Chapter 2

$\pi^g$-Closed Sets in Topological Space

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2.1 Introduction.
2.2 $\pi^g$-Closed Sets and $\pi^g$ -Open Sets.
2.3 $\pi^g$ -Interior and $\pi^g$-Closure Operator.
2.4 $\pi^g$- Continuous Functions.
CHAPTER 2. \( \pi^*G \)-CLOSED SETS IN TOPOLOGICAL SPACE

2.1 Introduction

Closed sets are fundamental objects in a topological space. For example, one can define the topology on a set by using either the axioms for the closed sets or the Kuratowski closure axioms. General topology is important in many fields of applied sciences as well as branches of mathematics. In reality it is used in data mining, computational topology for geometric design and molecular design, computer-aided design, computer-aided geometric design, digital topology, information systems, particle physics and quantum physics etc.

Levine [105] introduced the concept of generalized closed sets which formed a strong tool in the characterization of topological spaces. This notion has been studied extensively in recent years by many topologists[46,93-96] because generalized closed sets are not only natural generalizations of closed sets, more importantly, they also suggest several new properties of topological spaces. Most of these new properties are separation axioms weaker than \( T_1 \), some of which have been found to be useful in computer science and digital topology. The study of generalized closed sets has produced some new separations which are between \( T_0 \) and \( T_1 \) such as \( T_{\frac{1}{2}} \), \( T_{\frac{3}{2}} \). For example, the well-known digital line is \( T_{\frac{3}{2}} \) but not \( T_1 \). J. Dontchev and T. Noiri [51] introduced the notions of \( \pi g \)-closed sets and studied some of their properties.

In this chapter we define and study the properties of \( \pi^*g \)-closed sets which is a weaker form of \( \pi g \)-closed sets but stronger than \( rwg \)-closed sets. Also the notions of \( \pi^*g \)-open sets, \( \pi^*g \)-neighbourhood, \( \pi^*g \)-closure and \( \pi^*g \)-interior, \( \pi^*g \)-continuous functions,\( \pi^*g \)-compact space and \( \pi^*g \)-connectedness are introduced and discussed.

2.2 \( \pi^*g \)-Closed Sets and \( \pi^*g \)-Open Sets

**Definition 2.2.1.** A subset \( A \) of \(( X, \tau )\) is called \( \pi^*g \)-closed set if \( cl(int(A) \subset U \) whenever \( A \subset U \) and \( U \) is \( \pi \)-open.

By \( \pi^*GC(X, \tau ) \), we mean the family of all \( \pi^*g \)-closed subsets of \(( X, \tau )\).
CHAPTER 2. \( \pi^*G \)-CLOSED SETS IN TOPOLOGICAL SPACE

Theorem 2.2.2.

1. Every closed set is \( \pi^*g \)-closed.
2. Every \( g \)-closed set is \( \pi^*g \)-closed.
3. Every \( \alpha \)-closed set is \( \pi^*g \)-closed.
4. Every \( \pi g \)-closed set is \( \pi^*g \)-closed.
5. Every \( wg \)-closed set is \( \pi^*g \)-closed.
6. Every preclosed set is \( \pi^*g \)-closed.
7. Every \( \pi^*g \)-closed set is rwg-closed.

Proof.

1. Obvious and straight forward.

2. Suppose \( A \) is \( g \)-closed in \( X \). Let \( A \subset U \) where \( U \) is \( \pi \)-open. Since \( A \) is \( g \)-closed set, \( \text{cl}(A) \subset U \) and as \( \text{cl}(\text{int}(A)) \subset \text{cl}(A) \), we get \( \text{cl}(\text{int}(A)) \subset U \). Using the fact that every \( \pi \)-open set is open, we have \( A \) is \( \pi^*g \)-closed set.

3. Let \( A \) be \( \alpha \)-closed set and \( A \subset U \) where \( U \) is \( \pi \)-open. Since \( A \) is \( \alpha \)-closed set, \( \text{cl}(\text{int}(\text{cl}(A))) \subset A \subset U \). Hence \( \text{cl}(\text{int}(A)) \subset \text{cl}(\text{cl}(A)) \subset U \). Therefore \( \text{cl}(\text{int}(A)) \subset U \) and \( A \) is \( \pi^*g \)-closed.

4. Let \( A \) be \( \pi g \)-closed and suppose \( A \subset U \) where \( U \) is \( \pi \)-open. Then \( \text{cl}(\text{int}(A)) \subset \text{cl}(A) \subset U \) and hence \( \text{cl}(\text{int}(A)) \subset U \). Therefore \( A \) is \( \pi^*g \)-closed.

5. Let \( A \) be \( wg \)-closed set. Let \( A \subset U \) where \( U \) is \( \pi \)-open. By definition of \( wg \)-closed set and the fact that every regular open set is \( \pi \)-open, \( A \) is \( \pi^*g \)-closed.

6. Obvious and straight forward.
7. Let $A$ be $π^g$-closed set and $A \subset U$ where $U$ is regular open. Since every regular open set is $π$-open and $A$ is $π^g$-closed, $\text{cl}(\text{int}(A)) \subset U$. Hence $A$ is $\text{rwg}$-closed.

The converses of the above theorem need not be true as seen from the following examples.

Example 2.2.3.

1. Consider the set $X = \{a, b, c\}$ with the topology $τ = \{X, \emptyset, \{a\}\}$. Here $\{a\}$ is $π^g$-closed but not closed, $g$-closed, $α$-closed.

2. Consider the set $X = \{a, b, c, d\}$ with the topology $τ = \{X, \emptyset, \{a, c, d\}, \{a, c, d\}\}$. Here $X = \{c\}$ is $π^g$-closed but not $πg$-closed.

