CHAPTER VI

EOQ MODEL FOR DETERIORATION OF ITEMS WITH CREDIT SYSTEM

6.1 AN EOQ MODEL WITH PERMISSIBLE DELAY IN PAYMENTS AND SPECIAL DISCOUNTS

The traditional economic order quantity model assumes that retailer capitals are adequate and must pay for the items as soon as the items are received. In practice, supplier will offer to retailer a delay period, which is the trade credit period, in paying for the amount of purchase. Before end of the trade credit period retailer can sell goods and accumulate revenue and earn interest. A higher interest is charged if the payment is not settled by the end of the trade credit period. In the real world, supplier often make use of this policy to promote commodities. Therefore it makes economic sense for the retailer to delay the settlement of the replenishment account up to the end of the delay period allowed by the supplier.

In the past decade, mathematical ideas have been used in different areas for controlling inventory. The important concerns of the management are to decide when and how much to order or to manufacture, so that the total cost associated with the inventory system should be minimum.

In the traditional inventory EOQ model, the purchaser pays for his items as soon as it is received. However in real competitive business world, the supplier may allow a credit period to encourage the customers. Delay in payments to the supplier is an alternative way of price discount. Hence paying later is directly reduces the purchase cost which attracts the customers to enhance their ordering quantity. It has been empirically observed that failure and life expectancy of many items can be expressed in terms of Weibull distribution function. Hence consideration of Weibull distribution function for rate of deterioration is more reasonable and practicable.
The deterioration of goods is a realistic phenomenon in many inventory systems. The controlling and regulating of deteriorating items is a measure problem in any inventory system. Number of researchers have been discussed inventory models for deteriorating and non-deteriorating items. However there are certain substances such as products like food stuff, pharmaceuticals, chemicals, volatile liquid, blood and the non deteriorating items like, wheat, rice, some types of dry fruits etc., in which deterioration plays a main role and commodities cannot be stored for a long time. While developing an optimal inventory policy for such products, the loss of inventory due to deterioration cannot be ignored. Several researchers discussed this topic and investigates inventory problems under varying conditions of such products.

The assumptions that the goods in inventory always preserve their physical characteristics is not true in general. There are some items which are subject to the risks of breakage, deterioration, evaporation, obsolescence and so forth. Managing inventory for deteriorating items is of great concern to the retailers, wholesalers, even to the production managers who are in the business of perishable items. The items that can deteriorate or lose values under normal conditions are meat, fish, sea food, poultry, dairy products, fruits and vegetables, some special type of medicines, radioactive substances etc. Often transportation of which are also needs special care. In most of the cases, these items should be used within a short period of time after delivery, as it may not be possible to preserve them in the same manner after delivery.

It is generally assumed that the buyer must pay for the items as soon as he receives them from the supplier, but in reality supplier will allow a certain fixed period called credit period for settling the amount the retailer owes to him for the items supplied. The credit period reduces the buyers cost of holding stock because it reduce the amount of capital invested in stock for the duration of the permissible period. Chung [10] used the DCF approach for studying the optimal inventory policy in the presence of the trade credit which permits an explicit recognition of
the exact timing of cash flows associated with the inventory system. A DCF approach permits a proper recognition of the financial implication of the opportunity cost and out of pocket costs in inventory system.

Generally in deterministic inventory models, the demand is either uniform or time dependent and has noting to do with the status of inventory at hand. However there are items whose demand depends upon the stock level. Hence Gupta and Lalit Rankawat [36] have derived an EOQ model of linear deterioration rate and stock dependent rate of consumption with permissible delay in payments. In this section, we have considered a model allowing credit facility with stock dependent demand and linearly time dependent deterioration rate allowing credit facility and shortages are not considered into consideration.

Notations and Assumptions

The model is developed under the following notations and assumptions:

- Replenishment is instantaneous, the lot size is of Z units per replenishment.
- Demand rate \( R(I) \) is linear function of \( I \) such that,
  \[
  R(I) = \alpha + I\beta, I \geq 0
  \]
  \[
  \alpha, \text{ otherwise.}
  \]
  where, \( \alpha > 0, 0 < \beta \leq 1 \) and \( I \) denotes the on hand inventory level at time \( t \).

- \( t_p \) is the time period allowed to the receiver for settling of the account.

  During time period, \( t_p \) interest is earned on the accumulate sales revenue at the rate of \( i_e \) per unit revenue per time unit. After period \( t_p \) interest is to be paid at the rate of \( i_p(i_p)^2i_e \) per unit investment per time unit on the capital tied in the remaining stock.
p is the purchasing price per unit.
• c is the selling price per unit.
• h is the holding cost per unit time.
• A is the ordering cost per order.
• shortages are not allowed.
• s special discount price per unit.
• Deterioration starts when the items arrive in stock and it is taken to be slowly varying with time linearly, i.e., \( \theta(t) = \gamma t, \) (0 \( \leq \) \( \gamma \) \( \leq \) 1).

