CHAPTER 2

g* b-CLOSED SETS IN TOPOLOGICAL SPACES

2.1 INTRODUCTION

Andrijevic [3] introduced a new class of generalized open sets in a topological space, the so-called b-open sets. This type of sets was discussed by Ekici and Caldas [43] under the name of \( \gamma \)-open sets. The class of b-open sets is contained in the class of semi-preopen sets and contains all semi-open sets and preopen sets. Ahmed et al [2] studied generalized b-closed sets in topological spaces. Iyappan and Nagaveni [60] introduced a new version of generalized b-closed sets and also discovered the properties, the similarities and the difference of this set with other form of closed sets.

The aim of this chapter is to continue the study of generalized closed sets to introduce the concept of g* b-closed sets and g*b-open sets in topological spaces. We obtain the relations between g*b-closed (g*b-open) sets with various closed (open) sets. A pictorial representation of those relations is given. Some important characterizations are obtained for a subset to be g*b-closed (open).

2.2 g* b-CLOSED SETS

Definition 2.2.1 A subset \( A \) of a space \( (X, \tau) \) is called

(i) b-generalized closed set (briefly bg-closed) if \( bcl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is b-open.

(ii) g*b-closed set in \( (X, \tau) \) if \( bcl(A) \subseteq G \) whenever \( A \subseteq G \) where \( G \) is g-open.

(iii) bg*-closed set in \( (X, \tau) \) if \( bcl(A) \subseteq G \) whenever \( A \subseteq G \) where \( G \) is gb-open.

Complement of bg-closed set (resp. g*b-closed, bg*-closed sets) is bg-open set resp. g*b-open, bg*-open).
Theorem 2.2.2 If a subset A of a topological space \((X,\tau)\) is closed, then it is \(g^*b\)-closed.

**Proof:** Let \(G\) be a \(g\)-open set containing \(A\). Since \(A\) is closed \(\text{cl}(A) = A \subseteq G\). Also \(\text{bcl}(A) \subseteq \text{cl}(A)\). Thus \(\text{bcl}(A) \subseteq G\). Hence \(A\) is a \(g^*b\)-closed set in \((X,\tau)\).

The converse of the above theorem need not be true as seen from the following example.

**Example 2.2.3** Let \(X = \{a,b,c\}\) and \(\tau = \{X,\emptyset,\{a\}\}\), then the subset \(\{c\}\) is \(g^*b\)-closed but not closed in \((X,\tau)\).

**Corollary 2.2.4** If a subset \(A\) of a topological space \((X,\tau)\) is regular closed, then it is \(g^*b\)-closed.

**Proof:** Since every regular closed set is closed [53] but not conversely. By Theorem 2.2.2, every closed set is \(g^*b\)-closed. Hence every regular closed set is \(g^*b\)-closed but the converse is not true.

**Theorem 2.2.5** If a subset \(A\) of a topological space \((X,\tau)\) is \(g^*b\)-closed, then it is \(gb\)-closed.

**Proof:** Let \(G\) be an open set containing \(A\). Then \(A \subseteq G\) and \(\text{bcl}(A) \subseteq G\) as \(A\) is \(g^*b\)-closed set. Hence \(A\) is \(gb\)-closed set in \((X,\tau)\).

The converse of Theorem 2.2.5 need not be true as seen from the following example.

**Example 2.2.6** Let \(X = \{a,b,c\}\) and \(\tau = \{X,\emptyset,\{a\},\{b\},\{a,b\},\{a,b,c\}\}\), then the subset \(\{b,c\}\) is \(gb\)-closed but not \(g^*b\)-closed set in \((X,\tau)\).

**Theorem 2.2.7** If a subset \(A\) of a topological space \((X,\tau)\) is \(b\)-closed, then it is \(g^*b\)-closed.

**Proof:** Let \(G\) be a \(g\)-open set containing \(A\). Then \(A \subseteq G = \text{bcl}(A)\), as \(A\) is \(b\)-closed. Thus \(\text{bcl}(A) \subseteq G\). Hence \(A\) is \(g^*b\)-closed in \((X,\tau)\)
The converse of the above Theorem need not be true and it can be seen from the following example.

**Example 2.2.8** Let $X = \{a,b,c\}$ and $\tau = \{X,\emptyset,\{a\},\{a,b\}\}$, then the subset $\{b\}$ is g*b-closed but not b-closed in $(X,\tau)$.

**Theorem 2.2.9** If a subset $A$ of a topological space $(X,\tau)$ is g*b-closed, then it is bg-closed.

**Proof:** Let $G$ be an open set containing $A$. Then $G$ is b-open set containing $A$, as every open set is b-open. Thus $bcl(A) \subseteq G$, as $A$ is g*b-closed set. Hence $A$ is bg-closed in $(X,\tau)$.

The converse of the above Theorem is not true and can be seen from the following example.

**Example 2.2.10** Let $X = \{a,b,c\}$ and $\tau = \{X,\emptyset,\{a\},\{a,b\}\}$, then the subset $\{a,c\}$ is bg-closed but not g*b-closed.

**Theorem 2.2.11** Let $A$ be a subset of a topological space $(X,\tau)$. If $A$ is g*b-closed, then it is gs-closed.

**Proof:** Let $G$ be an open set containing $A$. Then $bcl(A) \subseteq G$, as $A$ is g*b-closed. Thus $scl(A) \subseteq bcl(A) \subseteq G$. Therefore $A$ is gs-closed in $(X,\tau)$.

The following example shows that the converse of the above Theorem need not be true.

**Example 2.2.12** Let $X = \{a,b,c\}$ and $\tau = \{X,\emptyset,\{a\},\{a,b\}\}$, then the subset $\{a\}$ is gs-closed but is not g*b-closed in $(X,\tau)$.

**Remark 2.2.13** The following example shows that every g-closed set is g*b-closed but not conversely.

**Example 2.2.14** Let $X = \{a,b,c\}$ and $\tau = \{X,\emptyset,\{b\},\{a,b\}\}$, then the subset $\{a\}$ is g*b closed but not g-closed.
Remark 2.2.15 The following examples show that the concept of semi-closed and g*\(b\)-closed sets are independent.

Example 2.2.16 Let \(X = \{a,b,c\} \) and \(\tau = \{X,\phi,\{a\},\{a,b\}\}\), then the subset \(\{a,c\}\) is g*\(b\)-closed but not semi-closed.

Example 2.2.17 Let \(X = \{a,b,c,d\} \) and \(\tau = \{X,\phi,\{a\},\{a,b\},\{a,b,c\}\}\), then the subset \(\{b\}\) is semi-closed but not g*\(b\)-closed.