3. Consider the set $X = \{a, b, c\}$ with the topology $τ = \{X, \emptyset, \{a\}\}$. Here $\{a\}$ is $π^g$-closed but not $\text{wg}$-closed.

4. Consider the set $X = \{a, b\}$ with the topology $τ = \{\emptyset, X, \{a\}, \{a, c\}\}$. Here $\{a, b\}$ is $π^g$-closed but not $\text{preclosed}$.

5. Consider the set $X = \{a, b\}$ with the topology $τ = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Here $\{a, b\}$ is $\text{rwg}$-closed but not $π^g$-closed.

Remark 2.2.4. The Union and Intersection of two $π^g$-closed sets need not be $π^g$-closed.

Example 2.2.5. The following examples show that remark 2.2.4 holds good.

1. Consider $X = \{a, b, c, d\}$ and $τ = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}\}$. Here $\{c\}$ and $\{d\}$ are $π^g$-closed but their union $\{c, d\}$ is not $π^g$-closed.
2. Consider $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Here $\{a, b, d\}$ and $\{a, b, c\}$ are $\pi^*g$-closed but their intersection $\{a, b\}$ is not $\pi^*g$-closed.

Remark 2.2.6. The following examples prove that $\pi^*g$-closed sets and semi-closed sets are independent of each other.

1. Consider $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$. Here $\{a\}$ is $\pi^*g$-closed set but not semi-closed.

2. Consider $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Here $\{b\}$ is semi closed but not $\pi^*g$-closed.

Remark 2.2.7. The following examples prove that $\pi^*g$-closed sets and $sg$-closed sets are independent of each other.

1. Consider $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$. Here $\{a\}$ is $\pi^*g$-closed set but not $sg$-closed.

2. Consider $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Here $\{b\}$ is $sg$-closed but not $\pi^*g$-closed.

Remark 2.2.8. The following examples prove that $\pi^*g$-closed sets and $gs$-closed sets are independent of each other.

1. Consider $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$. Here $\{a\}$ is $\pi^*g$-closed but not $gs$-closed.

2. Consider $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Here $\{a\}$ is $gs$-closed but not $\pi^*g$-closed set.
Remark 2.2.9. The following examples prove that $\pi^*g$-closed sets and $\beta$-closed sets are independent of each other.

1. Consider $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$. Here $\{a\}$ is $\pi^*g$-closed but not $\beta$-closed.

2. Consider $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Here $\{b\}$ is $\beta$-closed but not $\pi^*g$-closed set.

The above discussion is summarized in the following figure with respect to $\pi^*g$-closed sets.

Theorem 2.2.10. If $A$ is $\pi^*g$-closed and $A \subset B \subset cl(int(A))$ then $B$ is also $\pi^*g$-closed.

Proof. Let $B \subset U$ where $U$ is $\pi$-open. Then $A \subset B$ implies $A \subset U$ and $U$ is $\pi$-open. Since $A$ is $\pi^*g$-closed, $cl(int(A)) \subset U$. Using hypothesis, $cl(int(B)) \subset U$. Hence $B$ is $\pi^*g$-closed. 

\[\]
**Theorem 2.2.11.** If $A$ is both regular open and $\pi^*g$-closed then it is clopen.

**Proof.** Since $A$ is regular open, $A$ is open and $A = \operatorname{int}(A)$. $A$ is $\pi^*g$-closed implies $\operatorname{cl}(\operatorname{int}(A)) \subset A$. $\operatorname{cl}(A) = \operatorname{cl}(\operatorname{int}(A)) \subset A$ implies $\operatorname{cl}(A) = A$. Therefore $A$ is clopen. \qed

**Lemma 2.2.12.** The following properties are equivalent for a subset $A$ of $X$.

1. $A$ is clopen.
2. $A$ is regular open and $\pi^*g$-closed.
3. $A$ is $\pi$-open and $\pi^*g$-closed.

**Proof.** Follows from Theorem 2.2.11 and the fact that every regular open set is $\pi$-open. \qed

**Theorem 2.2.13.** If $A$ is $\pi^*g$-closed, then $\operatorname{cl}(\operatorname{int}(A)) - A$ contains no non-empty $\pi$-closed set.

**Proof.** Suppose that $F$ is a non-empty $\pi$-closed subset of $\operatorname{cl}(\operatorname{int}(A)) - A$. Now $F \subset \operatorname{cl}(\operatorname{int}(A)) - A$ implies $F \subset \operatorname{cl}(\operatorname{int}(A)) \cap A^c$. Thus $F \subset \operatorname{cl}(\operatorname{int}(A))$. $F \subset A^c$ implies $A \subset F^c$. Since $F^c$ is $\pi$-open and $A$ is $\pi^*g$-closed. We have $\operatorname{cl}(\operatorname{int}(A)) \subset F^c$ and $F \subset (\operatorname{cl}(\operatorname{int}(A))^c$. Hence $F \subset \operatorname{cl}(\operatorname{int}(A)) \cap \operatorname{cl}(\operatorname{int}(A))^c = \phi$ (ie) $F = \phi$. This implies that $\operatorname{cl}(\operatorname{int}(A)) - A$ contains no non-empty $\pi$-closed set. \qed

**Theorem 2.2.14.** Suppose that $B \subset A \subset X$, $B$ is $\pi^*g$-closed relative to $A$ and $A$ is both regular open and $\pi^*g$-closed subset of $X$, then $B$ is $\pi^*g$-closed set relative to $X$. 

Proof. Let \( B \subset G \) and \( G \) be \( \pi \)-open in \( X \). Given \( B \subset A \subset X \). This implies \( B \subset A \cap G \). Since \( B \) is \( \pi^*g \)-closed relative to \( A \), \( \text{cl}(\text{int}_A(B)) \subset A \cap G \).