Model Development

The differential equation governing the stock status for the period (0, T) can be written as,

\[
\frac{dI}{dt} + \gamma I = -(\alpha + \beta I), \quad 0 \leq t \leq T
\]  

(6.1.1)

\[
\frac{dI}{dt} + (\gamma I + \beta) = -\alpha
\]

\[
I(t) = -\alpha e^{-(\gamma t/2 + \beta t)} \int_t^{T} e^{(\gamma t/2 + \beta t)} dt
\]

The solution of (6.1.1) is,

ie., \( I(t) = \alpha \left[ T + \frac{\gamma T^2}{6} + \frac{\beta T^2}{2} - t + \frac{\gamma^3}{6} + \frac{\beta t^2}{2} - \beta tT - \frac{\gamma T^2}{2} \right] \)  

(6.1.2)

At \( t=0, I(0)=Z, \) therefore,

\[
Z = \alpha \left[ T + \frac{\gamma^3}{6} + \frac{\beta T^2}{2} \right]
\]

Hence total amount of deteriorated units are,

\[
D = Z - \int_0^T (\alpha + \beta I) dt
\]

\[
= Z - \alpha \left[ T + \frac{\gamma T^3}{6} + \frac{\beta T^2}{2} - t - \frac{\gamma^3}{3} + \frac{\beta t^2}{2} - \beta tT - \frac{\gamma T^2}{2} \right] dt
\]
\[ Z = \alpha \left( T + \frac{\gamma T^3}{6} + \frac{\beta T^2}{2} \right) - \beta \left( T + \frac{\beta T^2}{2} \right) \]

\[ = \alpha \left( T + \frac{\gamma T^3}{6} + \frac{\beta T^2}{2} \right) - \alpha \left( T + \beta T - \frac{\beta T^2}{2} \right) \]

ie., \[ D = \frac{\alpha T^3}{6} \quad (6.1.3) \]

The average inventory per time unit of the system during the period T is,

\[ A(I) = \frac{1}{T} \int_0^T I(t) dt \]

\[ = \frac{\alpha T}{T} \left[ T + \frac{\gamma T^3}{6} + \frac{\beta T^2}{2} - T + \frac{\gamma T^3}{6} + \frac{\beta T^2}{2} - \beta T - \frac{\gamma T^3}{6} \right] dt \]

\[ = \frac{\alpha}{T} \left( T \frac{\gamma T^3}{6} - T + \frac{\gamma T^3}{6} - \frac{\beta T^3}{2} + \frac{\gamma T^4}{12} + \frac{\beta T^3}{2} - \frac{\gamma T^3}{6} \right) \]

\[ = \alpha \left( \frac{T^2}{2} + \frac{\gamma T^4}{12} + \frac{\beta T^3}{6} \right) \]

\[ = \alpha \left( \frac{T}{2} + \frac{\gamma T^3}{12} + \frac{\beta T^3}{6} \right) \quad (6.1.4) \]

There are three possibilities regarding the period of permissible delay in payments. Viz., (i) \( t_p < T \), (ii) \( t_p > T \), (iii) \( t_p = T \).

In cases (ii) and (iii) the time period allowed to the receiver for settling the account \( t_p \) is greater than or equal to the maximum permissible duration T. Hence in these cases there is no use for remaining stock. Since after period \( t_p \),
interest is to be paid at the rate of \( i_p(i_e) \) per unit investment per time unit on the
capital tied in the remaining stock. Hence the maintenance cost may not be
increased in both of these cases. But in case (i) there is a chance of remaining
stock because of the condition \( t_p < T \).

As per the case (i), to avoid the payment of interest on the capital tied in
the remaining stock, we allow special discount. Hence there is no need to pay the
interest for the remaining items. It will reduce the average cost of maintaining
inventory.

Let \( I_e \) denote the average accumulated sales per unit time of the system during \( T \).
Interest on which is earned during \( t_p \). Then \( I_e \) is given by,

\[
I_e = \frac{1}{T} \int_0^T \int_0^t \left( \alpha + \beta I \right) dx dt
\]

\[
= \frac{1}{T} \int_0^T \int_0^t \left( 1 + \beta(T + \frac{\gamma T^3}{6}) + \frac{\beta T^2}{2} - \gamma x^3 + \frac{\beta x^2}{2} \right) dx dt
\]

\[
= \frac{\alpha}{T} \left[ x + \beta Tx - \frac{x^2}{2} \right]_0^t dt
\]

\[
= \frac{\alpha}{T} \left[ t + \beta T - \frac{t^2}{2} \right] dt
\]

\[
= \frac{\alpha}{T} \left( \frac{t^2}{2} + \frac{\beta T t^2}{2} - \frac{\beta t^3}{6} \right)_0^t
\]

\[
I_e = \frac{\alpha}{T} \left( \frac{t_p^2}{2} + \frac{\beta T t_p^2}{2} - \frac{\beta t_p^3}{6} \right)
\]

(6.1.5)
The average remaining stock per unit time of the system during $T$ is given by,

$$ I_p = \frac{1}{T} \int_{I_p}^{T} I(t)dt $$

$$ = \frac{\alpha}{T} \int_{I_p}^{T} \left( T + \frac{\gamma T^3}{6} + \frac{\beta T^2}{2} - t + \frac{\gamma t^3}{3} + \frac{\beta t^2}{2} - \beta t - \frac{\gamma T^2}{2} \right) dt $$

$$ = \frac{\alpha}{T} \left[ T \left( T + \frac{\gamma T^3}{6} + \frac{\beta T^2}{2} - \frac{t^2}{2} + \frac{\gamma t^4}{12} + \frac{\beta t^3}{6} \right) \right]_{I_p}^{T} $$

$$ = \frac{\alpha}{T} \left[ T^2 + \frac{\beta T^3}{6} + \frac{\gamma T^4}{12} - T I_p - \frac{\gamma T^3}{6} I_p - \frac{\beta T^2}{2} I_p \right] $$

(6.1.6)