Remark 2.2.18 The following example shows that g*\(b\)-closed sets are pre-closed but not conversely.

Example 2.2.19 Let \(X = \{a,b,c,d\} \) and \(\tau = \{X,\phi,\{a\},\{a,b\},\{a,b,c\}\}\), then the subset \(\{a,c,d\}\) is g*\(b\) closed but not pre-closed.

Remark 2.2.20 The following examples show that the concept of g-closed and g*\(b\)-closed sets are independent.

Example 2.2.21 Let \(X = \{a,b,c\} \) and \(\tau = \{X,\phi,\{a\}\}\), then the subset \(\{a,b\}\) is g-closed but not g*\(b\)-closed.

Example 2.2.22 Let \(X = \{a,b,c\} \) and \(\tau = \{X,\phi,\{a\},\{a,b\}\}\), then the subset \(\{b\}\) is g*\(b\)-closed but not g-closed.

Remark 2.2.23 The following examples show that the concept of bg*\(\cdot\)-closed and g*\(b\)-closed sets are independent.

Example 2.2.24 Let \(X = \{a,b,c\} \) and \(\tau = \{X,\phi,\{a\},\{a,b\}\}\), then the subset \(\{a,c\}\) is g*\(b\)-closed but not bg*\(\cdot\)-closed in \((X,\tau)\).

Example 2.2.25 Let \(X = \{a,b,c\} \) and \(\tau = \{X,\phi,\{a\},\{b\},\{a,b\}\}\), then the subset \(\{a,b\}\) is bg*\(\cdot\)-closed but not g*\(b\)-closed in \((X,\tau)\).

Remark 2.2.26 Every bg-closed set is bg* \(\cdot\) closed but not conversely.

Example 2.2.27 Let \(X = \{a,b,c,d\} \) and \(\tau = \{X,\phi,\{a\},\{a,b\},\{a,b,c\}\}\), then the subset \(\{a,c,d\}\) is bg*\(\cdot\)-closed but not bg-closed in \((X,\tau)\).
**Remark 2.2.28** Intersection of two bg* closed sets need not be a bg*-closed set.

**Example 2.2.29** Let $X=\{a,b,c\}$ and $\tau=\{X,\emptyset,\{a\}\}$, then the subsets $\{a,c\}$ is bg*-closed and $\{a,c\}$ is bg*-closed but their intersection $\{a\}$ is not bg*-closed.

**Remark 2.2.30** The following diagram shows the relationship between g*b-closed sets with various closed sets.

![Diagram showing the relationship between different types of closed sets]

where $A \rightarrow B$ (resp $A \rightarrow B$) represents $A$ implies $B$ (resp $A$ and $B$ are independent of each other).

### 2.3. PROPERTIES OF g*b-CLOSED SETS IN TOPOLOGICAL SPACES

**Theorem 2.3.1**

(i) Let $A \subseteq (X,\tau)$ be $g^*b$ - closed. Then $\text{cl}(A) \setminus A$ contains no non empty bg-closed set.

(ii) If $A$ is $g^*b$ - closed $A \subseteq B \subseteq \text{cl}(A)$ , then $\text{cl}(B) \setminus B$ contains no non empty bg closed sets.

**Proof:** (i) Let us suppose that $A$ is $g^*b$ closed and $F$ is any bg closed subset of $\text{cl}(A) \setminus A$. Then $F \subseteq X \setminus A \Rightarrow A \subseteq X \setminus F$ is bg open. Since $A$ is $g^*b$ closed, $\text{bcl}(A) \subseteq X \setminus F$. That is $F \subseteq X \setminus \text{bcl}(A)$. We already have $F \subseteq \text{bcl}(A)$ . So $F \subseteq \text{bcl}(A) \cap X \setminus \text{bcl}(A) = \emptyset$. Thus $F = \emptyset$. Hence $\text{cl}(A) \setminus A$ contains no non empty bg- closed set.
(ii) Let $A$ be $g^*b$-closed and $A \subseteq B \subseteq \text{cl}(A)$, then we have $\text{cl}(B) \cap X \setminus B \subseteq \text{cl}(A) \cap X \setminus A$. That is $\text{cl}(B) \setminus B \subseteq \text{cl}(A) \setminus A$. By (i) $\text{cl}(A) \setminus A$ has no nonempty $b$-closed set. Hence $\text{cl}(B) \setminus B$ contains no nonempty $b$-closed set.

**Theorem 2.3.2** A $g^*b$-closed set $A$ is $b$-closed if and only if $\text{bcl}(A) - A$ is $bg$-closed.

**Proof:** **Necessity** Since $A$ is $b$-closed, we have $\text{bcl}(A) = A$. Then $\text{bcl}(A) - A = \emptyset$ is $b$-closed and hence $bg$-closed.

**Sufficiency** By Theorem 2.3.1, $\text{bcl}(A) - A$ contains no non empty $bg$-closed set. That is $\text{bcl}(A) - A = \emptyset$. Therefore $\text{bcl}(A) = A$. Hence $A$ is $b$-closed.

**Remark 2.3.3** Let $A$ be an open set, then

(i) $\text{bcl}(A) \subseteq \text{cl}(A)$

(ii) $\text{bcl}(A \cap B) \subseteq \text{cl}(A \cap B)$

(iii) $\text{bcl}(A \cap B) \subseteq \text{bcl}(A) \cap \text{bcl}(B)$

**Theorem 2.3.4** If $A$ is a $g^*b$-closed set and $B$ is any set such that $A \subseteq B \subseteq \text{cl}(A)$, then $B$ is a $g^*b$-closed set.

**Proof:** Let $B \subseteq U$ where $U$ is $g$-open set. Since $A$ is $g^*b$-closed set and $A \subseteq U$, then $\text{bcl}(A) \subseteq U$ and also $\text{bcl}(A) = \text{bcl}(B)$. Therefore $\text{bcl}(B) \subseteq U$ and hence $B$ is a $g^*b$-closed set.

**Theorem 2.3.5**

(i) The intersection of a $g^*b$-closed set and a $b$ closed set is always a $g^*b$-closed set.

(ii) If $A$ is a $g^*b$-closed set and $A \subseteq B \subseteq \text{cl}(A)$, then $B$ is $g^*b$-closed set.