\( A \cap \text{cl}(\text{int}(B)) \subset A \cap G \). Consequently \( A \cap \text{cl}(\text{int}(B)) \subset G \). Since \( A \) is regular open and \( \pi^*g \)-closed we have \( A = \text{cl}(A) \). \( \text{cl}(\text{int}(B)) \subset \text{cl}(B) \subset \text{cl}(A) = A \). Hence \( \text{cl}(\text{int}(B)) \cap A = \text{cl}(\text{int}(B)) \) and \( \text{cl}(\text{int}(B)) \subset G \). Therefore \( B \) is \( \pi^*g \)-closed set relative to \( X \).

Corollary 2.2.15. Let \( A \) be both regular open and \( \pi^*g \)-closed in \( X \) and suppose that \( F \) is closed, then \( (A \cap F) \) is \( \pi^*g \)-closed.

Proof. We show that \( \text{cl}(\text{int}(A \cap F)) \subset U \) whenever \( (A \cap F) \subset U \) and \( U \) is \( \pi \)-open. Since \( F \) is closed, \( (A \cap F) \) is closed in \( A \) and hence \( \pi^*g \)-closed in \( A \). Therefore \( (A \cap F) \) is \( \pi^*g \)-closed in \( X \).

Theorem 2.2.16. Let \( A \subset Y \subset X \). Suppose that \( A \) is \( \pi^*g \)-closed in \( X \) and \( Y \) is open in \( X \) then \( A \) is \( \pi^*g \)-closed relative to \( Y \).

Proof. Given \( A \subset Y \subset X \) and \( A \) is \( \pi^*g \)-closed in \( X \). Let \( A \subset (Y \cap G) \) where \( G \) is \( \pi \)-open in \( X \). Since \( A \) is \( \pi^*g \)-closed in \( X \), \( A \subset G \) implies that \( \text{cl}(\text{int}(A)) \subset G \). \( Y \cap \text{cl}(\text{int}(A)) \subset Y \cap G \). Therefore \( A \) is \( \pi^*g \)-closed relative to \( Y \).

Proposition 2.2.17. For a space \( X \), the following are equivalent.

1. \( X \) is extremally disconnected.
2. Every subset of \( X \) is \( \pi^*g \)-closed.
3. The topology on \( X \) generated by \( \pi^*g \)-closed sets is the discrete one.
CHAPTER 2. $\pi^*G$-CLOSED SETS IN TOPOLOGICAL SPACE

Proof. 1 $\Rightarrow$ 2. Assume $X$ is extremally disconnected. Let $A \subset U$ where $U$ is $\pi$-open in $X$. Since $U$ is the finite union of regular open sets and $X$ is extremally disconnected, $U$ is finite union of clopen sets and hence $U$ is clopen. Therefore $\text{cl}(\text{int}(A)) \subset \text{cl}(A) \subset \text{cl}(U) \subset U$ implies $A$ is $\pi^*g$-closed.

$2 \Rightarrow 1$. Let $A$ be regular open set of $X$. By assumption $A$ is $\pi^*g$-closed and hence $A$ is clopen by Lemma 2.2.12. Hence $X$ is extremally disconnected.

$2 \Leftrightarrow 3$. This is immediate.

The following result characterizes hyperconnected spaces in terms of $\pi^*g$-closed sets.

Proposition 2.2.18. For a space $(X, \tau)$ the following conditions are equivalent.

1. $X$ is hyperconnected.

2. Every subset of $X$ is $\pi^*g$-closed and $X$ is connected.

Proof. 1 $\Rightarrow$ 2. We know that if $X$ is hyperconnected then, as mentioned in [87], the only regular open subset of $X$ are trivial ones. Hence every subset of $X$ is $\pi^*g$-closed. Also every hyperconnected space is trivially connected.

$2 \Rightarrow 1$. Let $A$ be a non-empty proper regular open subset of $X$. By assumption, $A$ is $\pi^*g$-closed and hence $A$ is clopen, by Lemma 2.2.12, which is a contradiction to our assumption that $X$ is connected.

Theorem 2.2.19. If $X$ is locally indiscrete, then $\pi^*GC(X, \tau) = P(X)$.

Proof. Let $A$ be any subset of $X$ and $U$ be any $\pi$-open subset of $X$ such that $A \subset U$. Since every $\pi$-open set is open, $U$ is open. From hypothesis, every open set is closed. Hence $\text{cl}(A) \subset \text{cl}(U)$, i.e., $\text{cl}(\text{int}(A)) \subset \text{cl}(A) \subset \text{cl}(U) \subset U$ and this implies $A$ is $\pi^*g$-closed. Thus $\pi^*GC(X, \tau) = P(X)$.
Definition 2.2.20. A subset $A$ of a topological space $(X, \tau)$ is called $\pi^*g$-open set if its complement is a $\pi^*g$-closed set.

We denote the family of all $\pi^*g$-open sets in $X$ by $\pi^*GO(X)$.

Theorem 2.2.21. A subset $A$ of a topological space is $\pi^*g$-open iff $F \subset \text{int}(\text{cl}(A))$ whenever $F$ is $\pi$-closed and $F \subset A$.

Proof. Assume $A$ is $\pi^*g$-open then $A^c$ is $\pi^*g$-closed. Let $F$ be a $\pi$-closed set in $X$ contained in $A$. Then $F^c$ is a $\pi$-open set in $X$ containing $A^c$. Since $A^c$ is $\pi^*g$-closed, $\text{cl}(\text{int}(A^c)) \subset F^c$. Consequently $F \subset \text{int}(\text{cl}(A))$.

Conversely, let $F \subset \text{int}(\text{cl}(A))$ whenever $F \subset A$ and $F$ is $\pi$-closed in $X$. Let $G$ be $\pi$-open set containing $A^c$ then $G^c \subset \text{int}(\text{cl}(A))$. Hence $\text{cl}(\text{int}(A^c)) \subset G$ implies $A$ is $\pi^*g$-open.

