CHAPTER 6

(1,2)*-g*b-SPACES

6.1 INTRODUCTION

The systematic study of bitopological spaces was initiated by Kelly [69] in 1963. The notion of generalized closed sets and $T_{1/2}$ spaces in bitopological spaces were introduced by Fukutake[45,46]. Also several authors [23,46,72,73,80,85] turned their attention to generalization of various concepts of topology by considering bitopological spaces.

In this chapter, we introduce and investigate $(1,2)^*-g^*b$-open sets and $(1,2)^*$ semi pre b-$T_{1/2}$ spaces in bitopological spaces. We also introduce the concept of generalized b-homeomorphisms in bitopological spaces and establish its relations with other homeomorphisms.

Throughout this chapter $(X,\tau_1,\tau_2)$ and $(Y,\sigma_1,\sigma_2)$ denote two nonempty bitopological spaces on which no separation axioms are assumed, unless otherwise mentioned.

6.2 $(1,2)^*-g^*b$-OPEN SETS

**Definition 6.2.1** A subset $A$ of a bitopological space $(X,\tau_1,\tau_2)$ is said to be a $(1,2)^*$-b-closed space if every $(1,2)^*$ b-closed set in $X$ is $\tau_{1,2}$-closed in $X$.

**Definition 6.2.2** A subset of a bitopological space $X$ is called

(i) $(1,2)^*$- g* b-closed if $(1,2)^*$bcl($A$) $\subseteq G$, whenever $A \subseteq G$ and $G$ is $(1,2)^*$-g-open in $X$.

(ii) $(1,2)^*$- g* b-open (briefly $(1,2)^*$- g*b-open) if $A^c$ is $(1,2)^*$- g* b-closed.

**Theorem 6.2.3** A subset $A$ of a bitopological space $X$ is $(1,2)^*$- g* b-open if and only if $F \subseteq (1,2)^*$- bint($A$) whenever $F$ is $(1,2)^*$-g-closed and $F \subseteq A$. 

102
**Proof:** Assume that $A$ is $(1,2)^*\text{-}g^*\text{-}b$-open in $X$. Let $F$ be $(1,2)^*\text{-}g$-closed and $F \subseteq A$. This implies $F^c$ is $(1,2)^*\text{-}g$-open and $A^c \subseteq F^c$. Since $A^c$ is $(1,2)^*\text{-}g^*\text{-}b$-closed, $(1,2)^*\text{-}bcl(A^c) \subseteq F^c$. Since $(1,2)^*\text{-}bcl(A^c) = ((1,2)^*\text{-}b\text{-}int(A))^c$, $((1,2)^*\text{-}b\text{-}int(A))^c \subseteq F^c$. Therefore $F \subseteq (1,2)^*\text{-}b\text{int}(A)$.

Conversely, assume that $F \subseteq (1,2)^*\text{-}b\text{int}(A)$ whenever $F$ is $(1,2)^*\text{-}g$-closed and $F \subseteq A$. Let $G$ be a $(1,2)^*\text{-}g$-open set containing $A^c$. Therefore $G^c$ is $(1,2)^*\text{-}g^*\text{-}b$-closed in $X$. Hence $A$ is $(1,2)^*\text{-}g^*\text{-}b$-open in $X$.

**Remark 6.2.4** Intersection of two $(1,2)^*\text{-}g^*\text{-}b$-open sets need not be a $(1,2)^*\text{-}g^*\text{-}b$-open set.

**Example 6.2.5** Let $\tau_1 = \{X, \phi, \{a\}, \{a,b\}\}$, $\tau_2 = \{X, \phi, \{b\}\}$, then $\tau_1 \tau_2$-open sets are $\{X, \phi, \{a\}, \{b\}, \{a,b\}\}$. Here the subset $\{a,c\}$ is $(1,2)^*\text{-}g^*\text{-}b$-open but their intersection $\{c\}$ is not $(1,2)^*\text{-}g^*\text{-}b$-open.

**Remark 6.2.6** If $A$ is $(1,2)^*\text{-}g^*\text{-}b$-closed then $(1,2)^*\text{-}bcl(A) - A$ contains no nonempty $(1,2)^*\text{-}g$-closed sets.

**Theorem 6.2.7** If a set $A$ is $(1,2)^*\text{-}g^*\text{-}b$-closed, then $(1,2)^*\text{-}bcl(A) - A$ is $(1,2)^*\text{-}g^*\text{-}b$-open.

**Proof:** If $A$ is $(1,2)^*\text{-}g^*\text{-}b$-closed, by the Remark 6.2.6 $(1,2)^*\text{-}bcl(A) - A$ contains no nonempty $(1,2)^*\text{-}g$-closed set. Therefore, by Theorem 6.2.3 $(1,2)^*\text{-}bcl(A) - A$ is $(1,2)^*\text{-}g^*\text{-}b$-open.

**Theorem 6.2.8** If a set $A$ is $(1,2)^*\text{-}g^*\text{-}b$-open in $X$, then $G = X$ whenever $G$ is $(1,2)^*\text{-}g$-open and $(1,2)^*\text{-}b\text{int}(A) \cup A^c \subseteq G$.