The average cost per unit time is,

$$ C = \frac{pD}{T} + hA(I) + \frac{A}{T} - s I_p - c I_e $$

$$ = \frac{p\alpha \gamma T^2}{6} + h\alpha \left( \frac{T}{2} + \frac{\gamma T^3}{12} + \frac{\beta T^2}{6} \right) + \frac{A}{T} $$

$$ - s \alpha \left( \frac{T^2}{2} + \frac{\beta T^3}{6} + \frac{\gamma T^4}{12} - T I_p - \frac{\gamma T^3}{6} I_p - \frac{\beta T^2}{2} I_p \right) $$

$$ + c I \left( \frac{t_p^2}{2} + \frac{\beta T I_p^2}{2} - \frac{\beta t_p^3}{6} \right) $$

(6.1.7)

$$ \frac{dC}{dT} = \frac{p\gamma \alpha T}{3} + h\alpha \left( \frac{1}{2} + \frac{\gamma T^2}{4} + \frac{\beta T}{3} \right) - \frac{A}{T^2} + $$
\[ \alpha s \left( \frac{1}{2} + \frac{\gamma T^2}{4} + \frac{\beta T}{3} - \frac{\gamma T t_p}{3} - \frac{\beta t_p}{2} - \frac{t_p^2}{2 T^2} + \frac{\gamma t_p^4}{12 T^2} + \frac{\beta t_p^3}{6 T^2} \right) \]

\[ - \frac{\alpha c i_s}{T^2} \left( \frac{t_p^2}{2} - \frac{\beta t_p^3}{6} \right) = 0 \]

and the second derivative \( \frac{d^2C}{dT^2} \) is non-negative.

Hence the optimality is found.

**Conclusion**

We have discussed a model with deterioration of time which is linearly time dependent. Demand rate is a linear function of stock level at any time. Shortages are not considered. The optimal equations of \( T \) and \( C \) are derived. Further study could be taken up with backlogging of demand and finite production rate.
6.2 INVENTORY MODEL FOR DETERIORATING ITEMS USING DCF APPROACH

Demand also depends on the retailers sales efforts. Tripathi and Misra [87] presented the credit financing in economic ordering policy of non-deteriorating item with time dependent demand rate. Tripathi et. al., [88] have presented the economic ordering policies of time dependent deteriorating items in presence of trade credit using discounted cash flow approach. Neetu and Arun Kumar Tomer [56] have developed a model for discounted cash flow approach for deteriorating items to find the optimal present values of all future cash flows with demand rate is time dependent and follows the power demand pattern, shortages are not allowed.

In this section, we are using discounted cash flow approach for deteriorating items in the presence of trade credit period. Demand rate is stock dependent and follows power demand pattern, shortages are not allowed.

Mathematical models are derived for three different cases:
Case (i) Instantaneous cash flows,
Case (ii) Credit only on unit in stock,
Case (iii) Fixed credit period.

Our approach is to find the optimal value of all future cash flows for three cases. Numerical examples are also given to illustrate the results. Sensitivity analysis are incorporated.

Assumptions and Notations

- Deterioration of items starts after a definite time.
- Deterioration rate varies with time and follows a two parameter Weibull distribution.
- Replenishment is instantaneous.
• Lead time is zero.
• Shortages are not allowed.
• There is no repair or replacement of deteriorating items during the period.
• Inflation and time value of money is considered.
• The demand rate is stock-dependent.
• The demand rate \( D(t) \) at time \( t \) is, \( D(t) = a + bI(t), \quad I(t) > 0 \)
  \[ a, \quad I(t) \leq 0 \]
  where \( a \) and \( b \) are positive constants \( 0 \leq b \leq 1 \).
• ‘\( C \)’ is unit cost of the item.
• ‘\( Q \)’ is the order quantity.
• \( D(t) \) is the demand rate at time ‘\( t \)’.
• ‘\( i \)’ is inventory holding cost fraction.
• ‘\( IC \)’ is the out-of–pocket inventory carrying cost per unit time.
• ‘\( R \)’ is constant representing the difference between the discount rate and inflation rate.
• ‘\( H \)’ is the ordering cost per unit.
• \( I(t) \) is the inventory level at time \( t \).
• \( T_1 \) is the optimal cycle time for case I.
• \( T_2 \) is the optimal cycle time for case II.
• \( T_3 \) is the optimal cycle time for case III.
• \( Z_1(T) \) is the present value of all future cash-flows for case I.
• \( Z_2(T) \) is the present value of all future cash-flows for case II.
• \( Z_3(T) \) is the present value of all future cash-flows for case III.
• \( Z_1(T_1) \) is the optimum value of all future cash-flows for case I.
• \( Z_2(T_2) \) is the optimum value of all future cash-flows for case II.
• \( Z_3(T_3) \) is the optimum value of all future cash-flows for case III.
• \( T \) is the inventory cycle time.
MATHEMATICAL FORMULATION

The level of inventory $I(t)$ at time ‘t’ is depleted due to both market demand and deterioration. The differential equation describing the inventory system over $(0,T)$ is given by,

$$\frac{dI(t)}{dt} + \alpha \beta t^{\beta-1} I(t) = -(a + b I(t)), \quad 0 \leq t \leq T$$

(6.2.1)

With the boundary condition $I(T)=0$,

The solution of (6.2.1) is,

$$I(t) = a \left[ (T-t) + \frac{\alpha}{\beta+1} (T^{\beta+1} - t^{\beta+1}) + \frac{b}{2} (T^2 - t^2) \right] \left( 1 - \alpha t^{\beta} - bt \right)$$

(6.2.2)

Assuming a very small value of $\alpha (0 \leq \alpha \ll 1)$, the approximate solution is obtained by neglecting the second and higher order terms of $\alpha$.