Remark 2.2.22. Both Union and Intersection of two $\pi^*g$-open sets need not be $\pi^*g$-open as seen in the following example.

Example 2.2.23. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. The union of $\pi^*g$-open sets $\{b, c\}$ and $\{d\}$ is not a $\pi^*g$-open set.

Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}\}$. The intersection of $\pi^*g$-open sets $\{a, b, d\}$ and $\{b, c\}$ is not a $\pi^*g$-open set.

2.3 $\pi^*g$-Interior and $\pi^*g$-Closure Operator

Definition 2.3.1. Let $(X, \tau)$ be a topological space

1. Let $x \in X$. A subset $S$ of $X$ is called a $\pi^*g$-neighbourhood (briefly, $\pi^*g$-nbhd) of $x$ if there exist a $\pi^*g$-open set $U$ such that $x \in U \subset S$. 

2. Let $A$ be a subset of $X$. A point $x \in A$ is said to be \( \pi^g \)-interior point of $A$ if $A$ is a \( \pi^g \)-neighbourhood of $x$. The set of all \( \pi^g \)-interior points of $A$ is called the \( \pi^g \)-interior of $A$ and it is denoted by \( \pi^g \text{-int}(A) \).

**Definition 2.3.2.** Let \((X, \tau)\) be a topological space and $E \subset X$. Then \( \pi^g \text{-int}(E) \) is the union of all \( \pi^g \)-open sets contained in $E$.

**Proposition 2.3.3.** Let $A$ and $B$ be subsets of \((X, \tau)\). Then

1. \( \pi^g \text{-int}(\emptyset) = \emptyset \) and \( \pi^g \text{-int}(X) = X \).

2. \( \pi^g \text{-int}(A) \subset A \).

3. If $B$ is any \( \pi^g \)-open set contained in $A$, then $B \subset \pi^g \text{-int}(A)$.

4. If $A \subset B$, then \( \pi^g \text{-int}(A) \subset \pi^g \text{-int}(B) \).

5. \( \pi^g \text{-int}(\pi^g \text{-int}(A)) = \pi^g \text{-int}(A) \).

**Proof.**

1. Obvious.

2. Let $x \in \pi^g \text{-int}(A)$. Then $x$ is a \( \pi^g \)-interior point of $A$ and $A$ is a \( \pi^g \)-nbhd of $x$. Hence $x \in A$ and we get \( \pi^g \text{-int}(A) \subset A \).

3. Let $B$ be any \( \pi^g \)-open set such that $B \subset A$. Let $x \in B$, by def, $x$ is a \( \pi^g \)-interior point of $A$ and $x \in \pi^g \text{-int}(A)$. Hence $B \subset \pi^g \text{-int}(A)$.

4. Let $A$ and $B$ be subsets of $X$ such that $A \subset B$. Let $x \in \pi^g \text{-int}(A)$. Then $A$ is \( \pi^g \)-nbhd of $x$. Since $A \subset B$, $B$ is also a \( \pi^g \)-nbhd of $x$. This implies that $x \in \pi^g \text{-int}(B)$ and \( \pi^g \text{-int}(A) \subset \pi^g \text{-int}(B) \).

5. Let $A$ be any subset of $X$ then \( \pi^g \text{-int}(A) = \bigcup \{ F : F \in \pi^g O(X), F \subset A \} \) and if $A \subset F \in \pi^g O(X)$, then \( \pi^g \text{-int}(A) \subset F \). Since $F$ is \( \pi^g \)-open set containing \( \pi^g \text{-int}(A) \), using (3) above \( \pi^g \text{-int}(\pi^g \text{-int}(A)) \subset F \). Hence \( \pi^g \text{-int}(\pi^g \text{-int}(A)) \subset \bigcup \{ F : A \subset F \in \pi^g O(X) \} = \pi^g \text{-int}(A) \). Hence \( \pi^g \text{-int}(\pi^g \text{-int}(A)) = \pi^g \text{-int}(A) \).

\( \square \)
CHAPTER 2. $\pi^g$-CLOSED SETS IN TOPOLOGICAL SPACE

Theorem 2.3.4. If a subset $A$ of space $X$ is $\pi^g$-open, then $\pi^g$-int $(A) = A$.

Proof. Let $A$ be $\pi^g$-open subset of $X$ and we know that $\pi^g$-int $(A) \subset A$. Since $A$ is $\pi^g$-open set contained in $A$, using proposition 2.3.3. (3), $A \subset \pi^g$-int $(A)$ and hence we get $\pi^g$-int $(A) = A$.

The converse of the above theorem need not be true as seen in the following example.

Example 2.3.5. Let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$.

Here $\pi^g$-int$(\{c,d\}) = \{c,d\}$ but $\{c,d\}$ is not a $\pi^g$-open set.

Theorem 2.3.6. If $A$ and $B$ are subsets of $X$, then $\pi^g$-int $(A) \cup \pi^g$-int $(B) \subset \pi^g$-int $(A \cup B)$.

Proof. Since $A \subset A \cup B$ and $B \subset A \cup B$, Using Theorem 2.3.3. (4), $\pi^g$-int $(A) \subset \pi^g$-int $(A \cup B)$ and $\pi^g$-int $(B) \subset \pi^g$-int $(A \cup B)$ which implies $\pi^g$-int $(A) \cup \pi^g$-int $(B) \subset \pi^g$-int $(A \cup B)$.

Proposition 2.3.7. If $A$ is a subset of $X$, then int$(A) \subset \pi^g$-int $(A)$.

Definition 2.3.8. For a subset $A$ of $(X, \tau)$ the $\pi^g$-closure of $A$ is the intersection of all $\pi^g$-closed sets containing $A$.