**Proof:** Suppose that $G$ is $(1,2)^*\text{-}g$-open and $(1,2)^*\text{-}b\text{int}(A) \cup A^c \subseteq G$. Now $G^c \subseteq (1,2)^*\text{-}bcl(A) \cap A = (1,2)^*\text{-}bcl(A^c) - A^c$. Since $G^c$ is $(1,2)^*\text{-}g$-closed and $A^c$ is $(1,2)^*\text{-}g^*\text{-}b$-closed, $G^c = \phi$ and hence $G = X$. 
**Theorem 6.2.9** For each \( x \in X \), \( \{x\} \) is \((1,2)^*\)-g-closed or \((1,2)^*\)-g**b-open.

**Proof:** If \( \{x\} \) is not \((1,2)^*\)-g-closed, then the only \((1,2)^*\)-g-open set containing \( X-\{x\} \) is \( X \). Thus \( X-\{x\} \) is \((1,2)^*\)-g**b-closed and \( \{x\} \) is \((1,2)^*\)-g**b-open.

### 6.3. \((1,2)^*\)-STRONGLY b-T\(_{1/2}\) SPACES

In this section we introduce \((1,2)^*\)-strongly b-T\(_{1/2}\) spaces and study some of their properties.

**Definition 6.3.1** A subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is said to be an \((1,2)^*\)-b space if every \((1,2)^*\)-b closed set in \( X \) is \( \tau_{1,2} \)-closed in \( X \).

**Definition 6.3.2** A subset of a bitopological space \((X, \tau_1, \tau_2)\) is said to be a \((1,2)^*\)-b T\(_{1/2}\) space if every \((1,2)^*\)-gb closed set in \( X \) is \( \tau_{1,2} \)-b closed in \( X \).

**Definition 6.3.3** A subset of a bitopological space \((X, \tau_1, \tau_2)\) is said to be a \((1,2)^*\)-strongly b T\(_{1/2}\) space if every \((1,2)^*\)-gb closed set in \( X \) is \( \tau_{1,2} \)-gb closed in \( X \).

**Remark 6.3.4** \((1,2)^*\)-strongly b- T\(_{1/2}\) bitopological space and \((1,2)^*\)-T\(_{1/2}\) space are independent.

**Example 6.3.5** Let \( X=\{a, b, c\} \); \( \tau_1=\{X, \phi, \{a\}\} \); \( \tau_2=\{X, \phi, \{a, b\}\} \); \( \tau_{1,2} \)-open sets are \( \{X, \phi, \{a\}, \{a, b\}\} \). This bitopological space is \((1,2)^*\)-strongly b- T\(_{1/2}\) space but not \((1,2)^*\)-T\(_{1/2}\) space.

**Example 6.3.6** Let \( X=\{a, b, c, d\} \); \( \tau_1=\{X, \phi, \{a\}, \{a, b, d\}\} \); \( \tau_2=\{X, \phi, \{a, d\}, \{a, c, d\}\} \); \( \tau_{1,2} \)-open sets are \( \{X, \phi, \{a\}, \{a, d\}, \{a, c, d\}, \{a, b, d\}\} \). This bitopological space is \((1,2)^*\)-T\(_{1/2}\) space but not \((1,2)^*\)-strongly b- T\(_{1/2}\) space.

**Theorem 6.3.7** Every \((1,2)^*\)-b space is \((1,2)^*\)-strongly b- T\(_{1/2}\) space.

**Proof:** Let \((X, \tau_1, \tau_2)\) be a \((1,2)^*\)-b-space. Let \( A \) be a \((1,2)^*\)-gb-closed set in \( X \). Since \( X \) is \((1,2)^*\)-b- T\(_{1/2}\), \( A \) is b-closed in \( X \). Then it follows from the definition 6.3.2 that \( X \) is \((1,2)^*\)-strongly b- T\(_{1/2}\).
Lemma 6.3.8 If a set $A$ in bitopological space $X$ is $(1,2)^*\text{-gb}$ -closed then $(1,2)^*\text{-bcl}(A) - A$ does not contain non-empty $\tau_{1,2}$ -closed set.

**Proof:** Let $F$ be a $\tau_{1,2}$ -closed set of $(1,2)^*\text{-bcl}(A) - A$. Then $A \subseteq X - F$ where $A$ is $(1,2)^*\text{-gb}$-closed and $X - F$ is $\tau_{1,2}$-open. Therefore $(1,2)^*\text{-bcl}(A) \subseteq X - F$ or equivalently $F \subseteq X - (1,2)^*\text{-bcl}(A)$. Thus $F \subseteq (1,2)^*\text{-bcl}(A) \cap (1,2)^*\text{-bcl}(A))^c = \emptyset$ or $F = \emptyset$.

Lemma 6.3.9 If every singleton subset in a bitopological space $X$ is $\tau_{1,2}$ -closed or $(1,2)^*\text{-b}$-open then $X$ is $(1,2)^*\text{-b}$- $T_{1/2}$.