**Proof:** (i) Let $A$ be $g^*b$-closed set and let $F$ be a $b$-closed set. Suppose $G$ is a $g$-open set with $A \cap F \subseteq G$. Then $A \subseteq G \cup F^c$ where $G \cup F$ is $b$-open. Therefore $\text{bcl}(A) \subseteq G \cup F$. Now $\text{bcl}(A \cap F) \subseteq \text{bcl}(A) \cap \text{bcl}(F) = \text{bcl}(A) \cap F \subseteq G$. Hence $A \cap F$ is a $g^*b$-closed set.
(ii) Let $A$ be $g*b$-closed and $B \subseteq G$ where $G$ is a $g$-open set. Then $A \subseteq G$. Since $A$ is $g*b$-closed, $bcl(A) \subseteq G$. Hence by assumption $bcl(B) \subseteq bcl(A) \subseteq G$. Thus $bcl(B) \subseteq G$ implies that $B$ is $g*b$-closed.

**Theorem 2.3.6** Let $\{A_i : i \in I\}$ be a locally finite family of $g*b$-closed sets. Then $A = \bigcup A_i$ is $g*b$-closed for every $i \in I$.

**Proof:** Since $\{A_i : i \in I\}$ is locally finite, $cl(\bigcup A_i) = \bigcup cl(A_i)$. Assume that for some $b$-open set we have $A = \bigcup A_i \subseteq U$. Then $cl(\bigcup A_i) = \bigcup cl(A_i) \subseteq U$, since each $A_i$ is $g*b$-closed. Thus $A$ is $g*b$-closed.

**Remark 2.3.7** In a $T_{1/2}$-space, the concepts of $g*b$-closed, and $gb$-closed sets coincide.

**Theorem 2.3.8** Let $(X, \tau)$ be a topological space and $B \subseteq A \subseteq X$. If $B$ is a $g*b$-closed relative to $A$ and $A$ is a $g*b$-closed subset of $X$. Then $B$ is $g*b$-closed relative to $X$.

**Proof:** Let $B \subseteq G$ and $G$ be $g$-open set in $(X, \tau)$. Then $B \subseteq A \cap G$. Since $B$ is $g*b$-closed relative to $A$, $bcl(B) \subseteq A \cap G$. That is $A \cap bcl(B) \subseteq A \cap G$. We have $A \cap bcl(B) \subseteq G$ and hence $A \cap bcl(B) \cup (bcl(B))^c \subseteq G \cup (bcl(B))^c$. Since $A$ is $g*b$-closed in $(X, \tau)$, we have $bcl(A) \subseteq G \cup (bcl(B))^c$. Also $B \subseteq A$ implies $bcl(B) \subseteq bcl(A)$. Thus $bcl(B) \subseteq bcl(A) \subseteq g \cup (bcl(B))^c$. Therefore $bcl(B) \subseteq G$, since $bcl(B)$ is not contained in $(bcl(B))^c$. Thus $B$ is $g*b$-closed relative to $X$.

**Corollary 2.3.9** If $A$ is $g*b$-closed and $F$ is closed in a topological space $(X, \tau)$, then $A \cap F$ is $g*b$-closed in $(X, \tau)$.

**Proof:** Clearly, $A \cap F$ is closed in $A$. Therefore $Cl_A(A \cap F) = A \cap F$ in $A$. Let $A \cap F \subseteq G$, where $G$ is $g$-open in $A$.

Then $(bcl)_A (A \cap F) \subseteq cl_A (A \cap F) \subseteq G$. Thus $(bcl)_A (A \cap F) \subseteq G$ and hence $A \cap F$ is $g*b$-closed in $A$. Therefore $A \cap F$ is $g*b$-closed in $(X, \tau)$. 

22
**Theorem 2.3.10** Let $B \subseteq A \subseteq X$ where $A$ is $g*b$-closed and open set. Then $B$ is $g*b$-closed relative to $A$ if and only if $B$ is $g*b$-closed in $X$.

**Proof:** We first note that since $B \subseteq A$ and $A$ is both a $g*b$-closed and open set, then $\text{bcl}(A) \subseteq A$ and thus $\text{bcl}(B) \subseteq \text{bcl}(A) \subseteq A$. Now from the fact that $A \cap \text{bcl}(B) = \text{bcl}_A(B)$, we have $\text{bcl}(B) \subseteq \text{bcl}(A) \subseteq A$. If $B$ is $g*b$-closed relative to $A$ and $U$ is open subset of $X$ such that $B \subseteq U$, then $B=B \cap A \subseteq U \cap A$ where $U \cap A$ is open in $A$. Hence as $B$ is $g*b$-closed relative to $A$, $\text{bcl}(B) = \text{bcl}_A(B) \subseteq U \cap A \subseteq U$. Therefore $B$ is $g*b$-closed in $X$.

Conversely if $B$ is $g*b$-closed in $X$ and $U$ is an open subset of $A$ such that $B \subseteq U$, then $U=V \cap A$ for some open subset $V$ of $X$. As $B \subseteq V$ and $B$ is $g*b$-closed in $X$, $\text{bcl}(B) \subseteq V$. Thus $\text{bcl}_A(B)=\text{bcl}(B) \cap A \subseteq V \cap A = U$. Therefore $B$ is $g*b$-closed relative to $A$.

**Theorem 2.3.11** For a subset $A$ of a semi-regular space $(X,\tau)$, the following are equivalent. (i) $A$ is $g$-closed. (ii) $A$ is $g*b$-closed.

**Proof:** By remark 2.2.13 (i) $\Rightarrow$ (ii).

To prove (ii) $\Rightarrow$ (i). In a semi regular space we have $\tau = \tau_s = \tau_b$. Let $U$ be an open set such that $A \subseteq U$. Open set is $b$-open in semi regular space. Since $A$ is $g*b$-closed, $\text{bcl}(A) \subseteq U$. Therefore $A$ is $g$-closed.

**Remark 2.3.12**

(i) The intersection of two $g*b$-closed sets need not be always $g*b$-closed.

(ii) Countable union of $g*b$-closed sets need not be $g*b$-closed.

**Corollary 2.3.13** Let $A$ be open $g*b$-closed set. Then $A \cap F$ is $g*b$-closed whenever $F \in \text{bC}(X)$.

**Proof:** Since $A$ is $g*b$-closed and open, then $\text{bcl}(A) \subseteq A$ and thus $A$ is $b$-closed. Hence $A \cap F$ is $b$-closed in $X$ which implies that $A \cap F$ is $g*b$-closed in $X$ by theorem 2.3.5.
Theorem 2.3.14 Let $A \subset Y \subset X$, and suppose that $A$ is $g^{*}b$-closed in $X$. Then $A$ is $g^{*}b$-closed relative to $Y$ provided $Y$ is open or dense in $X$.

Proof: Let $A \subset Y \cap G$ where $G$ is $b$-open in $X$. Then $A \subset G$. Since $A$ is $g^{*}b$-closed in $X$, then $bcl(A) \subset G$. This implies that $bcl(A) \cap Y \subset G \cap Y$. Thus $A$ is $g^{*}b$-closed relative to $Y$.