In symbols we have $\pi^g$-cl$(A) = \cap \{F: A \subset F, F \text{ is } \pi^g$-closed in $X\}$.

Proposition 2.3.9. Let $A$ and $B$ be subsets of $(X, \tau)$. Then

1. $\pi^g$-cl$(\phi) = \phi$ and $\pi^g$-cl$(X) = X$.
2. $A \subset \pi^g$-cl$(A)$. 
3. If B is any \( \pi^*g \)-closed set containing A, then \( \pi^*g \)-cl(A) ⊂ B.

4. If A ⊂ B, then \( \pi^*g \)-cl(A) ⊂ \( \pi^*g \)-cl(B).

5. \( \pi^*g \)-cl(A) = \( \pi^*g \)-cl(\( \pi^*g \)-cl(A))

**Proposition 2.3.10.** If A ⊂ X is \( \pi^*g \)-closed, then \( \pi^*g \)-cl (A) = A.

The converse of above proposition is not true as seen in the following example.

**Example 2.3.11.** Let X = \{a,b,c,d\} and \( \tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\} \). Here \( \pi^*g \)-cl(\{a\}) = \{a\} but \{a\} is not \( \pi^*g \)-closed.

**Proposition 2.3.12.** If A and B are subsets of a space X, then \( \pi^*g \)-cl (A ∩ B) ⊂ \( \pi^*g \)-cl (A) \( \pi^*g \)-cl (B).

**Proof.** Let A and B be subsets of X. Clearly (A ∩ B) ⊂ A and (A ∩ B) ⊂ B. Therefore \( \pi^*g \)-cl (A ∩ B) ⊂ \( \pi^*g \)-cl (A) and \( \pi^*g \)-cl (A ∩ B) ⊂ \( \pi^*g \)-cl (B). Hence \( \pi^*g \)-cl (A ∩ B) ⊂ \( \pi^*g \)-cl (A) \( \pi^*g \)-cl (B).

**Proposition 2.3.13.** If A is subset of a space X, then \( \pi^*g \)-cl(A) ⊂ cl(A).

**Proof.** By definition, cl(A) = \( \cap \{ F : A ⊂ F ∈ C(X) \} \). If A ⊂ F ∈ C(X) then A ⊂ F ∈ \( \pi^*GC \) (X), since every closed set is \( \pi^*g \)-closed. That is \( \pi^*g \)-cl(A) ⊂ F. Therefore \( \pi^*g \)-cl(A) ⊂ \( \cap \{ F : A ⊂ F ∈ C(X) \} = cl(A) \). Hence \( \pi^*g \)-cl(A) ⊂ cl(A).

**Lemma 2.3.14.** Let A be a subset of (X, \( \tau \)) and \( x \in X \). Then \( x \in \pi^*g \)-cl(A) if and only if \( V ∩ A ≠ \phi \) for every \( \pi^*g \)-open set V containing x.
Proof. Necessity. Suppose there exist a \( \pi^*g \)-open set \( V \) containing \( x \) such that \( V \cap A = \phi \). Since \( A \subset X - V, \pi^*g\text{-cl}(A) \subset X - V \) and this implies \( x \notin \pi^*g\text{-cl}(A) \), a contradiction.

Sufficiency. Suppose that \( x \notin \pi^*g\text{-cl}(A) \). Then there exist a \( \pi^*g \)-closed set \( F \) containing \( A \) such that \( x \notin F \). Then \( x \in X - F \) and \( X - F \) is \( \pi^*g \)-open. Also \( (X - F) \cap A = \phi \), a contradiction.

\[ \square \]

Lemma 2.3.15. Let \( A \) be a subset of a space \( X \). Then \( X - \pi^*g\text{-int}(A) = \pi^*g\text{-cl}(X - A) \).

Proof. Let \( x \in X - \pi^*g\text{-int}(A) \). Then \( x \notin \pi^*g\text{-int}(A) \). That is, every \( \pi^*g \)-open set \( B \) containing \( x \) is such that \( B \not\subset A \). This implies every \( \pi^*g \)-open set \( B \) containing \( x \) intersects \( X - A \). Hence \( x \in \pi^*g\text{-cl}(X - A) \).

Conversely, let \( x \in \pi^*g\text{-cl}(X - A) \). Then every \( \pi^*g \)-open set \( B \) containing \( x \) intersects \( X - A \). That is, every \( \pi^*g \)-open set \( B \) containing \( x \) is such that \( B \not\subset A \). This implies \( x \in \pi^*g\text{-int}(A) \). Thus \( \pi^*g\text{-cl}(X - A) \subset X - \pi^*g\text{-int}(A) \) and \( X - \pi^*g\text{-int}(A) = \pi^*g\text{-cl}(X - A) \).

Similarly we have \( X - \pi^*g\text{-cl}(X - A) = \pi^*g\text{-int}(X - A) \).

\[ \square \]

Proposition 2.3.16. Let \( A \) be a \( \pi^*g \)-open set and \( B \) be any set in \( X \). If \( A \cap B = \phi \), then \( A \cap \pi^*g\text{-cl}(B) = \phi \).

Proof. Suppose \( A \cap \pi^*g\text{-cl}(B) \neq \phi \) and \( x \in A \cap \pi^*g\text{-cl}(B) \). Then \( x \in A \) and \( x \in A \cap \pi^*g\text{-cl}(B) \). By Lemma 2.3.14, \( A \cap B \neq \phi \) which is contrary to the hypothesis. Hence \( A \cap \pi^*g\text{-cl}(B) = \phi \).

\[ \square \]

Definition 2.3.17. A topological Space \( (X, \tau) \) is called

1. \( \pi^*g \)-\( T_1 \) space if every \( \pi^*g \)-closed set of \( X \) is \( g \)-closed in \( X \).