**Theorem 6.3.10** Every $(1,2)^*\text{-} T_{1/2}$ bitopological space is $(1,2)^*\text{-b}$- $T_{1/2}$.

**Proof:** Let $(X, \tau_1, \tau_2)$ be a $(1,2)^*\text{-} T_{1/2}$ bitopological space. First let us prove that for each $x \in X$, $\{x\}$ is $\tau_{1,2}$ -open or closed. Let $x \in X$. If $\{x\}$ is not $\tau_{1,2}$ – closed, then $X - \{x\}$ is not $\tau_{1,2}$ -open. Therefore the only $\tau_{1,2}$-open set containing $X - \{x\}$ is $X$ and hence $X - \{x\}$ is $(1,2)^*\text{-g}$-closed. Since $X$ is $(1,2)^*\text{-} T_{1/2}$, $X - \{x\}$ is $\tau_{1,2}$ -closed or $\{x\}$ is $\tau_{1,2}$ -open. Thus for each $x \in X$, $\{x\}$ is $\tau_{1,2}$ -open or $\tau_{1,2}$ -closed. This implies for each $x \in X$, $\{x\}$ is $(1,2)^*\text{-b}$-open or $\tau_{1,2}$ -closed. Then by lemma 6.3.9, $X$ is $(1,2)^*\text{-b}$- $T_{1/2}$.

**Theorem 6.3.11** Every $(1,2)^*\text{-} T_{1/2}$ bitopological space is $(1,2)^*\text{-}$ strongly b-$T_{1/2}$ but not conversely.

**Proof:** Let $(X, \tau_1, \tau_2)$ be a $(1,2)^*\text{-} T_{1/2}$ -space. Then by Theorem 6.3.9, $X$ is $(1,2)^*\text{-b}$- $T_{1/2}$. Let $A$ be a $(1,2)^*\text{-gb}$-closed set in $X$. Then $A$ is $(1,2)^*\text{-b}$-closed in $X$ and hence $A$ is $(1,2)^*\text{-g}$-b-closed set. Hence $X$ is $(1,2)^*\text{-}$-strongly b $T_{1/2}$.

The bitopological space given in example 6.2.6 is $(1,2)^*\text{-}$ strongly b- $T_{1/2}$ but not $(1,2)^*\text{-}$- $T_{1/2}$ space.

**Theorem 6.3.12** Every singleton set in a $(1,2)^*\text{-}$ strongly b - $T_{1/2}$ bitopological space $X$ is $\tau_{1,2}$ – closed or $(1,2)^*\text{-g}$-b-closed.
**Proof:** Let $x \in X$. If the set $\{x\}$ is not $\tau_{1,2}$-closed, then the only $\tau_{1,2}$-open set containing $\{x\}^c$ is $X$. Since $X$ is $(1,2)^*$-strongly b-$T_{1/2}$, $\{x\}^c$ is $(1,2)^*$-g*b-closed. Therefore $\{x\}$ is $(1,2)^*$-g*b-open.

**Corollary 6.3.13** Every singleton set in a $(1,2)^*$-strongly b-$T_{1/2}$ space $X$ is $\tau_{1,2}$-closed or $(1,2)^*$-gb-open.

From the above results we have the following diagram

![Diagram](image)

**6.4. $(1,2)^*$-GENERALIZED b-HOMEOMORPHISMS**

**Definition 6.4.1** A subset $S$ of $X$ is said to be $(1,2)^*$ generalized b-closed (briefly $(1,2)^*$ gb-closed) if $(1,2)^*$bcl$(S) \subseteq F$ whenever $S \subseteq F$ and $F$ is $\tau_{1,2}$-open. The complement of $(1,2)^*$gb-closed set is said to be $(1,2)^*$gb-open.

**Definition 6.4.2** A subset $S$ of $X$ is said to be b-generalized closed (briefly $(1,2)^*$ bg-closed) if and only if $(1,2)^*$bcl$(S) \subseteq F$ whenever $S \subseteq F$ and $F$ is $(1,2)^*$ b-open.

**Definition 6.4.3** A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be

(i) $(1,2)^*$ gb-open if $f(U)$ is $(1,2)^*$ gb-open set in $Y$ for every $\tau_{1,2}$-open sets in $X$.

(ii) $(1,2)^*$ gb-closed if $f(U)$ is $(1,2)^*$ gb-closed set in $Y$ for every $\tau_{1,2}$-closed sets in $X$. 

106
(iii) \((1,2)^*\) gb-continuous if \(f^{-1}(V)\) is \((1,2)^*\) gb-closed set in X for every \(\sigma_{1,2}\) closed set V in Y.

(iv) \((1,2)^*\) gb-irresolute if \(f^{-1}(V)\) is \((1,2)^*\) gb-closed set in X for every \((1,2)^*\) gb-closed set V in Y.

**Definition 6.4.4** A bijection \(f : X \to Y\) is called

(i) \((1,2)^*\) generalized b-homeomorphism (briefly \((1,2)^*\) gb-homeomorphism) if \(f\) is both \((1,2)^*\) gb-continuous and \((1,2)^*\) gb-open.

(ii) \((1,2)^*\) gbc-homeomorphism if \(f\) is \((1,2)^*\) gb-irresolute and its inverse \(f^{-1}\) is \((1,2)^*\) gb-irresolute.

(iii) \((1,2)^*\) - irresolute if \(f^{-1}(V)\) is \((1,2)^*\) b-open in X for every \((1,2)^*\) b-open set V in Y.

**Example 6.4.5** Let \(X = \{a, b, c\}\), \(\tau_1 = \{\phi, X, \{a\}\}\) and \(\tau_2 = \{\phi, X\}\). Let \(Y = \{p, q, r\}\), \(\sigma_1 = \{\phi, X, p\}\) and \(\sigma_2 = \{\phi, Y, \{p, q\}\}\). Then the \(\tau_{1,2}\) open sets are \(\{X, \phi, \{a\}\}\) and \(\sigma_{1,2}\) open sets are \(\{\phi, Y, \{p\}, \{p, q\}\}\). Define a mapping \(f : X \to Y\) by \(f(a) = p, f(b) = r, f(c) = q\). Then \(f\) is both \((1,2)^*\)-gb-open and \((1,2)^*\)-gb-continuous and hence \((1,2)^*\)-gb-homeomorphism.