Theorem 2.3.15 Every paracompact subset of regular space $(X, \tau)$ is $g$-closed and hence $g^{*}b$-closed but not necessarily closed.

Proof: Let $A \subset U$ where $A$ is paracompact and $U$ is open. By regularity, choose for each $x \in A$ an open set $U_x$ such that $x \in U_x$ and $U_x \subset cl(U_x) \subset U$. Clearly \[ \{U_x: x \in A\} \] is an open cover of $A$. Let $\{V_i: i \in I\}$ be a locally finite open refinement of $\{U_x: x \in A\}$. For every $V_i$, choose an $U_x$ from $\{U_x: x \in A\}$ such that $V_i \subset U_x$. Now $cl(A) \subset cl(\bigcup V_i : i \in I) = \bigcup_{i \in I} cl(V_i) \subset \bigcup_{i \in I} cl(U_x) \subset U$. This shows that $A$ is $g$ closed and hence $A$ is $g^{*}b$-closed. On the other hand, the real line with the minimal topology shows that $g$-closedness cannot be replaced by closedness.

Theorem 2.3.16

(i) The intersection of a $g^{*}b$-closed set and a $gb$-closed set is always a $g^{*}b$-set.

(ii) If $A$ is $g^{*}b$-closed set and $A \subset B \subset bcl(A)$, then $B$ is $g^{*}b$-closed set.

Proof: (i) Let $A$ be a $g^{*}b$-closed set and let $F$ be a $gb$-closed set. Suppose $G$ is a $gb$-open set with $A \cap F \subset G$, then $A \subset G \cup F^c$ where $G \cup F^c$ is $gb$-open. Therefore, $bcl(A) \subset G \cup F^c$. Now $bcl(A \cap F) \subset cl(A) \cap cl(F) = cl(A) \cap F \subset G$. Hence $A \cap F$ is a $g^{*}b$-closed set.

(ii) Let $A$ be $g^{*}b$-closed and $B \subset G$ where $G$ is a $b$-open set. Then $A \subset G$. Since $A$ is $g^{*}b$-closed, $bcl(A) \subset G$. Hence by assumption $bcl(B) \subset bcl(A) \subset G$. Thus $bcl(B) \subset G$ implies that $B$ is $g^{*}b$-closed.
Theorem 2.3.17 For a subset A of a topological space \( (X, \tau) \) the following conditions are equivalent

(i) A is clopen

(ii) A is b-open and g*b-closed.

Proof: (i) \( \Rightarrow \) (ii). Obvious from the definition 2.2.1

(ii) \( \Rightarrow \) (i) Since A is b-open and g*b-closed, \( \text{cl}(A) \subset A \). But \( A \subset \text{cl}(A) \). So \( A = \text{cl}(A) \) implies that A is closed. Every b-open set is open. Hence A is open and closed. Thus A is clopen.

Corollary 2.3.18 For a space \( (X, \tau) \) the following conditions are equivalent.

(i) \( X \) is hyperconnected,

(ii) Every subset of \( X \) is g*b-closed and \( X \) is connected.

Proof: (i) \( \Rightarrow \) (ii) Since \( X \) is hyperconnected, then the only regular open subsets of \( X \) are the trivial ones. Thus the only b-open sets of \( X \) are trivial ones. Hence every subset of \( X \) is trivially g*b-closed. On the other hand, every hyperconnected space is connected.

(ii) \( \Rightarrow \) (i) Let \( A \) be a non-empty proper b-open subset of \( X \). By (ii) \( A \) is g*b-closed. Thus from theorem 1.2.14 \( A \) is clopen. Also by assumption \( X \) must be connected. Thus the contradiction shows that \( X \) is hyperconnected.

2.4. g*b-OPEN SETS IN TOPOLOGICAL SPACES

In this section the concept of g*b-open sets in topological spaces and some of their properties are studied.

Definition 2.4.1 A subset of a topological space \( (X, \tau) \) is called g*b-open set if its complement \( A^c \) is g*b-closed in \( (X, \tau) \).

Theorem 2.4.2 If a subset \( A \) of a topological space \( (X, \tau) \) is open, then it is g*b-open in \( (X, \tau) \).
Proof: Let A be an open set in (X,τ). Then is A^c closed in (X,τ). By Theorem 2.2.2 A^c is g*b-closed in (X,τ). Hence A is g*b-open in (X,τ).

The converse need not be true as seen from the following example.

Example 2.4.3 Let X= {a,b,c,d} and τ = {X,ϕ, {a}, {b}, {a,b}}, then the subset {b,c} is g*b-open but not open in (X,τ).

Theorem 2.4.4

(i) Every b-open set in (X,τ) is g*b-open in (X,τ).
(ii) Every g*b-open set in (X,τ) is gb-open in (X,τ).
(iii) Every g*b-open set in (X,τ) is bg-open in (X,τ).
(iv) Every g*b-open set in (X,τ) is gs-open in (X,τ).
(v) Every g*b-open set in (X,τ) is gpr-open in (X,τ).

By means of simple examples, it can be shown that the converses of (i),(ii),(iii),(iv) and (v) need not hold.

Theorem 2.4.5 A subset A of a topological space (X,τ) is g*b-open if and only if G ⊆ b-int(A) whenever G ⊆ A and G is g-closed.

Proof: Assume that A is g*b-open. Then A^c is g*b-closed. Let G be g-closed set in (X,τ) contained in A. Then G^c is g-open set in (X,τ) containing A^c. Since A^c is g*b-closed, bcl(A^c) ⊆ G^c. Equivalently G ⊆ (b-int(A)).

Conversely assume that G is contained in b-int(A), whenever G is contained in A and G is g-closed in (X,τ). Then G^c is a g-open set containing A^c. Now b-int(A) ⊆ G is equivalent to bcl(A^c). Therefore A^c is g*b-closed. Hence A is g*b-open.

2.5. PLACATIONS OF g*b-CLOSED SETS

In this section we introduce and study five new spaces viz. b_T, b_Tg, b_Tg^*, b_Tg^*T_c, b_Tg^*T_b, b_Tg^*T_g -space, g*b_T -space, g*b_Tg -space, g*b_Tg^* -space, using g*b-closed sets.
**Definition 2.5.1** A space $\langle X, \tau \rangle$ is said to be

(i) $g^*bT_c$ -space if every $g^*b$-closed subset of $\langle X, \tau \rangle$ is closed in $\langle X, \tau \rangle$.

(ii) $g^*bT_b$ -space if every $g^*$-closed subset of $\langle X, \tau \rangle$ is $b$-closed in $\langle X, \tau \rangle$.