Order quantity, $Q = I(0) = a \left[ T + \frac{\alpha}{\beta+1} T^{\beta+1} + \frac{b}{2} T^2 \right]$ 

(6.2.3)

The number of deteriorating units during one cycle is,

$$D(T) = Q - \int_0^T D(t) dt$$

$$= Q - \int_0^T (a + bt) dt$$

$$= a \left[ T + \frac{\alpha}{\beta+1} T^{\beta+1} + \frac{b}{2} T^2 \right] - \left[ aT + \frac{b}{2} T^2 \right]$$

$$= a \left[ \frac{\alpha}{\beta+1} T^{\beta+1} \right]$$

(6.2.4)

Using DCF approach, we have three cases on the trade credit terms.

Case I: Instantaneous cash-flows
In this case, we present the DCF approach to the inventory model of stock-dependent deteriorating items under instantaneous inventory holding cost. Hence at the beginning of each cycle, there will be cash out flows of ordering cost and purchasing cost. Since the inventory carrying cost is proportional to the value of the inventory, the out-of-pocket inventory carrying cost per unit time at ‘t’ is $iCI(t)$.

The present value of cash flow for the first order cycle $Z_1(T)$ is,

$$Z_1(T) = H + CQ + iC \int_0^T I(t) e^{-Rt} dt$$

$$= H + C a \left[ T + \frac{\alpha}{\beta + 1} T^{\beta+1} + \frac{b}{2} T^2 \right] +$$

$$iC \int_0^T a \left[ (T-t) + \frac{\alpha}{\beta + 1} (T^{\beta+1} - t^{\beta+1}) + \frac{b}{2} (T^2 - t^2) \right] (1 - \alpha t^\beta - bt) e^{-Rt} dt$$

$$= H + C a \left[ T + \frac{\alpha}{\beta + 1} T^{\beta+1} + \frac{b}{2} T^2 \right] +$$

$$iC \int_0^T a \left[ (T-t) + \frac{\alpha}{\beta + 1} (T^{\beta+1} - t^{\beta+1}) \right] e^{-Rt} dt$$

$$= H + C a \left[ T + \frac{\alpha}{\beta + 1} T^{\beta+1} + \frac{b}{2} T^2 \right] +$$

$$iCa \int_0^T \left[ (T-t) + \frac{\alpha}{\beta + 1} (T^{\beta+1} - t^{\beta+1}) + \frac{b}{2} (T^2 - t^2) \right.$$

$$- \alpha(T-t) t^\beta - b(T-t) t \left. \right] (1 - Rt) dt$$

$$= H + C a \left[ T + \frac{\alpha}{\beta + 1} T^{\beta+1} + \frac{b}{2} T^2 \right] +$$
\[ iCa \left[ Tt - \frac{t^2}{2} + \frac{\alpha}{\beta+1} T^{\beta+1}t - \frac{\alpha \beta+2}{(\beta+1)(\beta+2)} \frac{b}{2} T^2t \right. \]

\[ - \frac{bt}{6} - \frac{\alpha Tt^{\beta+1}}{\beta+1} - \frac{\beta t^2}{2} - \frac{RT}{3} t^2 + \frac{RT^3}{2} \left]_0^T \right. \]

\[ = H + C a \left[ T + \frac{\alpha}{\beta+1} T^{\beta+1} + \frac{b}{2} T^2 \right] + \]

\[ iCa \left[ \frac{T^2}{2} - \frac{\alpha T^\beta+2}{\beta+1} \frac{\beta+2}{(\beta+1)(\beta+2)} - \frac{RT^3}{2} - \frac{bT^3}{2} \right] \]

ie., \[ Z_1(T) = \frac{1}{RT} \left[ H + C a \left[ T + \frac{\alpha}{\beta+1} T^{\beta+1} + \frac{b}{2} T^2 \right]\right. + \]

\[ iCa \left[ \frac{T^2}{2} - \frac{\alpha T^\beta+2}{\beta+1} \frac{\beta+2}{(\beta+1)(\beta+2)} - \frac{RT^3}{2} - \frac{bT^3}{2} \right] \]

\[ = \frac{1}{R} \left[ H + C a \left[ 1 + \frac{\alpha}{\beta+1} T^{\beta} + \frac{b}{2} T^1 \right]\right. + \]

\[ iCa \left[ \frac{T^2}{2} - \frac{\alpha T^\beta+1}{\beta+2} - \frac{\alpha T^\beta+1}{(\beta+1)(\beta+2)} - \frac{RT^1}{2} - \frac{bT^2}{2} \right] \]

\[ = \frac{1}{R} \left[ \frac{H}{T} + C a \left[ 1 + \frac{\alpha T^\beta}{\beta+1} + \frac{b}{2} T \right]\right. + \]

\[ iCa \left[ \frac{T^2}{2} - \frac{\alpha T^\beta+1}{\beta+2} - \frac{\alpha T^\beta+1}{(\beta+1)(\beta+2)} - \frac{bT^2}{2} - \frac{RT^2}{2} \right] \]

(6.2.5)

