}{r}) g(t) dt \\
\end{array} \right\} = 0 \)

After simplification, we get, \( G\left( \frac{s}{r} \right) = \frac{d}{d + r\mu h} < 1, \)

(2.1.2)

for all \( r, \mu, h \) and \( d \). This is an improved version of (2.1.1) with no restrictions.

An extension of the above model is developed into ‘n’ machines in series.

First, let us verify the system for ‘3’ machines in series using notations defined below:

In this model, 3 machines \( M_1, M_2 \) and \( M_3 \) in series is considered and the optimal value of the reserve inventory between \( M_1 \) and \( M_2 \), \( M_2 \) and \( M_3 \) for the system is discussed.

- \( h_1, h_2 \) denote inventory holding costs of \( M_1, M_2 \) respectively.
- \( r_1, r_2 \) denote consumption rates.
- \( d_1, d_2 \) denote the idle time costs.
• $t_1, t_2$ are random variables denote the repair time durations.

• $s_1, s_2$ are reserve inventory between $M_1, M_2$ and $M_2, M_3$ respectively.

If $T_1, T_2$ are the random variables denoting the idle time of $M_2, M_3$ respectively, then, it may be noted that

$$T_1 = 0, \quad \text{if } t_1 \leq \frac{s_1}{r_1}$$

$$= \left( t_1 - \frac{s_1}{r_1} \right) \text{if } t_1 > \frac{s_1}{r_1}$$

$$T_2 = 0, \quad \text{if } t_2 \leq \frac{s_2}{r_2}$$

$$= \left( t_2 - \frac{s_2}{r_2} \right) \text{if } t_2 > \frac{s_2}{r_2}$$

The average level of inventory is

$$\int_{0}^{\frac{s_1}{r_1}} (s_1 - r_1 t_1) g(t_1) dt_1 \quad \text{and} \quad \int_{0}^{\frac{s_2}{r_2}} (s_2 - r_2 t_2) g(t_2) dt_2,$$

assuming that the rate of consumption of $M_2$ and $M_3$ are $r_1, r_2$ per unit of time,

where $\frac{1}{\mu_1}, \frac{1}{\mu_2}$ denote the number of breakdowns of $M_1$ and $M_2$ per unit time.

The expected total cost is,

$$E(C) = h_1 r_1 \int_{0}^{\frac{s_1}{r_1}} \left( \frac{s_1}{r_1} - t_1 \right) g(t_1) dt_1 + \frac{d_1}{\mu_1} \int_{\frac{s_1}{r_1}}^{\infty} \left( t_1 - \frac{s_1}{r_1} \right) g(t_1) dt_1 +$$

$$h_2 r_2 \int_{0}^{\frac{s_2}{r_2}} \left( \frac{s_2}{r_2} - t_2 \right) g(t_2) dt_2 + \frac{d_2}{\mu_2} \int_{\frac{s_2}{r_2}}^{\infty} \left( t_2 - \frac{s_2}{r_2} \right) g(t_2) dt_2$$

To find optimal ‘s’, we find $\frac{dE(C)}{ds} = 0$.
\[
\frac{d}{ds}\left[ h_1 r_1 \int_{0}^{s_1} g(t_1) dt_1 + \frac{d_1}{\mu_1} \int_{0}^{\infty} \left( t_1 - \frac{s_1}{r_1} \right) g(t_1) dt_1 \right] + \\
\frac{d}{ds}\left[ h_2 r_2 \int_{0}^{s_2} g(t_2) dt_2 + \frac{d_2}{\mu_2} \int_{0}^{s_2} \left( t_2 - \frac{s_2}{r_2} \right) g(t_2) dt_2 \right] = 0
\]

ie.,
\[
\frac{d}{ds}\left[ h_1 r_1 \int_{0}^{s_1} g(t_1) dt_1 + \frac{d_1}{\mu_1} \int_{0}^{\infty} \left[ t_1 - \frac{s_1}{r_1} \right] g(t_1) dt_1 \right] + \\
\frac{d}{ds}\left[ h_2 r_2 \int_{0}^{s_2} g(t_2) dt_2 + \frac{d_2}{\mu_2} \int_{0}^{s_2} \left[ t_2 - \frac{s_2}{r_2} \right] g(t_2) dt_2 \right] = 0
\]

ie.,
\[
\frac{d}{ds}\left[ \left( \frac{h_1 r_1 g(s_1)}{r_1} \right) + \frac{d_1}{\mu_1} \int_{0}^{\infty} \left[ t_1 - \frac{s_1}{r_1} \right] g(t_1) dt_1 \right] + \\
\frac{d}{ds}\left[ \left( \frac{h_2 r_2 g(s_2)}{r_2} \right) + \frac{d_2}{\mu_2} \int_{0}^{s_2} \left[ t_2 - \frac{s_2}{r_2} \right] g(t_2) dt_2 \right] = 0
\]