**Example 6.4.6** Consider the same space as in example 6.4.5 and define the mapping \(f : X \to Y\) as \(f(a) = p, f(b) = q\) and \(f(c) = r\), the mapping \(f\) is both \((1,2)^*\)-g*gb-continuous and \((1,2)^*\)-g*gb-open and hence \((1,2)^*\)-g*gb-homeomorphism.

**Example 6.4.7** Let \(X = \{a, b, c\}\), \(\tau_1 = \{\phi, X, \{b\}\}\) and \(\tau_2 = \{\phi, X\}\). Let \(Y = \{p, q, r\}\), \(\sigma_1 = \{\phi, X, r\}\) and \(\sigma_2 = \{\phi, Y, \{r\}\}\). Then the \(\tau_{1,2}\) open sets are \(\{X, \phi, \{b\}\}\) and \(\sigma_{1,2}\) open sets are \(\{\phi, Y, \{r\}\}\). Define a mapping \(f : X \to Y\) by \(f(a) = p, f(b) = r, f(c) = q\). Then \(f\) is both \((1,2)^*\)-gb-irresolute and \(f^{-1}\) is \((1,2)^*\)-gb-irresolute and hence \((1,2)^*\)-gbc-homeomorphism.
**Result 6.4.8** Every \((1,2)^*\)continuous function is \((1,2)^*\) b-continuous but not conversely.

**Theorem 6.4.9** For any bijection \(f : X \to Y\) the following statements are equivalent.

(i) \(f^{-1} : Y \to X\) is \((1,2)^*\) gb- continuous

(ii) \(f\) is \((1,2)^*\) gb-open

(iii) \(f\) is \((1,2)^*\) gb-closed.

**Proof:** (i) \(\Rightarrow\) (ii). Let \(F\) be any \(\tau_{1,2}\) -open set in \(X\). Then \(X - F\) is \(\tau_{1,2}\) - closed in \(X\). Since \(f^{-1}\) is gb-continuous, \((f^{-1})^{-1}(X - F) = f(X - F) = Y - f(F)\) is \((1,2)^*\) gb-closed in \(Y\). Then \(f(F)\) is \((1,2)^*\) gb-open in \(Y\). Hence \(f\) is \((1,2)^*\) gb-open.

(ii) \(\Rightarrow\) (iii) Let \(F\) be ant \(\tau_{1,2}\) -closed set in \(X\). Then \(X - F\) is \(\tau_{1,2}\) - open in \(X\). Since \(f\) is \((1,2)^*\) gb-open, \(f(X - F) = Y - f(F)\) is \((1,2)^*\) gb-closed in \(Y\). Then \(f(F)\) is \((1,2)^*\) gb-closed in \(Y\). Hence \(f\) is \((1,2)^*\) gb-closed.

(iii) \(\Rightarrow\) (i) Let \(V\) be any \(\tau_{1,2}\) - closed set in \(X\). Since \(f : X \to Y\) is \((1,2)^*\) gb-closed, \(f(V)\) is \((1,2)^*\) gb-closed in \(Y\). That is, \((f^{-1})^{-1}(V)\) is \((1,2)^*\) gb-closed in \(Y\). Hence \(f^{-1}\) is \((1,2)^*\) gb-continuous.

**Theorem 6.4.10** Let \(f : X \to Y\) be a bijective and \((1,2)^*\) gb-continuous function, then the following statements are equivalent.

(i) \(f\) is \((1,2)^*\) gb-open.

(ii) \(f\) is \((1,2)^*\) gb-homeomorphism.

(iii) \(f\) is \((1,2)^*\) gb-closed.

**Proof:** (i) \(\Rightarrow\) (ii). Given \(f\) is bijective, \((1,2)^*\) gb continuous and \((1,2)^*\) gb-open. Hence \(f\) is \((1,2)^*\) gb homeomorphism.
(ii) \(\Rightarrow\) (iii). Let \(f\) be \((1,2)^*\) gb-homeomorphism. Hence \(f\) is \((1,2)^*\) gb-open. By Theorem 6.4.9 \(f\) is \((1,2)^*\) gb-closed.

(iii) \(\Rightarrow\) (i) Follows from Theorem 6.4.9

**Remark 6.4.11** The composition of two \((1,2)^*\)-gb-homeomorphism need not be a \((1,2)^*\)-gb-homeomorphism as the following example shows

**Example 6.4.12** Let \((X,\tau_1,\tau_2)\), \((Y,\sigma_1,\sigma_2)\) and \((Z,\eta_1,\eta_2)\) be defined as \(X=\{a,b,c\}, \tau_1=\{\phi, X, \{a\}\}\) and \(\tau_2 = \{\phi, X\}\). Let \(Y=\{a,b,c\}, \sigma_1=\{\phi, Y, \{a\}\}\) and \(\sigma_2 = \{\phi, Y, \{a, b\}\}\), \(Z=\{a, b, c\}, \eta_1 = \{Z, \phi, \{b\}\}\) and \(\eta_2 = \{Z, \phi, \{a, b\}\}\). Then the \(\tau_{1,2}\) open sets are \(\{X, \phi, \{a\}\}\) and \(\sigma_{1,2}\) open sets are \(\{\phi, Y, \{a\}, \{a, b\}\}\) and \(\eta_{1,2}\) open sets are \(\{Z, \phi, \{b\}, \{a, b\}\}\). Define a function \(f: X \rightarrow Y\) by \(f(a) = b\), \(f(b) = a\), \(f(c) = c\) and the function \(g: Y \rightarrow Z\) by \(g(a)=b\), \(g(b)=a\), \(g(c)=c\). Then \(f\) is \((1,2)^*\)-gb-homeomorphism and \(g\) is \((1,2)^*\)-gb-homeomorphism but their composition is not \((1,2)^*\)-gb-homeomorphism.