(iii) $g^*bT_g^*b$ -space if every $wbg$-closed subset of $\langle X, \tau \rangle$ is closed in $\langle X, \tau \rangle$.

(iv) $b^*T_g^*b$ -space if every $bg$-closed subset of $\langle X, \tau \rangle$ is $g^*$-closed in $\langle X, \tau \rangle$.

(v) $wbg^*bT_g^*$ -space if every $wbg$-closed subset of $\langle X, \tau \rangle$ is $g^*$-closed in $\langle X, \tau \rangle$.

(vi) $wgb^*bT_g^*$ -space if every $wgb$-closed subset of $\langle X, \tau \rangle$ is $g^*$-closed in $\langle X, \tau \rangle$.

(vii) $g^*bT_g^*$ -space if every $g^*$-closed subset of $\langle X, \tau \rangle$ is $g^*$-closed in $\langle X, \tau \rangle$.

(viii) $b$-space if every $b$-closed set in $\langle X, \tau \rangle$ is closed in $\langle X, \tau \rangle$.

**Theorem 2.5.2**

(i) Every $g^*bT_c$ -space is $g^*bT_b$ -space.

(ii) Every $wbg^*bT_g^*$ -space is $wgb^*bT_g^*$ -space.

(iii) Every $wgb^*bT_g^*$ -space is $g^*T_g^*$ -space.

(iv) Every $wbg^*bT_g^*$ -space is $b^*T_g^*$ -space.

(v) Every $wbg^*bT_g^*$ -space is $g^*bT_b$ -space.

(vi) Every $b^*T_g^*$ -space is $g^*bT_g^*$ -space.

**Remark 2.5.3** For the above spaces we have the following implications.
Theorem 2.5.4 If \((X, \tau)\) is a \(g^*_b T_b\)−space and a b-space, then it is a \(g^*_b T_c\)−space.

**Proof:** Let A be a \(g^*_b\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is a \(g^*_b T_b\)−space, A is b-closed in \((X, \tau)\). Since \((X, \tau)\) is a b-space, every b-closed is closed and hence A is closed in \((X, \tau)\). Hence \((X, \tau)\) is a \(g^*_b T_c\)−space.

**Remark 2.5.5** The spaces \(g^*_b T_{g^*}\) and space \(g^*_b T_b\) are independent as seen from the following examples.

**Example 2.5.6** Let \(X= \{a,b,c\}\) with topology \(\tau= \{X,\phi,\{a,b\}\}\), then \((X, \tau)\) is a \(g^*_b T_{g^*}\)−space but not a \(g^*_b T_b\)−space, since \(\{a,c\}\) is \(g^*\)-closed but not b-closed in \((X, \tau)\).

**Example 2.5.7** Let \(X=\{a,b,c\}\) with topology \(\tau= \{X,\phi,\{a\},\{a,b\}\}\). Then \((X, \tau)\) is a \(g^*_b T_b\)−space but not a \(g^*_b T_{g^*}\)−space, since \(\{b\}\) is \(g^*_b\)-closed but not \(g^*\)-closed in \((X, \tau)\).

**Theorem 2.5.8** If \((X, \tau)\) is both b-space and \(g^*_b T_b\)−space, then it is a \(g^*_b T_{g^*}\)−space.

**Proof:** Let A be a \(g^*_b\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is a \(g^*_b T_b\)−space, A is b-closed in \((X, \tau)\). Since \((X, \tau)\) is a b-space, every b-closed set is closed and hence A is closed in \((X, \tau)\). We know that every closed set is \(g^*\)-closed in \((X, \tau)\), A is \(g^*\)-closed. Hence \((X, \tau)\) is a \(g^*_b T_{g^*}\)−space.
Theorem 2.5.9 If \((X,\tau)\) is both \(T^*_{1/2}\)-space and \(g^\ast_b T^*_g\)-space, then it is a \(g^\ast_b T_b\)-space.

Proof: Let \(A\) be a \(g^\ast_b\)-closed set in \((X,\tau)\). Since \((X,\tau)\) is a \(g^\ast_b T^*_g\)-space, \(A\) is \(g^\ast\)-closed. Since \((X,\tau)\) is a \(T^*_{1/2}\)-space, \(A\) is closed in \((X,\tau)\). Since every closed set is \(b\)-closed, \(A\) is \(b\)-closed in \((X,\tau)\). Hence it is a \(g^\ast_b T_b\)-space.

2.6 \(g^\ast_b\)-Interior and \(g^\ast_b\)-Closure

Definition 2.6.1 Let \(X\) be a topological space and let \(x \in X\). A subset \(N\) of \(X\) is said to be \(g^\ast_b\)-neighborhood of \(x\) if there exists a \(g^\ast_b\)-open set \(G\) such that \(x \in G \subseteq N\).

Definition 2.6.2 Let \(A\) be a subset of \(X\). A point \(x \in A\) is said to be \(g^\ast_b\)-interior point of \(A\) if \(A\) is a \(g^\ast_b\)-nbhd of \(x\). The set of all \(g^\ast_b\)-interior points of \(A\) is called the \(g^\ast_b\)-interior of \(A\) and is denoted by \(g^\ast_b\)-int\((A)\).

Theorem 2.6.3 If \(A\) be a subset of \(X\). Then \(g^\ast_b\)-int\((A) = \bigcup \{G: G\ is \ g^\ast_b\text{-open, } G \subseteq A\}\).

Proof: Let \(A\) be a subset of \(X\).

\[
x \in g^\ast_b\text{-int}(A) \iff x \text{ is a } g^\ast_b\text{-interior point of } A.
\]

\[
\iff A \text{ is a } g^\ast_b\text{-nbhd of point } x.
\]

\[
\iff \text{there exists } g^\ast_b\text{-open set } G \text{ such that } x \in G \subseteq A.
\]

\[
\iff x \in \bigcup \{G: G\ is \ g^\ast_b\text{-open, } G \subseteq A\}.
\]

Hence \(g^\ast_b\text{-int}(A) = \bigcup \{G: G\ is \ g^\ast_b\text{-open, } G \subseteq A\}\).

Theorem 2.6.4 Let \(A\) and \(B\) be subsets of \(X\). Then

(i) \(g^\ast_b\text{-int}(X) = X\) and \(g^\ast_b\text{-int}(\emptyset) = \emptyset\).

(ii) \(g^\ast_b\text{-int}(A) \subseteq A\).

(iii) If \(B\) is any \(g^\ast_b\)-open set contained in \(A\), then \(B \subseteq g^\ast_b\text{-int}(A)\).

(iv) If \(A \subseteq B\), then \(g^\ast_b\text{-int}(A) \subseteq g^\ast_b\text{-int}(B)\).