(2.1.3)

On differentiating the above equation partially with respect to \(s_1\) we get,
\[
\frac{d}{ds_1}\left\{ h_1 g\left( \frac{s_1}{r_1} \right) - \frac{d_1}{\mu_1} \left[ \int_{0}^{s_1} g(t_1) dt_1 - \int_{0}^{\infty} g(t_1) dt_1 \right] \right\} = 0
\]

ie.,
\[
h_1 g\left( \frac{s_1}{r_1} \right) - \frac{d_1}{\mu_1} \left( 1 + g\left( \frac{s_1}{r_1} \right) \right) = 0
\]
Hence we get,

\[
G\left(\frac{s_1}{r_1}\right) = \frac{d_1}{d_1 + r_1 \mu_1 h_1} < 1 \quad (2.1.4)
\]

for all \( r_1, h_1, \mu_1 \) and \( d_1 \).

On differentiating equation (2.1.3) partially with respect to \( s_2 \), we get

\[
\frac{d}{ds_2} \left\{ h_2 g\left(\frac{s_2}{r_2}\right) - \frac{d_2}{\mu_2} \left[ \frac{1}{r_2} \int_0^{r_2} g(t_2) dt_2 - \frac{r_2}{r_2} \int_0^{r_2} g(t_2) dt_2 \right] \right\} = 0
\]

ie.,

\[
h_2 g\left(\frac{s_2}{r_2}\right) - \frac{d_2}{\mu_2} \left[ 1 + g\left(\frac{s_2}{r_2}\right) \right] = 0
\]

and hence, we get

\[
G\left(\frac{s_2}{r_2}\right) = \frac{d_2}{d_2 + r_2 \mu_2 h_2} < 1 , \quad (2.1.5)
\]

for all \( r_2, h_2, \mu_2 \) and \( d_2 \).

From the equations (2.1.4) and (2.1.5) we get the optimal reserve inventory for \( s_1 \) & \( s_2 \) between the machines \( M_1, M_2 \) and \( M_2, M_3 \) in series.

Let us prove the result for the system of ‘n’ machines in series by the method of mathematical induction hypothesis.

Let us assume that the result is true for \( k \), the equation for the reserve inventory ‘s’ becomes,

\[
G\left(\frac{s_k}{r_k}\right) = \frac{d_k}{d_k + r_k \mu_k h_k} < 1
\]

It is clear that the equation is true for \( k = 1 \), then it is true for \( k = 2 \),

Let us verify the result for \( n = k+1 \),
\[ E(C) = h_{k+1} r_{k+1} \int_{0}^{\infty} \left( \frac{s_{k+1}}{r_{k+1}} - t_{k+1} \right) g\left(t_{k+1}\right) dt_{k+1} + \frac{d_{k+1}}{\mu_{k+1}} \int_{0}^{\infty} \left( t_{k+1} - \frac{s_{k+1}}{r_{k+1}} \right) g\left(t_{k+1}\right) dt_{k+1} \]

From this, the optimal reserve inventory \( \frac{dE(C)}{ds} = 0 \) reduces and is given by,

\[ G \left( \frac{s_{k+1}}{r_{k+1}} \right) = \frac{d_{k+1}}{d_{k+1} + r_{k+1} \mu_{k+1} h_{k+1}} \bigg\langle 1, \bigg(2.1.6) \bigg\rangle \]

for all values of \( r_{k+1}, h_{k+1}, \mu_{k+1} \) and \( d_{k+1} \). Hence it is true for \( k+1 \) also.

Therefore we concluded that this is true for all \( n \).

ie.,

\[ E(C) = h_{1} r_{1} \int_{0}^{s_{1}} \left( \frac{s_{1}}{r_{1}} - t_{1} \right) g\left(t_{1}\right) dt_{1} + \frac{d_{1}}{\mu_{1}} \int_{0}^{\infty} \left( t_{1} - \frac{s_{1}}{r_{1}} \right) g\left(t_{1}\right) dt_{1} \]

\[ = h_{2} r_{2} \int_{0}^{s_{2}} \left( \frac{s_{2}}{r_{2}} - t_{2} \right) g\left(t_{2}\right) dt_{2} + \frac{d_{2}}{\mu_{2}} \int_{0}^{\infty} \left( t_{2} - \frac{s_{2}}{r_{2}} \right) g\left(t_{2}\right) dt_{2} + \ldots + \]

\[ = h_{n} r_{n} \int_{0}^{s_{n}} \left( \frac{s_{n}}{r_{n}} - t_{n} \right) g\left(t_{n}\right) dt_{n} + \frac{d_{n}}{\mu_{n}} \int_{0}^{\infty} \left( t_{n} - \frac{s_{n}}{r_{n}} \right) g\left(t_{n}\right) dt_{n} . \]