**Theorem 6.4.13** Every \((1,2)^*\)-bg-closed set is \((1,2)^*\)-gb-closed.

**Proof:** Let \(F\) be \(\tau_{1,2}\)-open set of \(X\) such that \(S \subset F\). Then \(F\) is \((1,2)^*\) - b-open such that \(S \subset F\). Since \(S\) is \((1,2)^*\)-bg-closed, \((1,2)^*\)-bcl(S) \(\subset F\). Thus \(S\) is \((1,2)^*\)-gb-closed.

**Remark 6.4.14** A bijection \(f: X \rightarrow Y\) is Pre-(1,2)* - b-open if and only if \(f\) is pre-(1,2)*- b-closed.

**Theorem 6.4.15** If a function \(f: X \rightarrow Y\) is \((1,2)^*\)-b-irresolute and pre- \((1,2)^*\)-b-closed, then

(i) For every \((1,2)^*\)-bg-closed set \(A\) of \(Y\), \(f^{-1}(A)\) is \((1,2)^*\)-bg-closed set in \(A\)

(ii) For every \((1,2)^*\)-bg- closed set \(B\) of \(X\), \(f(B)\) is \((1,2)^*\)-bg closed set in \(Y\).

**Proof:** Let \(A\) be \((1,2)^*\)-bg closed set of \(Y\). Suppose that \(f^{-1}(A) \subset O\) where \(O\) is \((1,2)^*\)-b-open in \(X\). Since \(f\) is \((1,2)^*\)-b-irresolute, we have \(f(1,2)^*-\text{bcl}(f^{-1}(A)) \cap (X \setminus O) \subset (1,2)^*-\text{bcl}(f^{-1}(Y/A)) \subset f^{-1}(Y/A) \subset (1,2)^*-\text{bcl}(A) \setminus A\). This means
that \((1,2)^*\)-bcl\((A)\) \(\setminus A\) contains a \((1,2)^*\)-b-closed subset \(f(1,2)^*\)-bcl\((f^{-1}(A) \cap (X/O))\). Since \(f\) is pre-(1,2)^*-b-closed. We have \(f((1,2)^*\)-bcl\((f^{-1}(A) \cap (X/O)) = \emptyset\) and hence \((1,2)^*\)-bcl\((f^{-1}(A))\) \(\subseteq O\). This implies that \(f^{-1}(A)\) is \((1,2)^*\)-bg closed in \(X\).

(ii) Let \(B\) be a \((1,2)^*\)-bg-closed set in \(X\). Let \(f(B) \subseteq O\) where \(O\) is any \((1,2)^*\)-bopen set of \(Y\). Then, \(B \subseteq f^{-1}(O)\) holds, and \(f^{-1}(O)\) is \((1,2)^*\)-b-open in \(X\) because \(f\) is \((1,2)^*\)-b-irresolute. Since \(B\) is \((1,2)^*\)-bg-closed, \((1,2)^*\)-bcl\((B)\) \(\subseteq f^{-1}(O)\), and hence \(f((1,2)^*\)-bcl\((B))\) \(\subseteq O\). Since \((1,2)^*\)-bcl\((B)\) is \((1,2)^*\)-b-closed set in \(X\) and \(f\) is pre-(1,2)^*-b-closed, \(f((1,2)^*\)-bcl\((B))\) is \((1,2)^*\)-b-closed in \(Y\). Then \((1,2)^*\)-bcl\((f((1,2)^*\)-bcl\((B))) = f((1,2)^*\)-bcl\((B))\). Therefore, we have \((1,2)^*\)-bclf\((B) \subseteq f((1,2)^*\)-bcl\((B))\) \(\subseteq O\). Hence \(f(B)\) is \((1,2)^*\)-bg-closed in \(Y\).

**Theorem 6.4.16** If \(f : X \to Y\) is \((1,2)^*\)-b-irresolute and pre-(1,2)^*-b-closed, then for every \((1,2)^*\)-bg-closed set \(A\) of \(Y\), \(f^{-1}(A)\) is \((1,2)^*\)-gb-closed.

**Proof:** Let \(A\) be \((1,2)^*\)-bg-closed set in \(Y\). By Theorem 6.4.15, \(f^{-1}(A)\) is \((1,2)^*\)-bg-closed set in \(X\) and hence \(f^{-1}(A)\) is \((1,2)^*\)-gb-closed set in \(X\).

**Theorem 6.4.17** If \(f : X \to Y\) is \((1,2)^*\)-continuous and pre-(1,2)^*-b-closed, then for every \((1,2)^*\)-gb-closed set \(A\) of \(Y\), \(f^{-1}(A)\) is \((1,2)^*\)-bg-closed.