(v) \(g^\ast_b\text{-int}(g^\ast_b\text{-int}(A)) = g^\ast_b\text{-int}(A)\).
Proof: (i) Since $X$ and $\phi$ are $g*b$-open sets, by Theorem 2.6.3 $g*b$-int$(X) = \bigcup \{G : G$ is $g*b$-open, $G \subset X\} = X \cup \{A : A$ is a $g*b$-open set$\} = X$. That is $g*b$-int$(X) = X$.

Since $\phi$ is the only $g*b$-open set contained in $\phi$, $g*b$-int$(\phi) = \phi$.

(ii) Let $x \in g*b$-int$(A) \Rightarrow x$ is a $g*b$-interior point of $A$.

$\Rightarrow A$ is a $g*b$-nbhd of $x$.

$\Rightarrow x \in A$.

Thus $x \in g*b$-int$(A) \Rightarrow x \in A$.

Hence $g*b$-int$(A) \subset A$.

(iii) Let $B$ be any $g*b$-open sets such that $B \subset A$. Let $x \in B$, then since $B$ is a $g*b$-open set contained in $A$, $x$ is $g*b$-interior point of $A$. That is $B$ is a $g*b$-int$(A)$. Hence $B \subset g*b$-int$(A)$.

(iv) Let $A$ and $B$ be subsets of $X$ such that $A \subset B$. Let $x \in g*b$-int$(A)$. Then $x$ is a $g*b$-interior point of $A$ and so $A$ is $g*b$-nbhd of $x$. Since $A \subset B$, $B$ is also a $g*b$-nbhd of $x$. This implies that $x \in g*b$-int$(B)$. Thus we have shown that $x \in g*b$-int$(A) \Rightarrow x \in g*b$-int$(B)$. Hence $g*b$-int$(A) \subset g*b$-int$(B)$.

(v) Let $A$ be any subset of $X$. By definition of $g*b$-interior, $g*b$-int$(A) = \bigcap \{A \subset F \in g*bC(X)\}$, if $A \subset F \in g*bC(X)$, then $g*b$-int$(A) \subset F$. Since $F$ is $g*b$-closed set containing $g*b$-int$(A)$, by (iii) $g*b$-int$(g*b$-int$(A)) \subset F$.

Hence $g*b$-int$(g*b$-int$(A)) \subset \bigcap \{A \subset F \in g*bC(X)\} = g*b$-cl$(A)$. That is $g*b$-int$(g*b$-int$(A)) = g*b$-int$(A)$.

Theorem 2.6.5 If a subset $A$ of space $X$ is $g*b$-open, then $g*b$-int$(A) = A$.

Proof: Let $A$ be $g*b$-open subset of $X$. We know that $g*b$-int$(A) \subset A$. Also, $A$ is $g*b$-open set contained in $A$. From Theorem 2.6.4 (iii) $A \subset g*b$-int$(A)$. Hence $g*b$-int$(A) = A$.

The converse of the above Theorem need not be true and can be seen from the following example.
Example 2.6.6 Let $X=\{a,b,c\}$, $\tau=\{X,\emptyset,\{b\},\{c\},\{b,c\}\}$. Let $A=\{a\}$ be a subset of $X$. Then $g^*\text{int}(A) = \text{int}(A)$, but $A$ is not $g^*$-open.

**Theorem 2.6.7** If $A$ and $B$ are subsets of $X$, then $g^*\text{int}(A) \cup g^*\text{int}(B) \subset g^*\text{int}(A \cup B)$.

**Proof:** We know that $A \subset A \cup B$ and $B \subset A \cup B$. We have, by Theorem 2.6.4 (iv), $g^*\text{int}(A) \subset g^*\text{int}(A \cup B)$ and $g^*\text{int}(B) \subset g^*\text{int}(A \cup B)$. This implies that $g^*\text{int}(A) \cup g^*\text{int}(B) \subset g^*\text{int}(A \cup B)$.

**Theorem 2.6.8** If $A$ and $B$ are subsets of space $X$, then $g^*\text{int}(A \cap B) = g^*\text{int}(A) \cap g^*\text{int}(B)$.

**Proof:** We know that $A \cap B \subset A$ and $A \cap B \subset B$. We have, by Theorem 2.6.4 (iv) $g^*\text{int}(A \cap B) \subset g^*\text{int}(A)$ and $g^*\text{int}(A \cap B) \subset g^*\text{int}(B)$. This implies that $g^*\text{int}(A \cap B) \subset g^*\text{int}(A) \cap g^*\text{int}(B) \to (1)$. Again, let $x \in g^*\text{int}(A) \cap g^*\text{int}(B)$. Then $x \in g^*\text{int}(A)$ and $x \in g^*\text{int}(B)$. Hence $x$ is a $g^*$-interior point of each sets $A$ and $B$. It follows that $A$ and $B$ are $g^*$-nbhds of $x$, so that their intersection $A \cap B$ is also $g^*$-nbhds of $x$. Hence $x \in g^*\text{int}(A \cap B)$. Therefore $g^*\text{int}(A \cap B) \subset g^*\text{int}(A \cap B) \to (2)$. From (1) and (2), we get $g^*\text{int}(A) \cup g^*\text{int}(B) \subset g^*\text{int}(A \cup B)$.

**Theorem 2.6.9** If $A$ is a subset of $X$, then $\text{int}(A) \subset g^*\text{int}(A)$.

**Proof:** Let $A$ be a subset of a space $X$.

Let $x \in \text{int}(A)$ $\Rightarrow x \in \bigcup \{G: G \text{ is } g^*\text{-open, } G \subset A\}$.

$\Rightarrow$ there exists an open set $G$ such that $x \in G \subset A$.

$\Rightarrow$ there exist an $g^*\text{-open}$ set $G$ such that $x \in G \subset A$, as every open set is a $g^*\text{-open}$ set in $X$.

$\Rightarrow x \in \bigcup \{G: G \text{ is } g^*\text{-open, } G \subset A\}$.

$\Rightarrow x \in g^*\text{int}(A)$. 

31
Thus $x \in \text{int}(A) \Rightarrow x \in g^*b\text{-int}(A)$.

Hence $\text{int}(A) \subseteq g^*b\text{-int}(A)$.

**Theorem 2.6.10** If $A$ is a subset of $X$, then $gp\text{-int}(A) \subseteq g^*b\text{-int}(A)$, where $gp\text{-int}(A)$ is given by $gp\text{-int}(A) = \{x \in \bigcup \{G \subseteq X : G \text{ is gp-open, } G \subseteq A \} \}$. 