Differentiating the above equation with respect to ‘s’, we get,
\[
\frac{dE(C)}{ds} = \frac{d}{ds}
\left[
\begin{align*}
&\frac{s}{\eta}\int_{0}^{\eta} \left(\frac{s_1}{\eta} - t_1\right) g(t_1) dt_1 + \frac{d_1}{\mu_1} \int_{0}^{s_1/\eta} \left(t_1 - \frac{s_1}{\eta}\right) g(t_1) dt_1 + \\
&\frac{s_2}{r_2}\int_{0}^{r_2} \left(\frac{s_2}{r_2} - t_2\right) g(t_2) dt_2 + \frac{d_2}{\mu_2} \int_{0}^{s_2/r_2} \left(t_2 - \frac{s_2}{r_2}\right) g(t_2) dt_2 + \ldots \ldots \\
&\ldots \ldots + h_n r_n \int_{0}^{r_n} \left(\frac{s_n}{r_n} - t_n\right) g(t_n) dt_n + \frac{d_n}{\mu_n} \int_{0}^{s_n/r_n} \left(t_n - \frac{s_n}{r_n}\right) g(t_n) dt_n
\end{align*}\right]
\]

Solving the above equation as above, we get,

\[
G\left(\frac{s}{r}\right) = \sum_{i=0}^{n} \frac{d_{i+1}}{d_{i+1} + r_{i+1}\mu_{i+1}h_{i+1}} \langle 1 \text{ for all values of } r, h, d \text{ and } \mu \rangle
\]

(2.1.7)

**CONCLUSION**

The optimal reserve inventory between ‘n’ machines in series assuming stochastic demand pattern for the input can be obtained from the equations (2.1.3), (2.1.4) and (2.1.5) for all values of r, h, d and \(\mu\) and the generalized equation (2.1.7) is derived.
2.2 RESERVE INVENTORY BETWEEN MACHINES IN PARALELL

In this section, an optimal reserve inventory between machines in parallel is attempted. The output of the first machine $M_1$ is the input for the second and third machines $M_2$, $M_3$ respectively. In between the Machines $(M_2, M_3)$ a separate inventory is maintained. A generalized equation is derived for ‘n’ machines in parallel.

Fig:2.2. Reserve inventory in parallel

In this model, 3 machines $M_1, M_2$ and $M_3$ are considered and the optimal value of the reserve inventory between $M_1$ and $(M_2, M_3)$ for the system is discussed.

\[
\frac{dE(C)}{ds} = \frac{d}{ds} \left[ h r \int_0^{\frac{t}{r}} \left( \frac{s}{r} - t \right) g(t) dt + \frac{d_1}{\mu} \int_{\frac{t}{r}}^{\infty} \left( t - \frac{s}{r} \right) g(t) dt \right] \\
+ \frac{d_2}{\mu} \int_{\frac{t}{r}}^{\infty} \left( t - \frac{s}{r} \right) g(t) dt 
\]
\[
= h r \int_{0}^{\frac{s}{r}} \frac{1}{r} g(t) \, dt + \frac{d_1}{\mu} \left\{ \frac{1}{r} \int_{0}^{\frac{s}{r}} g(t) \, dt \right\} + \\
\frac{d_2}{\mu} \left\{ \frac{d}{ds} \int_{s}^{\infty} \left( t - \frac{s}{r} \right) g(t) \, dt \right\}
\]

\[
= \frac{h r}{r} g\left( \frac{s}{r} \right) + \frac{d_1}{\mu} \left\{ \frac{1}{r} \int_{0}^{\frac{s}{r}} g(t) \, dt - \int_{0}^{\frac{s}{r}} g(t) \, dt \right\} + \\
\frac{d_2}{\mu} \left\{ \frac{1}{r} \int_{0}^{\frac{s}{r}} g(t) \, dt \right\}
\]