**Proof:** Let \(O\) be \(\sigma_{1,2}\)-open set of \(Y\) such that \(f(A) \subseteq O\). Then \(A \subseteq f^{-1}(O)\) implies \((1,2)^*\)-bcl\((A) \subseteq f^{-1}(O)\) since \(A\) is \((1,2)^*\)-gb-closed and \(f^{-1}(O)\) is \(\tau_{1,2}\)-open in \(X\). Since \(f\) is pre-(1,2)^*-b-closed, \((1,2)^*\)-bcl\((f(1,2)^*\)-bcl\((A)\)) = \(f(1,2)^*\)-bcl\((A)\) \(\subseteq O\) and hence \((1,2)^*\)-bcl\((f(A)) \subseteq O\). Therefore \(f(A)\) is \((1,2)^*\)-gb-closed set.

**Remark 6.4.18** The union of two disjoint \((1,2)^*\)-gb-open sets is not in general, \((1,2)^*\)-gb-open as seen from the following example.
Example 6.4.19 Let \((X, \tau_1, \tau_2)\) be defined as \(X = \{a, b, c\}, \ \tau_1 = \{\emptyset, X, \{a\}\}\) and \(\tau_2 = \{\emptyset, X\}\). Then the \(\tau_{1,2}\) open sets are \(\{X, \emptyset, \{a\}\}\). The sets \(\{b\}\) and \(\{c\}\) are disjoint \((1,2)^{\ast}\)-gb-open but their union \(\{b, c\}\) is not a \((1,2)^{\ast}\)-gb-open set.

Theorem 6.4.20 Let \(A\) be a subset of \(X\). Then \((1,2)^{\ast}\)-bcl\((A) = A \cup \tau_{1,2} \setminus \text{int}(\tau_{1,2} \setminus \text{cl}(A))\).

Theorem 6.4.21 If \(A\) is \(\tau_{1,2}\)-open and \((1,2)^{\ast}\)-gb-closed in \(X\), then \(A\) is \((1,2)^{\ast}\)-b-closed.

Proof: Since \(A\) is \(\tau_{1,2}\)-open and \((1,2)^{\ast}\)-gb-closed. By definition of \((1,2)^{\ast}\)-gb-closedness, \((1,2)^{\ast}\)-bcl\((A) \subset A\). By Theorem 6.4.20 \((1,2)^{\ast}\)-bcl\((A) = A \cup \tau_{1,2} \setminus \text{int}(\tau_{1,2} \setminus \text{cl}(A))\).

Thus \(\tau_{1,2} \setminus \text{int}(\tau_{1,2} \setminus \text{bcl}(A)) \subset A\). Hence \(A\) is \((1,2)^{\ast}\)-b-closed.

Theorem 6.4.22 [72] For any bijective function \(f : X \to Y\) the following statements are true.

(i) \(f^{-1} : Y \to X\) is \((1,2)^{\ast}\)-g-continuous.

(ii) \(f\) is \((1,2)^{\ast}\)-g-open

(iii) \(f\) is \((1,2)^{\ast}\)-g-closed.

Theorem 6.4.23 [72] Let \(f : X \to Y\) be a bijective and \((1,2)^{\ast}\)-g-continuous function. Then the following statements are equivalent.

(i) \(f\) is \((1,2)^{\ast}\)-g-open.

(ii) \(f\) is \((1,2)^{\ast}\)-g-homeomorphism.

(iii) \(f\) is \((1,2)^{\ast}\)-g-closed.

Remark 6.4.24 The composition of two \((1,2)^{\ast}\)-g-homeomorphism need not be a \((1,2)^{\ast}\)-g-homeomorphism.

Theorem 6.4.25 If \(A\) is \((1,2)^{\ast}\)-gb-closed set in \(X\) and \(A \subseteq B \subseteq (1,2)^{\ast}\)-bcl\((A)\), then \(B\) is \((1,2)^{\ast}\)-gb-closed in \(X\).
**Proof:** Let $B \subseteq U$ where $U$ is $\tau_{1,2}$-open in $X$. Since $A$ is $(1,2)^*\text{-gb}$-closed set and $A \subseteq U$, $(1,2)^*\text{-bcl}(A) \subseteq U$. Since $B \subseteq (1,2)^*\text{-bcl}(A)$, $(1,2)^*\text{-bcl}(B) \subseteq (1,2)^*\text{-bcl}(A) \subseteq U$. Hence $(1,2)^*\text{-bcl}(B) \subseteq U$ and so $B$ is $(1,2)^*\text{-gb}$-closed in $X$.

**Theorem 6.4.26** If $f : X \to Y$ is $(1,2)^*$-continuous, $(1,2)^*\text{-gb}$-closed and $A$ is $(1,2)^*$-g-closed set of $X$, then $f(A)$ is $(1,2)^*\text{-gb}$-closed in $Y$.