The optimal value of T can be found by solving equation (6.2.5) partially w.r.t. ‘T’ and equating to zero, we get

\[ \frac{\partial Z_1(T)}{\partial T} = 0, \]
ie., \[
\frac{1}{R} \left[ \frac{-H}{T} + Ca \left( \frac{\alpha T^{\beta - 1}}{\beta + 1} + \frac{b}{2} \right) \right] + iCa \left[ \frac{1}{2} - \frac{\alpha(\beta + 1)T^\beta}{(\beta + 2)} \right] - \frac{\alpha T^\beta}{(\beta + 2)} + bT - RT \right] = 0
\]
(6.2.6)

Now, \( \frac{\partial^2 Z_i(T)}{\partial T^2} \) \( \neq 0 \) gives,

ie., \[
\frac{1}{R} \left[ \frac{H}{T^3} + Ca \left( \frac{\alpha T^{\beta - 2}}{\beta + 1} \right) - iCa \left( \frac{\alpha(\beta + 1)T^\beta}{\beta + 2} \right) \right] + \frac{\alpha T^{\beta - 1}}{\beta + 2} + b + R \right] \neq 0
\]

Thus optimum value of \( T \) can be found from equation (6.2.6). Let it be “\( T_i \)”. Then, the optimum value of order quantity \( Q_i = a \left( T_i + \frac{\alpha}{\beta + 1} T_i^{\beta + 1} + \frac{b}{2} T_i^2 \right) \)
and minimum cost \( Z_i(T_i) \) can be found from equation (6.2.5).

**Case II: Credit only on unit in stock**

In this, payment is connected to the subsequent use of items. Here, there exists a credit period \( M \). During this period the customers make payment to the supplier immediately after the use of the items and the remaining balance is paid by the customer on the last day of the credit period. Here we have two cases depending on the value of “\( T \)” and credit period “\( M \)”.

**Sub case I:** If \( T \leq M \), the present value of cash flows for the first cycle is,

\[
Z_2(T) = H + C \int_0^T (a + bt) e^{-Rt} dt + CD(T)e^{-RM} + iC \int_0^T I(t)e^{-Rt} dt
\]
\[ H + C \int_0^T (a + bt)(1 - R t) dt + CD(T)(1 - RM) \]

\[ + iC \int_0^T \left[ (T - t) + \frac{\alpha}{\beta + 1} (T^{\beta + 1} - t^{\beta + 1}) + \frac{b}{2} (T^2 - t^2) \right] (1 - \alpha t^\beta - bt) e^{-R t} dt \]

\[ = H + C \int_0^T (a - a R t + b t - R b t^2) dt + C \frac{\alpha T^{\beta + 1}}{\beta + 1} (1 - RM) \]

\[ + iC \int_0^T \left[ (T - t) + \frac{\alpha}{\beta + 1} (T^{\beta + 1} - t^{\beta + 1}) + \frac{b}{2} (T^2 - t^2) \right] e^{-R t} dt \]

\[ = H + C \left[ a t - \frac{a R t^2}{2} + b t - \frac{R b t^3}{3} \right]_0^T + C \frac{\alpha T^{\beta + 1}}{\beta + 1} (1 - RM) \]

\[ + iC a \left[ \frac{T^2}{2} - \frac{\alpha T^{\beta + 2}}{\beta + 1} - \frac{\alpha T^{\beta + 2}}{(\beta + 1)(\beta + 2)} - \frac{R T^3}{2} - \frac{b T^3}{2} \right] \]

\[ = H + C \left[ a T - \frac{a R T^2}{2} + b T - \frac{R b T^3}{3} \right] + C \frac{\alpha T^{\beta + 1}}{\beta + 1} (1 - RM) \]

\[ + \ iC a \left[ \frac{T^2}{2} - \frac{\alpha T^{\beta + 2}}{\beta + 1} - \frac{\alpha T^{\beta + 2}}{(\beta + 1)(\beta + 2)} - \frac{R T^3}{2} - \frac{b T^3}{2} \right] \quad (6.2.7) \]

Sub case II:

If \( T > M \) the present value of cash flows for the first cycle is,

\[ Z_2(T) = H + C \int_0^M (a + bt)e^{-R t} dt + C \left( \int_0^M (a + bt) dt \right) e^{-RM} + iC \int_0^T I(t)e^{-R t} dt \]

\[ = H + C \left[ a t - \frac{a R t^2}{2} + b t - \frac{R b t^3}{3} \right]_0^M \]
\[
+ C \left( a \left[ T + \frac{\alpha}{\beta + 1} T^{\beta + 1} + \frac{b}{2} T^2 \right] - \int_0^M (a + bt) dt \right) e^{-RM}
\]

\[
+ iCa \left[ \frac{T^2}{2} - \frac{\alpha T^{\beta + 2}}{\beta + 1} - \frac{\alpha T^{\beta + 2}}{(\beta + 1)(\beta + 2)} - \frac{RT^3}{2} - \frac{bT^3}{2} \right]
\]

\[
= H + C \left[ aM - \frac{aRM^2}{2} + bM - \frac{RbM^3}{3} \right]
\]

\[
+ C \left( a \left[ T + \frac{\alpha}{\beta + 1} T^{\beta + 1} + \frac{b}{2} T^2 \right] - \left[ aM + \frac{bM^2}{2} \right] \right)(1 - RM)
\]

\[
+ iCa \left[ \frac{T^2}{2} - \frac{\alpha T^{\beta + 2}}{\beta + 1} - \frac{\alpha T^{\beta + 2}}{(\beta + 1)(\beta + 2)} - \frac{RT^3}{2} - \frac{bT^3}{2} \right]
\] (6.2.8)