**Proof:** Let $A$ be a subset of a space $X$.

Let $x \in gp\text{-int}(A) \Rightarrow x \in \bigcup \{G \subseteq X : G \text{ is gp-open, } G \subseteq A \}.

\Rightarrow$ there exists a gp-open set $G$ such that $x \in G \subseteq A$,

$\Rightarrow$ there exists a $g^*b$-open set $G$ such that $x \in G \subseteq A$.

as every gp-open set is $g^*b$-open set in $X$.

\Rightarrow $x \in \bigcup \{G \subseteq X : G \text{ is } g^*b\text{-open, } G \subseteq A \}.$

\Rightarrow $x \in g^*b\text{-int}(A)$.

Thus $x \in gp\text{-int}(A) \Rightarrow x \in g^*b\text{-int}(A)$.

Hence $gp\text{-int}(A) \subseteq g^*b\text{-int}(A)$.

Analogous to closure in a space $X$, we define $g^*b$-closure in a space $X$ as follows.

**Definition 2.6.11** Let $A$ be a subset of a space $X$, we define the $g^*b$-closure of $A$ to be the intersection of all $g^*b$-closed sets containing $A$. In symbols, $g^*b\text{-cl}(A) = \bigcap \{F : A \subseteq F \in g^*bC(X) \}$.

**Theorem 2.6.12** If $A$ and $B$ are subsets of a space $X$, then

(i) $g^*b\text{-cl}(X) = X$ and $g^*b\text{-cl}(\emptyset) = \emptyset$.

(ii) $A \subseteq g^*b\text{-cl}(A)$.

(iii) If $B$ is any $g^*b$-closed set containing $A$, then $g^*b\text{-cl}(A) \subseteq B$.

(iv) If $A \subseteq B$, then $g^*b\text{-cl}(A) \subseteq g^*b\text{-cl}(B)$.

(v) $g^*b\text{-cl}(A) = g^*b\text{-cl}(g^*b\text{-cl}(A))$. 
Proof: (i) By the definition of \(g^{*b}\)-closure, \(x\) is the only \(g^{*b}\)-closure set containing \(X\). Therefore \(g^{*b}\text{-cl}(X) = \text{Intersection of all } g^{*b}\text{-closed sets containing } X = \cap \{X\} = X\). That is \(g^{*b}\text{-cl}(X) = X\). By the definition of \(g^{*b}\)-closure, \(g^{*b}\text{-cl}(\emptyset) = \text{intersection of all } g^{*b}\text{-closed sets containing } \emptyset = \emptyset\). That is \(g^{*b}\text{-cl}(\emptyset) = \emptyset\).

(ii) By the definition of \(g^{*b}\)-closure of \(A\), it is obvious that \(A \subseteq g^{*b}\text{-cl}(A)\).

(iii) Let \(B\) be any \(g^{*b}\)-closed set containing \(A\). Since \(g^{*b}\text{-cl}(A)\) is the intersection of all \(g^{*b}\)-closed sets containing \(A\), \(g^{*b}\text{-cl}(A)\) is contained in every \(g^{*b}\)-closed sets containing \(A\). Hence in particular \(g^{*b}\text{-cl}(A) \subseteq B\).

(iv) Let \(A\) and \(B\) be the subsets of \(X\) such that \(A \subseteq B\). By definition of \(g^{*b}\)-closure, \(g^{*b}\text{-cl}(B) = \cap \{F : B \subseteq F \in g^{*b}C(X)\}\). If \(B \subseteq F \in g^{*b}C(X)\), then \(g^{*b}\text{-cl}(B) \subseteq F\). Since \(A \subseteq B\), \(A \subseteq B \subseteq F \in g^{*b}C(X)\), we have \(g^{*b}\text{-cl}(A) \subseteq F\). Therefore \(g^{*b}\text{-cl}(A) \subseteq \cap \{F : B \subseteq F \in g^{*b}C(X)\} = g^{*b}\text{-cl}(B)\). That is \(g^{*b}\text{-cl}(A) \subseteq g^{*b}\text{-cl}(B)\).

(v) Let \(A\) be any subset of \(X\). By the definition of \(g^{*b}\)-closure, \(g^{*b}\text{-cl}(A) = \cap \{F : A \subseteq F \in g^{*b}C(X)\}\), if \(A \subseteq F \in g^{*b}C(X)\), then \(g^{*b}\text{-cl}(A) \subseteq F\). Since \(F\) is \(g^{*b}\)-closed set containing \(g^{*b}\text{-cl}(A)\), by (iii) \(g^{*b}\text{-cl}(g^{*b}\text{-cl}(A)) \subseteq F\). Hence \(g^{*b}\text{-cl}(g^{*b}\text{-cl}(A)) \subseteq \cap \{F : A \subseteq F \in g^{*b}C(X)\} = g^{*b}\text{-cl}(A)\). That is \(g^{*b}\text{-cl}(A) = g^{*b}\text{-cl}(g^{*b}\text{-cl}(A))\).

Theorem 2.6.13 If \(A \subseteq X\) is \(g^{*b}\)-closed then \(g^{*b}\text{-cl}(A) = A\).

Proof: Let \(A\) be \(g^{*b}\)-closed set of \(X\). We know that \(A \subseteq g^{*b}\text{-cl}(A)\). Also \(A \subseteq A\) and \(A\) is \(g^{*b}\)-closed. By Theorem 2.6.12 (iii) \(g^{*b}\text{-cl}(A) \subseteq A\). Hence \(g^{*b}\text{-cl}(A) = A\).

Theorem 2.6.14 If \(A\) and \(B\) are subsets of a space \(X\), then \(g^{*b}\text{-cl} (A \cap B) \subseteq g^{*b}\text{-cl}(A) \cap g^{*b}\text{-cl}(B)\).

Proof: Let \(A\) and \(B\) be subsets of \(X\). Clearly \(A \cap B \subseteq A\) and \(A \cap B \subseteq B\). By Theorem 2.6.12 (iv) \(g^{*b}\text{-cl}(A \cap B) \subseteq g^{*b}\text{-cl}(A)\) and \(g^{*b}\text{-cl}(A \cap B) \subseteq g^{*b}\text{-cl}(B)\). Hence \(g^{*b}\text{-cl}(A \cap B) \subseteq g^{*b}\text{-cl}(A) \cap g^{*b}\text{-cl}(B)\).
Theorem 2.6.15 If $A$ and $B$ are subsets of a space $X$, then $g^*b\text{-cl}(A \cup B) = g^*b\text{-cl}(A) \cup g^*b\text{-cl}(B)$.