2. \( T_{\pi^*g} \) space if every \( \pi^*g \)-closed subset of \( X \) is closed in \( X \).
CHAPTER 2. \(\pi^*G\)-CLOSED SETS IN TOPOLOGICAL SPACE

Remark 2.3.18. It can be easily established that every T\(\pi^*g\)-space is \(\pi^*g\)-T\(_1\) but not conversely.

Proposition 2.3.19.

1. Every T\(\pi^*g\)-space is T\(_{wg}\)-space.
2. Every T\(\pi^*g\)-space is \(\alpha\)-space.
3. Every T\(\pi^*g\)-space is semi-pre T\(_1\frac{1}{2}\)-space.
4. Every T\(\pi^*g\)-space is T\(_w\)-space.
5. Every T\(\pi^*g\)-space is T\(_1\frac{1}{2}\)-space.
6. Every T\(\pi^*g\)-space is semi-T\(_1\)-space.

Proof. 1. Assume that \((X, \tau)\) is a T\(\pi^*g\)-space. Let A be \(wg\)-closed set in X, then A is \(\pi^*g\)-closed. Since X is T\(\pi^*g\)-space, A is closed and hence X is T\(_{wg}\)-space.

The proof of the other results are similar and straightforward.

The converse of the above theorem need not be true as seen in the following examples.

Example 2.3.20.

1. Let X = \{a, b, c\} with \(\tau = \{X, \phi, \{b\}, \{a, b\}, \{b, c\}\}\). This space is T\(_{wg}\)-space but not T\(\pi^*g\)-space as the \(\pi^*g\)-closed set \(\{b\}\) is not closed.
2. Let X = \{a, b, c\} with \(\tau = \{X, \phi, \{b\}, \{a, b\}, \{b, c\}\}\). This is a \(\alpha\)-space but not T\(\pi^*g\)-space as the \(\pi^*g\)-closed set \(\{b\}\) is not closed.
3. Let X = \{a, b, c\} with \(\tau = \{X, \phi, \{a\}\}\). This space is semi-pre T\(_1\frac{1}{2}\) but not T\(\pi^*g\) as the \(\pi^*g\)-closed set \(\{b\}\) is not closed.
4. Let X = \{a, b, c\} with \(\tau = \{X, \phi, \{a\}\}\). This is T\(_w\) but not T\(\pi^*g\) as the subset \(\{b\}\) is \(\pi^*g\)-closed but not closed.
5. Let \( X = \{a,b,c\} \) with \( \tau = \{ \emptyset, X, \{b\}, \{a,b\}, \{b,c\} \} \). Here this space is \( T\frac{1}{2} \)-space but not \( T_{\pi^*g} \)-space as the subset \( \{b\} \) is \( \pi^*g \)-closed but not closed.

6. Let \( X = \{a,b,c,d\} \) with \( \tau = \{X, \emptyset, \{a\}, \{a,c\}, \{a,c,d\}\} \). This space is semi-\( T_1 \) but not \( T_{\pi^*g} \), as the \( \pi^*g \)-closed set \( \{c\} \) is not closed.

The summary of the above discussion is drawn here.

**Proposition 2.3.21.** If a space \( X \) is \( \pi^*g-T\frac{1}{2} \) space then every singleton of \( X \) is either \( \pi \)-closed or \( g \)-open.

**Proof.** Let \( x \in X \) and assume that \( \{x\} \) is not \( \pi \)-closed. Then clearly \( X \setminus \{x\} \) is not \( \pi \)-open and \( X \setminus \{x\} \) is trivially \( \pi^*g \)-closed set. By our assumption, it is \( g \)-closed and thus \( \{x\} \) is \( g \)-open.

\( \square \)

**Proposition 2.3.22.**

1. For a space \( X \), \( GO(X, \tau) \subset \pi^*GO(X, \tau) \).
2. A space is $\pi^*g$-$T\frac{1}{2}$ space if and only if $\text{GO}(X, \tau) = \pi^*\text{GO}(X, \tau)$.

**Proof.**

1. Let $A$ be $g$-open. Then $X - A$ is $g$-closed. Hence $X - A$ is $\pi^*g$-closed and $A$ is $\pi^*g$-open.

2. Let $X$ be $\pi^*g$-$T\frac{1}{2}$ space. Let $A \in \pi^*\text{GO}(X, \tau)$. Then $X - A$ is $\pi^*g$-closed. By hypothesis $X - A$ is $g$-closed and thus $A \in \text{GO}(X, \tau)$ and therefore $\text{GO}(X, \tau) = \pi^*\text{GO}(X, \tau)$.

Conversely let $\text{GO}(X, \tau) = \pi^*\text{GO}(X, \tau)$. Let $A$ be $\pi^*g$-closed. Then $X - A$ is $\pi^*g$-open. By assumption $X - A$ is $g$-open and $A$ is $g$-closed. Hence $X$ is $\pi^*g$-$T\frac{1}{2}$ space.

\[\square\]

### 2.4 $\pi^*g$- Continuous Functions

**Definition 2.4.1.** A function $f : (X, \tau) \to (Y, \sigma)$ is called $\pi^*g$-continuous if $f^{-1}(V)$ is $\pi^*g$-closed ($\pi^*g$-open) in $(X, \tau)$ for every closed (open) $V$ of $(Y, \sigma)$.

**Theorem 2.4.2.** Consider a map $f : (X, \tau) \to (Y, \sigma)$ from a topological space $X$ into a topological space $Y$.