The present value of all future cash flows are,

\[
Z_2(T) = \frac{1}{RT} \left[ H + C \left[ aT - \frac{aRT^2}{2} + bT - \frac{RbT^3}{3} \right] + \frac{C\alpha T^{\beta + 1}}{\beta + 1} (1 - RM) \right]
\]

\[
+ iCa \left[ \frac{T^2}{2} - \frac{\alpha T^{\beta + 2}}{\beta + 1} - \frac{\alpha T^{\beta + 2}}{(\beta + 1)(\beta + 2)} - \frac{RT^3}{2} - \frac{bT^3}{2} \right]
\]

\[
= \frac{1}{R} \left[ \frac{H}{T} + C \left[ a - \frac{aRT}{2} + b - \frac{RbT^2}{3} \right] + \frac{C\alpha T^\beta}{\beta + 1} (1 - RM) \right]
\]

\[
+ iCa \left[ \frac{T^1}{2} - \frac{\alpha T^{\beta + 1}}{\beta + 1} - \frac{\alpha T^{\beta + 1}}{(\beta + 1)(\beta + 2)} - \frac{RT^2}{2} - \frac{bT^2}{2} \right]
\] for \( T \leq M \) (6.2.9)

and

\[
Z_2(T) = \frac{1}{RT} \left[ H + C \left[ aM - \frac{aRM^2}{2} + bM - \frac{RbM^3}{3} \right] \right]
\]
The necessary condition for $Z_2(T)$ to be minimum is

$$\frac{\partial Z_2(T)}{\partial T} = 0,$$ for $T \leq M$,

$$\frac{\partial Z_2(T)}{\partial T} = \frac{1}{R} \left[ -\frac{H}{T^2} + \frac{C}{T^2} \left[ aM - \frac{aRM^2}{2} + bM - \frac{RbM^3}{3} \right] + C\alpha T^{\beta-1} \left( \frac{1}{\beta+1} (\beta+2) - \frac{RT}{2} - \frac{bT^2}{2} \right) \right],$$

(6.2.11)

and

$$\frac{\partial Z_2(T)}{\partial T} = \frac{1}{R} \left[ -\frac{H}{T^2} - \frac{C}{T^2} \left[ aM - \frac{aRM^2}{2} + bM - \frac{RbM^3}{3} \right] \right] + C\left( a \left( \frac{\beta \alpha}{\beta+1} T^{\beta-1} + \frac{b}{2} \right) + \frac{1}{T^2} \left[ aM + \frac{bM^2}{2} \right] \right)(1-RM).$$
\[ + iC \left[ \frac{1 - \alpha T^\beta}{1} - \frac{\alpha T^\beta}{1} \frac{RT}{(\beta + 2)} - \frac{bT}{1} \right] = 0 \text{ for } T \leq M. \]

\[ \frac{\partial^2 Z_2(T)}{\partial T^2} = \frac{1}{R} \left[ \frac{H}{T^3} + C \left[ -\frac{Rb^2}{3} \right] + \frac{C \alpha \beta (\beta - 1) T^{\beta - 2}}{\beta + 1} (1 - RM) \right] \]

\[ + iC \left[ -\frac{\alpha \beta T^{\beta - 1}}{1} - \frac{\alpha \beta T^{\beta - 1}}{1} \frac{R - b}{(\beta + 2)} \right] \]

\[ \gg 0 \]

for \( T \leq M, \)

and

\[ \frac{\partial^2 Z_2(T)}{\partial T^2} = \frac{1}{R} \left[ \frac{H}{T^3} + C \left[ aM - \frac{aRM^2}{2} + bM - \frac{RbM^3}{3} \right] \right] \]

\[ + C \left( a \frac{\beta (\beta - 1) \alpha}{\beta + 1} T^{\beta - 2} \right) - \frac{1}{T^3} \left[ aM + \frac{bM^2}{2} \right] (1 - RM) \]

\[ + iC \left[ -\frac{\alpha \beta T^{\beta - 1}}{1} - \frac{\alpha \beta T^{\beta - 1}}{1} \frac{R - b}{(\beta + 2)} \right] \]

\[ \gg 0 \]

for \( T \geq M \)

and the optimum value of \( T = T_2 \) can be found from equations (6.2.11) and (6.2.12).

Hence if the payments to the supplier is done immediately after the use of materials and if the credit period \( M \) is longer than cycle length \( T \), then only out of pocket cost and the discounted cost of deterioration are relevant in finding the optimal cycle length. When \( T \leq M \), there would be no opportunity cost in the expression of total cash flows because in this case, the firm finances the inventory investment with the trade credit offered by its supplier.
Case III: Fixed Credit Period

In this case the credit period is fixed and hence the customer pays the full purchase amount on the last day of the credit period. The present value of cash-flows for one cycle, \( Z_3(T) \) is

\[
Z_3(T) = H + CQe^{-RM} + iC \int_0^T I(t)e^{-Rt} dt
\]

\[
= H + Ca \left[ T + \frac{\alpha}{\beta + 1} T^{\beta + 1} + \frac{b}{2} T^2 \right] e^{-RM} + iC \int_0^T I(t)e^{-Rt} dt
\]

\[
= H + Ca \left[ T + \frac{\alpha}{\beta + 1} T^{\beta + 1} + \frac{b}{2} T^2 \right] (1 - RM)
\]

\[
+ iCa \left[ \frac{T}{2} - \frac{\alpha T^{\beta + 1}}{\beta + 1} - \frac{\alpha T^{\beta + 1} (\beta + 1)(\beta + 2)}{2} - \frac{RT^3}{2} - \frac{bT^3}{2} \right]
\]