Simplifying, we get

\[
= h g\left( \frac{s}{r} \right) - \frac{d_1}{\mu} \left\{ \frac{1}{r} \left[ \int_{0}^{\frac{s}{r}} g(t) \, dt - \int_{0}^{\frac{s}{r}} g(t) \, dt \right] \right\} - \\
\frac{d_2}{\mu} \left\{ \frac{1}{r} \left[ \int_{0}^{\frac{s}{r}} g(t) \, dt - \int_{0}^{\frac{s}{r}} g(t) \, dt \right] \right\}
\]

i.e., \( h g\left( \frac{s}{r} \right) - \frac{d_1}{\mu} \left( 1 + g\left( \frac{s}{r} \right) \right) - \frac{d_2}{\mu} \left( 1 + g\left( \frac{s}{r} \right) \right) = 0 \)

Further,

\[
h g\left( \frac{s}{r} \right) - \frac{d_1 + d_2}{\mu} \left( 1 + g\left( \frac{s}{r} \right) \right)
\]

\[
G\left( \frac{s}{r} \right) = \frac{d_1 + d_2}{d_1 + d_2 + r \mu h} \quad 1 \quad (2.2.1)
\]

for all values of \( r, h, \mu \) and \( d_1, d_2 \).
Which is the optimal reserve inventory ‘s’ between the machines M_1 and (M_2, M_3).

We prove the result is true for the system of ‘n’ machines in parallel by using the method of mathematical induction hypothesis.

The equation becomes,

\[ G \left( \frac{s}{r} \right) = \frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} d_i + r \mu h} \quad \langle 1 \]

Assume the result is true for \( i = 1 \).

\[ G \left( \frac{s}{r} \right) = \frac{d_1}{d_1 + r \mu h} \quad \langle 1 \]

It is trivial and true for \( i=1 \).

Let us verify the equation for \( i+1 \),

\[
E(C) = h g \left( \frac{s}{r} \right) - \frac{d_1}{\mu} \left[ \int_0^\infty g(t)dt - \int_0^{\frac{s}{r}} g(t)dt \right]
\]

\[
= \frac{d_{i+1}}{\mu} \left[ \int_0^\infty g(t)dt - \int_0^{\frac{s}{r}} g(t)dt \right]
\]

From this, the optimal reserve inventory \( \frac{dE(C)}{ds} = 0 \) is obtained and is given by,

\[ G \left( \frac{s_{i+1}}{r_{i+1}} \right) = \frac{\sum_{i=0}^{n} d_{i+1}}{\sum_{i=0}^{n} d_{i+1} + r \mu h} \quad \langle 1, \]

for all \( r, h, \mu \quad \& \quad d_{i+1} \).

Hence it is true for \( i+1 \) also.
Therefore we concluded that it is true for all \( n \).

\[
   \text{ie.,} \quad E(C) = \frac{hr}{r} g\left(\frac{s}{r}\right) + \frac{d_1}{\mu} \left( \frac{d}{ds} \int_s^\infty \left[ t - \frac{s}{r} \right] g(t) \, dt \right) + 
\]

\[
   \frac{d_2}{\mu} \left( \frac{d}{ds} \int_s^\infty \left[ t - \frac{s}{r} \right] g(t) \, dt \right) + \frac{d_3}{\mu} \left( \frac{d}{ds} \int_s^\infty \left[ t - \frac{s}{r} \right] g(t) \, dt \right) + \ldots \ldots + 
\]

\[
   \frac{d_n}{\mu} \left( \frac{d}{ds} \int_s^\infty \left[ t - \frac{s}{r} \right] g(t) \, dt \right)
\]

Differentiating the above equation,

\[
   \frac{dE(C)}{dt} = \frac{d}{ds} \left( \frac{hr}{r} g\left(\frac{s}{r}\right) + \frac{d_1}{\mu} \left( \frac{d}{ds} \int_s^\infty \left[ t - \frac{s}{r} \right] g(t) \, dt \right) + 
\]

\[
   \frac{d_2}{\mu} \left( \frac{d}{ds} \int_s^\infty \left[ t - \frac{s}{r} \right] g(t) \, dt \right) + \ldots + \frac{d_n}{\mu} \left( \frac{d}{ds} \int_s^\infty \left[ t - \frac{s}{r} \right] g(t) \, dt \right)
\]

Solving the above equation, we get,

\[
   G\left(\frac{s}{r}\right) = \frac{\sum_{i=0}^{n} d_{i+1}}{\sum_{i=0}^{n} d_{i+1} + r \mu h} \langle 1 ,
\]

for all values of \( r, h, d_1, \ldots, d_n \) (where \( i = 1, 2, 3, \ldots, n \)) and \( \mu \).

**CONCLUSION**

The optimal reserve inventory between ‘n’ machines in parallel assuming stochastic demand pattern for the input can be obtained for all values of \( r, h, d, \mu \) and derived the generalized equation for optimality is derived.