1. If $f$ is continuous then $f$ is $\pi^*g$-continuous.
2. If $f$ is $g$-continuous then $f$ is $\pi^*g$-continuous.
3. If $f$ is $wg$-continuous then $f$ is $\pi^*g$-continuous.
4. If $f$ is $\alpha$-continuous then $f$ is $\pi^*g$-continuous.
5. If $f$ is $\alpha$-irresolute then $f$ is $\pi^*g$-continuous.
6. If $f$ is $w$-continuous then $f$ is $\pi^*g$-continuous.
Proof. 1. Let \( V \) be a closed set in \( Y \). Since \( f \) is continuous, \( f^{-1}(V) \) is closed in \( X \). As every closed set is \( \pi^*g \)-closed, \( f^{-1}(V) \) is \( \pi^*g \)-closed in \( X \). Hence \( f \) is \( \pi^*g \)-continuous.

The proof of remaining results are immediate. \( \square \)

The converse of the above theorem need not be true as seen from the following examples.

**Example 2.4.3.**

1. Let \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{X, \phi, \{a\}\} \) and \( \sigma = \{Y, \phi, \{b\}, \{b, c\}\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be defined by \( f(a) = b, f(b) = c, f(c) = a \) then \( f \) is \( \pi^*g \)-continuous but not continuous as the inverse image of open set \( \{b, c\} \) in \( Y \) is \( \{a, b\} \) in \( X \) which is not open.

2. Let \( X = Y = \{a, b, c\} \) with the topologies \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{b, c\}\} \) and \( f : (X, \tau) \to (Y, \sigma) \) be the identity function. Then the inverse image of \( \{b, c\} \) is not \( g \)-open in \( X \) and hence \( f \) is \( \pi^*g \)-continuous but not \( g \)-continuous.

3. Let \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{X, \phi, \{a\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a, b\}\} \) and \( f : (X, \tau) \to (Y, \sigma) \) be defined as \( f(a) = c, f(b) = b \) and \( f(c) = a \) then the inverse image of open set \( \{a, b\} \) in \( Y \) is \( \{b, c\} \) which is not \( wg \)-open and hence \( f \) is \( \pi^*g \)-continuous but not \( wg \)-continuous.

4. Let \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{X, \phi, \{a\}, \{b, c\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a, b\}\} \) and \( f : (X, \tau) \to (Y, \sigma) \) be the identity function. Here \( f \) is \( \pi^*g \)-continuous but not \( \alpha \)-continuous since the inverse image of \( \{a, b\} \) is not \( \alpha \)-open in \( X \).

5. Let \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{X, \phi, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a, c\}\} \) and \( f : (X, \tau) \to (Y, \sigma) \) be the identity map then the inverse image of \( \alpha \)-open set \( \{a, c\} \) is not \( \alpha \)-open in \( X \) and hence \( f \) is \( \pi^*g \)-continuous but not \( \alpha \)-irresolute.
6. Let \( X = Y = \{a, b, c\} \) with the topologies \( \tau = \{X, \emptyset, \{a\}, \{a, b\}\} \) and \( \sigma = \{Y, \emptyset, \{a, c\}\} \) and \( f : (X, \tau) \to (Y, \sigma) \) be the identity function. Then the inverse image of \( \{a, c\} \) is not \( w \)-open in \( X \) and hence \( f \) is \( \pi^*g \)-continuous but not \( w \)-continuous.

**Remark 2.4.4.** Composition of two \( \pi^*g \)-continuous function need not be \( \pi^*g \)-continuous as seen in the following example.

**Example 2.4.5.** Let \( X = Y = Z = \{a,b,c\}, \tau = \{\emptyset, X,\{a\},\{b\},\{a,b\}\}, \sigma = \{\emptyset, X, \{a\},\{a,b\}\}, \eta = \{\emptyset, X, \{b,c\}\}. \)

Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be the identity functions. Then \( f, g \) are \( \pi^*g \)-continuous functions. But \( g \circ f \) is not \( \pi^*g \)-continuous since \( \{a\} \) is closed in \( (Z, \eta) \) but \( (g \circ f)^{-1}(\{a\}) = f^{-1}(g^{-1}(\{a\})) = \{a\} \) is not \( \pi^*g \)-closed in \( (X, \tau) \).

**Proposition 2.4.6.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a function, then the following are equivalent

1. \( f \) is \( \pi^*g \)-continuous.
2. The inverse image of every open set in \( Y \) is \( \pi^*g \)-open in \( X \).

**Proof.** Straight forward.

**Theorem 2.4.7.** If \( f : (X, \tau) \to (Y, \sigma) \) is a \( \pi^*g \)-continuous function, then \( f(\pi^*g\text{-}\text{cl}(A)) \subset \text{cl}(f(A)) \) for every subset \( A \) of \( X \).

**Proof.** Let \( A \subset X \). Since \( f \) is \( \pi^*g \)-continuous and \( A \subset f^{-1}(\text{cl}(f(A))) \) we obtain \( \pi^*g\text{-}\text{cl}(A) \subset f^{-1}(\text{cl}(f(A))) \) and \( f(\pi^*g\text{-}\text{cl}(A)) \subset \text{cl}(f(A)) \).
The converse of above theorem need not be true as seen from the following example.

**Example 2.4.8.** Let \( X = Y = \{a,b,c,d\} \), \( \tau = \{ \emptyset, X, \{a\}, \{b\}, \{a,b\} \) and \( \sigma = \{\emptyset, X, \{a\}, \{c,d\}, \{a,c,d\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity function.

Then \( f (\pi^*g\text{-cl}(A)) \subseteq \text{cl}(f(A)) \) for every subset \( A \) of \( X \), since \( \pi^*g\text{-cl}(A) = A \) for every subset \( A \) of \( (X, \tau) \). But \( f \) is not \( \pi^*g\)-continuous since, for the closed set \( \{a,b\} \) of \( (Y, \sigma) \), \( f^{-1}(\{a,b\}) \) is not \( \pi^*g\)-closed in \( (X, \tau) \).