(6.2.13)

ie.,

\[
\frac{1}{RT} \left[ H + Ca \left[ T + \frac{\alpha}{\beta + 1} T^{\beta + 1} + \frac{b}{2} T^2 \right] (1 - RM)
\]

\[
+ iCa \left[ \frac{T}{2} - \frac{\alpha T^{\beta + 1}}{\beta + 1} - \frac{\alpha T^{\beta + 1} (\beta + 1)(\beta + 2)}{2} - \frac{RT^3}{2} - \frac{bT^3}{2} \right]
\]

\[
= \frac{1}{R} \left[ \frac{H}{T} + Ca \left[ 1 + \frac{\alpha}{\beta + 1} T^{\beta} + \frac{b}{2} T \right] (1 - RM)
\]

\[
+ iCa \left[ \frac{1}{2} - \frac{\alpha T^{\beta}}{\beta + 1} - \frac{\alpha T^{\beta} (\beta + 1)(\beta + 2)}{2} - \frac{RT^2}{2} - \frac{bT^2}{2} \right]
\]

(6.2.14)

For optimum value of \( T \), we have \( \frac{\partial Z_3(T)}{\partial T} = 0 \) and \( \frac{\partial^2 Z_3(T)}{\partial T^2} > 0 \),
ie.,  
\[
\frac{1}{R} \left[ \frac{H}{T^2} + Ca \left( \frac{\alpha \beta}{\beta + 1} T^{\beta-1} + \frac{b}{2} \right) (1 - RM) \right] 
\]
\[+ iCa \left[ -\frac{\alpha \beta T^{\beta-1}}{\beta + 1} - \frac{\alpha \beta T^{\beta-1}}{(\beta + 1)(\beta + 2)} - \frac{RT}{1} - \frac{bT}{1} \right] \] = 0 \hspace{1cm} (6.2.15)

and

ie.,  
\[
\frac{1}{R} \left[ \frac{H}{T^3} + Ca \left( \frac{\alpha \beta (\beta - 1)}{\beta + 1} T^{\beta-2} \right) (1 - RM) \right] 
\]
\[+ iCa \left[ -\frac{\alpha \beta (\beta - 1)T^{\beta-2}}{\beta + 1} - \frac{\alpha \beta (\beta - 1)T^{\beta-2}}{(\beta + 1)(\beta + 2)} - R - b \right] \] > 0 \hspace{1cm} (6.2.16)

Hence, the optimum value of \( T = T_3 \) can be obtained from equation (6.2.15) and the corresponding optimal order quantity is  
\[ Q = Q_3 = a \left[ T_3 + \frac{\alpha}{\beta + 1} T_3^{\beta+1} + \frac{b}{2} T_3^2 \right] \]

The corresponding optimal present value of all future cash-flows \( Z_3(T) = Z_3(T_3) \) is obtained from equations (6.2.15), (6.2.16) contains trade credit, the correct opportunity cost and the cost of deterioration, which are the discounted cost of deterioration. This reveals that effective capital cost should be less than that of the instantaneous payments.

**Numerical example**

We have considered the solution by taking a set of values of the parameters. The results of calculations are presented in the following table for the three cases. Sensitivity analysis could also be done to see the effect of such variations in the values of parameters.
Case I: Instantaneous cash-flows

Let $\alpha = 0.001, \beta = 1, i = 0.20, C = 20, H = 250, a = 1.1\ b = 0.5$

$T = 50\text{days} = 0.1369\ \text{year}, M = 100\ \text{days} = 0.27399\ \text{years},$

The results of calculations are presented in the following table:

**Sensitivity analysis on R**

<table>
<thead>
<tr>
<th>R</th>
<th>$Z_1(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>12327.86</td>
</tr>
<tr>
<td>0.18</td>
<td>10273.21</td>
</tr>
<tr>
<td>0.21</td>
<td>8805.6</td>
</tr>
<tr>
<td>0.24</td>
<td>7704.9</td>
</tr>
<tr>
<td>0.27</td>
<td>6848.79</td>
</tr>
<tr>
<td>0.3</td>
<td>6163.91</td>
</tr>
</tbody>
</table>

Comparing the different values in the above table, it is concluded that, when the increase in ‘R’ will decrease the present value. i.e., the decrease in present value of cash-flows with increase in ‘R’

**Sensitivity analysis on C**

Using the same values of parameters as like the previous example, and analyzing the present value on ‘C’ we get,

<table>
<thead>
<tr>
<th>C</th>
<th>$Z_1(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>12289.48</td>
</tr>
</tbody>
</table>
Comparing the different values in the above table, it is concluded that, when the increase in ‘C’ will increase the present value. ie., with increase in C, $Z_1(T)$ increases.

**Case II: Credit only on unit in stock**

There are two cases depending on the value of “T” and credit period “M”.

**Sub case I:** The present value of cash flows for the first cycle is calculated for $T \leq M$.

Let $\alpha = 0.001$, $\beta = 1$, $i = 0.20$, $C = 20$, $H = 250$, $a = 1.1$, $b = 0.5$

$T = 50$ days $= 0.1369$ year, $M = 100$ days $= 0.27399$ years,

The results of calculations are presented in the following table:

**Sensitivity analysis on R**

<table>
<thead>
<tr>
<th>$R$</th>
<th>$Z_2(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>254.39</td>
</tr>
<tr>
<td>0.18</td>
<td>254.38</td>
</tr>
<tr>
<td>0.21</td>
<td>254.37</td>
</tr>
</tbody>
</table>
Comparing the different values in the above table, it is concluded that, when the increase in ‘R’ will decrease the present value of future cash flows. i.e., the increase in ‘R’ will decrease in $Z_2(T)$.