**Proof:** Let $f(A) \subseteq O$ where $O$ is $\sigma_{1,2}$-open set in $Y$. Then $A \subseteq f^{-1}(O)$. Since $f$ is $(1,2)^*$-continuous, $f^{-1}(O)$ is $\tau_{1,2}$-open set in $X$. Hence $\tau_{1,2}\text{-bcl}(A) \subseteq f^{-1}(O)$ as $A$ is $(1,2)^*$-g-closed set. Therefore $f(\tau_{1,2}\text{-bcl}(A)) \subseteq O$. Since $f$ is $(1,2)^*\text{-gb}$-closed and $\tau_{1,2}\text{-bcl}(A)$ is $\tau_{1,2}$-closed in $X$, $f(\tau_{1,2}\text{-bcl}(A))$ is $(1,2)^*\text{-gb}$-closed in $Y$. Thus $(1,2)^*\text{-bcl}[f(\tau_{1,2}\text{-bcl}(A))] \subseteq O$. Since $f(A) \subseteq f(\tau_{1,2}\text{-bcl}(A))$, $f(A) \subseteq (1,2)^*\text{-bcl}[f(\tau_{1,2}\text{-bcl}(A))] \subseteq O$. Therefore $f(A)$ is $(1,2)^*$-g-closed in $Y$.

**Theorem 6.4.27** Every $(1,2)^*$-g-closed set is $(1,2)^*$-gb-closed.

**Proof:** Let $F$ be any $\tau_{1,2}$-open set of $X$ such that $S \subseteq F$. Since $S$ is $(1,2)^*$-g-closed, $\tau_{1,2}\text{-cl}(S) \subseteq F$. But $(1,2)^*\text{-bcl}(S) \subseteq \tau_{1,2}\text{-bcl}(S)$. Hence $(1,2)^*\text{-bcl}(S) \subseteq F$. Hence $S$ is $(1,2)^*$-gb-closed.

**Theorem 6.4.28** If $f : X \to Y$ is $(1,2)^*$-g-closed and $g : Y \to Z$ is $(1,2)^*$-continuous and $(1,2)^*$-gb-closed, then $g \circ f : X \to Z$ is $(1,2)^*$-gb-closed.

**Proof:** Let $F$ be any $\tau_{1,2}$-closed set of $X$. Since $f$ is $(1,2)^*$-g-closed, $f(F)$ is $(1,2)^*$-g-closed set in $Y$. Since $g$ is $(1,2)^*$-continuous and $(1,2)^*$-gb-closed and $f(F)$ is $(1,2)^*$-gb-closed set of $Y$, By **Theorem 6.4.26**, $g(f(F))$ is $(1,2)^*$-gb-closed in $Z$. Hence $g \circ f$ is $(1,2)^*$-gb-closed.

**Theorem 6.4.29** If $f : X \to Y$ is $(1,2)^*$-g-closed and $g : Y \to Z$ is $(1,2)^*$-gb-closed, then $g \circ f : X \to Z$ is $(1,2)^*$-gb-closed.
Proof: Let F be any $\tau_{1,2}$-closed set of X. Since $f$ is (1,2)*-closed, $f(F)$ is $\sigma_{1,2}$-closed in Y. Since $g$ is (1,2)*-gb-closed, $g(f(F)) = (g \circ f)(F)$ is (1,2)*-gb-closed in Z. Hence $g \circ f$ is (1,2)*-gb-closed.

6.5 COMPARISONS

We now compare the (1,2)*-gb-closed sets with various (1,2)*-closed sets.

Remark 6.5.1 A (1,2)*-g-closed set is (1,2)*-gb-closed but not conversely.

Example 6.5.2 Let $X = \{a,b,c\}$, $\tau_1 = \{X,\emptyset,\{a\}\}$, $\tau_2 = \{x,\emptyset,\{a,b\}\}$. Then the set $\{b\}$ is (1,2)*-gb-closed but not (1,2)*-g-closed.

Remark 6.5.3 A (1,2)*-bg-closed set is (1,2)*-gb-closed but not conversely.

Example 6.5.4 Let $X = \{a,b,c\}$, $\tau_1 = \{X,\emptyset,\{b\}\}$, $\tau_2 = \{x,\emptyset,\{a,b\}\}$. Then the set $\{b,c\}$ is (1,2)*-gb-closed but not (1,2)*-bg-closed.

Remark 6.5.5 The (1,2)*-g-closed sets and (1,2)*-bg-closed sets are independent and can be seen from the following example.

Example 6.5.6

(i) Let $X = \{a,b,c\}$, $\tau_1 = \{X,\emptyset,\{b\}\}$, $\tau_2 = \{x,\emptyset,\{a,b\}\}$. Then the set $\{a,b\}$ is (1,2)*-bg-closed but not (1,2)*-g-closed.

(ii) Let $X = \{a,b,c\}$, $\tau_1 = \{X,\emptyset,\{a\}\}$, $\tau_2 = \{x,\emptyset,\{a,b\}\}$. Then the set $\{a,c\}$ is (1,2)*-g-closed but not (1,2)*-bg-closed.

For the above sets we considered we have the following implications

\[(1,2)^*\text{-g-closed} \iff (1,2)^*\text{-gb-closed} \iff (1,2)^*\text{-bg-closed}\]

Theorem 6.5.7 If $f : X \to Y$ is (1,2)*-b-continuous and (1,2)*-open mapping then $f$ is (1,2)*-b-irresolute.
**Theorem 6.5.8** If \( f : X \to Y \) is \((1,2)^*\)-homeomorphism, then \( f \) and its inverse \( f^{-1} \) are pre- \((1,2)^*\)-b closed and also \((1,2)^*\)-b-irresolute.