**Proof:** Let $A$ and $B$ be subsets of $X$. Clearly $A \subset A \cup B$ and $B \subset A \cup B$. Hence $g^*b\text{-cl}(A) \cup g^*b\text{-cl}(B) \subset g^b(A \cup B) \rightarrow (1)$.

Now to prove $g^*b\text{-cl}(A \cup B) \subset g^*b\text{-cl}(A) \cup g^*b\text{-cl}(B)$. Let $x \in g^*b\text{-cl}(A \cup B)$ and suppose $x \notin g^*b\text{-cl}(A) \cup g^*b\text{-cl}(B)$, then there exists $g^*b$-closed sets $A_1$ and $B_1$ with $A \subset A_1$, $B \subset B_1$ and $x \notin A_1 \cup B_1$. We have $A \cup B \subset A_1 \cup B_1$ and $A_1 \cup B_1$ is $g^*b$-closed set such that $x \notin A_1 \cup B_1$. Thus $x \notin g^*b\text{-cl}(A \cup B)$ which is a contradiction to $x \in g^*b\text{-cl}(A \cup B)$. Hence $g^*b\text{-cl}(A \cup B) \subset g^*b\text{-cl}(A) \cup g^*b\text{-cl}(B) \rightarrow (2)$.

From (1) and (2), we have $g^*b\text{-cl}(A \cup B) = g^*b\text{-cl}(A) \cup g^*b\text{-cl}(B)$.

Theorem 2.6.16 For an $x \in X$, $x \in g^*b\text{-cl}(A)$ if and only if $V \cap A \neq \emptyset$ for every $g^*b$-open sets $V$ containing $x$.

**Proof:** Let $x \in X$ and $x \in g^*b\text{-cl}(A)$. To prove $V \cap A \neq \emptyset$ for every $g^*b$-open sets $V$ containing $x$, we prove the result by contradiction. Suppose there exist a $g^*b$-open set $V$ containing $x$ such that $V \cap A \neq \emptyset$. Then $A \subset X-V$ and $X-V$ is $g^*b$-closed. We have $g^*b\text{-cl}(A) \subset X-V$. This shows that $x \notin g^*b\text{-cl}(A)$, which is a contradiction. Hence $V \cap A \neq \emptyset$ for every $g^*b$-open set $V$ containing $x$.

Conversely, let $V \cap A \neq \emptyset$ for every $g^*b$-open set $V$ containing $x$. To prove $x \in g^*b\text{-cl}(A)$, we prove the result by contradiction. Suppose $x \notin g^*b\text{-cl}(A)$. Then there exists a $g^*b$-closed subset $F$ containing $A$ such that $x \notin F$. Then $x \in X-F$ and $X-F$ is $g^*b$-open. Also $(X-F) \cap A = \emptyset$, which is a contradiction. Hence $x \in g^*b\text{-cl}(A)$.

Theorem 2.6.17 If $A$ is subset of a space $X$, then $g^*b\text{-cl}(A) \subset \text{cl}(A)$.

**Proof:** Let $A$ be a subset of a space $X$. By definition of closure, $\text{cl}(A) = \bigcap \{ F \subset X : A \subset X \in C(X) \}$. If $A \subset F \in C(X)$, then $A \subset F \in g^*bC(X)$, because every closed set is $g^*b$-closed. That is $g^*b\text{-cl}(A) \subset \bigcap \{ F \subset X : A \subset X \in C(X) \} = \text{cl}(A)$. Hence $g^*b\text{-cl}(A) \subset \text{cl}(A)$. 34
**Theorem 2.6.18** If $A$ is a subset of a space $X$, then $\text{gp-cl}(A) \subseteq \text{g}*\text{b-cl}(A)$ where $\text{gp-cl}(A)$ is given by $\text{gp-cl}(A) = \cap \{ F \subseteq X : A \subseteq F \in GPC(X) \}$

**Proof:** Let $A$ be the subset of $X$. By the definition of $\text{g}*\text{b-closure}$, $\text{g}*\text{b-cl}(A) = \cap \{ F \subseteq X : A \subseteq F \in g^*b\text{C}(X) \}$. If $A \subseteq F \in g^*b\text{C}(X)$, then $A \subseteq F \in g^*b\text{C}(X)$, because every $g^*b$-closed set is $\text{gp}$-closed set. That is $\text{gp-cl}(A) \subseteq F$. Therefore $\text{gp-cl}(A) \subseteq \cap \{ F \subseteq X : A \subseteq F \in g^*b\text{C}(X) \} = g^*\text{b-cl}(A)$. Hence $\text{gp-cl}(A) \subseteq g^*\text{b-cl}(A)$.

**Definition 2.6.19** Let $\tau_{g^*b}$ be the topology on $X$ generated by $g^*\text{b-closure}$ in the usual manner. That is $\tau_{g^*b} = \{ \cup \subseteq X : g^*b-cl(U^c) = U^c \}$.

**Remark 2.6.20** For any topology $\tau$ on $X$, $\tau \subseteq \tau_{g^*b}$

**Theorem 2.6.21** Let $A$ be any subset of $X$. Then

(i) $\left( g^*\text{b-int}(A) \right)^c = g^*\text{b-cl}(A^c)$

(ii) $g^*\text{b-int}(A) = \left( g^*\text{b-cl}(A^c) \right)^c$

(iii) $g^*\text{b-cl}(A) = \left( g^*\text{b-int}(A^c) \right)^c$

**Proof:** (i) Let $x \in (g^*\text{b-int})$. Then $x \not\in (g^*\text{b-int}(A))^c$. That is every $g^*\text{b}$-open set $U$ containing $x$ is such that $U \not\subset A$. That is every $g^*\text{b}$-open set containing $x$ such that $U \cap A^c \neq \emptyset$. By Theorem 2.6.18, $x \in g^*\text{b-cl}(A^c)$ and therefore $(g^*\text{b-int}(A))^c \subseteq g^*\text{b-cl}(A^c)$. Conversely, let $x \in g^*\text{b-cl}(A^c)$. Then by Theorem 2.6.16 every $g^*\text{b}$-open set $U$ containing $x$ is such that $U \cap A^c \neq \emptyset$. That is every $g^*\text{b}$-open set containing $x$ is such that $U \not\subset A$. This implies by definition of $g^*\text{b}$-interior of $A$, $x \not\in (g^*\text{b-int}(A))^c$. That is $x \in (g^*\text{b-int}(A))^c$ and $g^*\text{b-cl}(A^c) \subseteq (g^*\text{b-int}(A))^c$. Thus $(g^*\text{b-int}(A))^c = g^*\text{b-cl}(A^c)$.

(ii) Follows by taking compliments in (i).

(iii) Follows by replacing $A$ by $A^c$ in (i)