**Sensitivity analysis on C**

Using the same values of parameters as like the previous example, and analyzing the value of $Z_2(T)$ on C, we get,

<table>
<thead>
<tr>
<th>C</th>
<th>$Z_2(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>253.2897</td>
</tr>
<tr>
<td>20</td>
<td>254.3863</td>
</tr>
<tr>
<td>25</td>
<td>255.4829</td>
</tr>
<tr>
<td>30</td>
<td>256.5795</td>
</tr>
<tr>
<td>35</td>
<td>257.6761</td>
</tr>
<tr>
<td>40</td>
<td>258.7727</td>
</tr>
</tbody>
</table>

Comparing the different values in the above table, it is concluded that, when the increase in ‘C’ will increase the present value of future cash flows.

**Sub case II:**

If $T > M$, then the present value of cash flows for the first cycle is analyzed as follows:
If all the parameters are same as the previous case, and let $T = 50$ days, $M = 30$ days we get,

**Sensitivity analysis on $R$**

<table>
<thead>
<tr>
<th>$R$</th>
<th>$Z_2(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>12364.92</td>
</tr>
<tr>
<td>0.18</td>
<td>10303.87</td>
</tr>
<tr>
<td>0.21</td>
<td>8831.7</td>
</tr>
<tr>
<td>0.24</td>
<td>7727.57</td>
</tr>
<tr>
<td>0.27</td>
<td>6868.8</td>
</tr>
<tr>
<td>0.3</td>
<td>6181.78</td>
</tr>
</tbody>
</table>

Comparing the different values in the above table, it is concluded that, when the increase in ‘$R$’ will decrease the present value of future cash flows. ie., when increase in ‘$R$’ there will be decrease in $Z_2(T)$.

Similarly,

**Sensitivity analysis on $C$**

All the parameters are same as the previous case., comparing the present value $Z_2(T)$ on ‘$C$’,

<table>
<thead>
<tr>
<th>$C$</th>
<th>$Z_2(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>12317.28</td>
</tr>
<tr>
<td>20</td>
<td>12364.92</td>
</tr>
<tr>
<td>25</td>
<td>12412.57</td>
</tr>
<tr>
<td>30</td>
<td>12460.21</td>
</tr>
</tbody>
</table>
From the above table, it is concluded that, when the increase in ‘C’ will increase the present value of future cash flows. ie., with increase in ‘C’, $Z_2(T)$ also increases.

**Case III: Fixed Credit Period**

In this case, credit period is fixed and hence, the customer pays the full purchase amount on the last day of the credit period.

Let $\alpha = 0.001$, $\beta = 1$, $i = 0.20$, $C = 20$, $H = 250$, $a = 1.1$, $b = 0.5$

$T = 50$ days = 0.1369 year, $M = 75$ days,

**Sensitivity analysis on R**

<table>
<thead>
<tr>
<th>$R$</th>
<th>$Z_3(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>12335.84</td>
</tr>
<tr>
<td>0.18</td>
<td>10279.08</td>
</tr>
<tr>
<td>0.21</td>
<td>8809.97</td>
</tr>
<tr>
<td>0.24</td>
<td>7708.13</td>
</tr>
<tr>
<td>0.27</td>
<td>6851.15</td>
</tr>
<tr>
<td>0.3</td>
<td>6165.56</td>
</tr>
</tbody>
</table>

With increase in ‘R’, the present value of cash-flow decreases.
Sensitivity analysis on C

All the parameters are same, and R = 0.15

<table>
<thead>
<tr>
<th>C</th>
<th>Z₃(T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>12295.47</td>
</tr>
<tr>
<td>20</td>
<td>12335.84</td>
</tr>
<tr>
<td>25</td>
<td>12376.22</td>
</tr>
<tr>
<td>30</td>
<td>12416.6</td>
</tr>
<tr>
<td>35</td>
<td>12456.97</td>
</tr>
<tr>
<td>40</td>
<td>12497.35</td>
</tr>
</tbody>
</table>

With increase in ‘C’, the present value of future cash-flow also increases.

Observations

Following observations can be made on the basis of the numerical example, whose results are presented in the tables with pre-assigned values.

Case I: Instantaneous cash-flows

The decrease in present value of cash-flows with increase in ‘R’

The increase in ‘C’ will increase in Z₁(T).

Case II: Credit only on unit in stock

Sub case I: If T ≤ M.

The increase in ‘R’ will decrease in Z₂(T).

The increase in ‘C’ will increase in Z₂(T).
**Sub case II:** If \( T > M \),

The increase in ‘\( R \)’, will decrease in \( Z_2(T) \).

The increase in ‘\( C \)’, will increase in \( Z_2(T) \).

**Case III: Fixed Credit Period**

The increase in ‘\( R \)’, will decrease in \( Z_3(T) \).

The increase in ‘\( C \)’, will increases in \( Z_3(T) \).

**CONCLUSION**

We have studied the discounted cash flow approach for deteriorating items in the presence of trade credit period. Demand rate is stock dependent and follows power demand pattern, shortages are not allowed. The mathematical models are derived for three different cases. Case (i) Instantaneous cash flows, Case (ii) Credit only on unit in stock, Case (iii) Fixed credit period. The optimal values are calculated. The effect of variation in the values of the other parameters on the total average cost of the system and time horizon are estimated by the analysis table.