**Proof:** Since \( f \) is \((1,2)^*\)-open and \((1,2)^*\)-continuous, by Theorem 6.5.7 \( f \) is \( b \)-irresolute. The bijective \( f \) implies that its inverse \( f^{-1} \) is pre-(\(1,2)^*\)-b-closed. Similarly since \( f^{-1} \) is \((1,2)^*\)-open and \((1,2)^*\)-continuous we have that \( f^{-1} \) is \((1,2)^*\)-b-irresolute and pre- \((1,2)^*\)-b-closed.

**Result 6.5.9** [72] Every \((1,2)^*\)-continuous mapping is \((1,2)^*\)-g-continuous but not conversely.

**Result 6.5.10** [72] If \( f : X \to Y \) is bijective, \((1,2)^*\)-open and \((1,2)^*\)-g-continuous mapping, then \( f \) is \((1,2)^*\)-gb-irresolute.

**Theorem 6.5.11** Every \((1,2)^*\)-homeomorphism is \((1,2)^*\)-gb-homeomorphism.

**Proof:** Let \( f \) be \((1,2)^*\)-homeomorphism. Then \( f \) and \( f^{-1} \) are continuous. By result 6.5.9 \( f \) and \( f^{-1} \) are \((1,2)^*\)-g-continuous. By result 6.5.10 \( f \) and \( f^{-1} \) are \((1,2)^*\)-gb-irresolute. Hence \( f \) is \((1,2)^*\)-gb-homeomorphism.

**Remark 6.5.12** The converse of the above Theorem need not be true and can be seen from the following example.

**Example 6.5.13** In example 6.5.2 \( f \) is \((1,2)^*\)-gb-homeomorphism but not \((1,2)^*\)-homeomorphism.

**Theorem 6.5.14** Every \((1,2)^*\)-g-continuous mapping is \((1,2)^*\)-gb-continuous.

**Proof:** Let \( A \) be \( \sigma_{1,2} \)-closed set of \( Y \). Let \( f : X \to Y \) be \((1,2)^*\)-g-continuous. Then \( f^{-1}(A) \) is \((1,2)^*\)-g-closed in \( X \). by Theorem 6.3.27 \( f^{-1}(A) \) is \((1,2)^*\)-gb-closed in \( X \). Hence \( f \) is \((1,2)^*\)-gb-continuous.

**Theorem 6.5.15** Every \((1,2)^*\)-g-open mapping is \((1,2)^*\)-gb-open.
Proof: Let $f : X \to Y$ be $(1,2)^*\text{-}g$-open and $F$ be any $\tau_{1,2}$-open set of $X$. Then $f(F)$ is $(1,2)^*\text{-}g$-open in $Y$ and $Y - f(F)$ is $(1,2)^*\text{-}g$-closed in $Y$. By Theorem 6.3.27 $Y - f(F)$ is $(1,2)^*\text{-}gb$-closed in $Y$. Therefore $f(F)$ is $(1,2)^*\text{-}gb$-open in $Y$. Hence $f$ is $(1,2)^*\text{-}gb$-open mapping.

Theorem 6.5.16 Every $(1,2)^*\text{-}g$-homeomorphism is $(1,2)^*\text{-}gb$-homeomorphism.

Proof: Follows from the Theorems 6.5.15 and 6.5.14

Remark 6.5.17 The converse of the above Theorem need not be true and can be seen from the following example.

Example 6.5.18 Let $X = \{a,b,c\}$, $\tau_1 = \{X, \emptyset, \{a\}\}$, $\tau_2 = \{X, \emptyset, \{a\}, \{a, b\}\}$. Consider a mapping $f : X \to Y$ defined as $f(a) = a$, $f(b) = b$, $f(c) = c$. Then $f$ is $(1,2)^*\text{-}gb$-homeomorphism but not $(1,2)^*\text{-}g$-homeomorphism.

Theorem 6.5.19 If $f$ is $(1,2)^*\text{-}g$-open and $(1,2)^*\text{-}gc$-irresolute, then $f$ is $(1,2)^*\text{-}gb$-irresolute.

Proof: It follows from the Theorem 6.3.27

Theorem 6.5.20 Every $(1,2)^*\text{-}g$-homeomorphism is $(1,2)^*\text{-}gbc$-homeomorphism.

Proof: Let $f : X \to Y$ be $(1,2)^*\text{-}g$-homeomorphism. By Theorem 6.5.11 $f$ is $(1,2)^*\text{-}gc$-homeomorphism. Hence $f$ and $f^{-1}$ are both $(1,2)^*\text{-}gc$-irresolute. By Theorem 6.5.19 $f$ and $f^{-1}$ are both $(1,2)^*\text{-}gb$-irresolute. Hence $f$ is $(1,2)^*\text{-}gbc$-homeomorphism.

Theorem 6.5.21 Every $(1,2)^*\text{-}gbc$-homeomorphism is $(1,2)^*\text{-}gb$-homeomorphism.

Proof: Let $f : X \to Y$ be $(1,2)^*\text{-}gbc$-homeomorphism. Hence $f$ and $f^{-1}$ are $(1,2)^*\text{-}gb$-irresolute and hence $(1,2)^*\text{-}gb$-continuous. Since $f^{-1}$ is $(1,2)^*\text{-}gb$-continuous, $f$ is $(1,2)^*\text{-}gb$-open. Hence $f$ is $(1,2)^*\text{-}gb$-homeomorphism.

We have the following diagram from the above Theorems and remarks
and

(1,2)*-continuous → (1,2)* g-continuous → (1,2)* gb-continuous