**Theorem 2.4.9.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function. The following statements are equivalent.

1. For each \( x \in X \) and each open set \( V \) containing \( f(x) \), there exists a \( \pi^*g\)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).
2. For every subset \( A \) of \( X \), \( f(\pi^*g\text{-cl}(A)) \subseteq \text{cl}(f(A)) \).

**Proof.** \( 1 \Rightarrow 2 \). Let \( y \in f(\pi^*g\text{-cl}(A)) \) and let \( V \) be any open neighborhood of \( y \). Then by (1) there exists an \( x \in X \) such that \( f(x) = y \) and a \( \pi^*g\)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \). By lemma 2.3.14, \( U \cap A \neq \emptyset \) and hence \( f(A) \cap V \neq \emptyset \). Hence \( y = f(x) \in \text{cl}(f(A)) \).

\( 2 \Rightarrow 1 \). Let \( x \in X \) and \( V \) be any open set containing \( f(x) \). Let \( A = f^{-1}(Y - V) \). Since \( f(\pi^*g\text{-cl}(A)) \subseteq \text{cl}(f(A)) \subseteq Y - V \), \( \pi^*g\text{-cl}(A) = A \). Since \( x \notin \pi^*g\text{-cl}(A) \), there exists a \( \pi^*g\)-open set \( U \) containing \( x \) such that \( U \cap A \neq \emptyset \) and hence \( f(U) \subseteq f(X - A) \subseteq V \).

**Remark 2.4.10.** Every \( \pi^*g \)-continuous function defined on a \( \pi^*g\text{-T}_1 \) space is \( g \)-continuous.
**Proposition 2.4.11.** Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be two functions. The composition \( g \circ f \) is \( \pi^*g \)-continuous if \( f \) is \( \pi^*g \)-continuous and \( g \) is continuous.

**Proof.** Let \( V \) be closed in \((Z, \eta)\). Since \( g \) is continuous, \( g^{-1}(V) \) is closed in \((Y, \sigma)\). Then \( f \) is \( \pi^*g \)-continuous implies \( f^{-1}(g^{-1}(V)) \) is \( \pi^*g \)-closed in \((X, \tau)\). Hence \( g \circ f \) is \( \pi^*g \)-continuous.

\( \square \)

**Definition 2.4.12.** Let \((X, \tau)\) be a topological space.

1. A collection \( \{ U_i : i \in \Lambda \} \) of \( \pi^*g \)-open subsets in \( X \) is called a \( \pi^*g \)-open cover of a subset \( A \) of \( X \) if \( A \subset \bigcup \{ U_i : i \in \Lambda \} \) holds.
2. \( X \) is called \( \pi^*gO \)-compact if every \( \pi^*g \)-open cover of \( X \) has a finite subcover.

**Definition 2.4.13.** A subset \( A \) of a space \((X, \tau)\) is said to be

1. \( \pi^*gO \)-compact relative to \( X \) if, for every collection \( \{ U_i : i \in \Lambda \} \) of \( \pi^*g \)-open subsets of \( X \) such that \( A \subset \bigcup \{ U_i : i \in \Lambda \} \), there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( A \subset \bigcup \{ U_i : i \in \Lambda_0 \} \)
2. \( \pi^*gO \)-compact if \( A \) is \( \pi^*gO \)-compact as a subspace of \( X \).

**Theorem 2.4.14.** Every \( \pi^*g \)-closed subset of a \( \pi^*gO \)-compact space \( X \) is \( \pi^*gO \)-compact relative to \( X \).

**Proof.** Let \( A \) be a \( \pi^*g \)-closed subset of a \( \pi^*gO \)-compact space \((X, \tau)\). Let \( \{ U_i : i \in \Lambda \} \) be a cover of \( A \) by \( \pi^*g \)-open subsets of \( X \). So \( A \subset \bigcup \{ U_i : i \in \Lambda \} \) and \((X - A) \cup \{ U_i : i \in \Lambda \} = X\). Since \( X \) is \( \pi^*gO \)-compact there exist a finite subset
Λ₀ of Λ such that \((X - A) \cup \{ U_i : i \in \Lambda₀ \} = X\). Then \(A \subset \cup \{ U_i : i \in \Lambda₀ \}\) and hence \(A\) is \(π^∗gO\)-compact relative to \(X\).

**Theorem 2.4.15.** The surjective \(π^∗g\)-continuous image of a \(π^∗gO\)-compact space is compact.

**Proof.** Let \(f : (X, τ) \rightarrow (Y, σ)\) be a \(π^∗g\)-continuous surjective map from \(π^∗gO\)-Compact space \(X\) on to \(Y\). Let \(\{A_i : i \in Λ\}\) be an open cover of \(Y\). Then \(\{f^{-1}(A_i) : i \in Λ\}\) is a \(π^∗g\)-open cover of \(X\). Since \(X\) is \(π^∗gO\)-compact, it has a finite sub cover, say \(\{f^{-1}(A_1), \ldots, f^{-1}(A_n)\}\). Since \(f\) is surjective, \(\{A_1, \ldots, A_n\}\) is a subcover of \(Y\) and \(Y\) is compact.

**Definition 2.4.16.** A topological space \((X, τ)\) is said to be \(π^∗g\)-connected if \(X\) cannot be written as the disjoint union of two non-empty \(π^∗g\)-open sets.

**Proposition 2.4.17.** For a space \((X, τ)\) the following are equivalent.

1. \(X\) is \(π^∗g\)-connected.
2. The only subsets of \(X\) which are both \(π^∗g\)-open and \(π^∗g\)-closed are the empty set \(φ\) and \(X\).
3. Every \(π^∗g\)-continuous function of \(X\) into a discrete space \(Y\) with at least two points is a constant map.

**Proof.** 1 \(\Rightarrow\) 2. Let \(U\) be a proper subset which is both \(π^∗g\)-open and \(π^∗g\)-closed in \(X\). Then \(X - U\) is both \(π^∗g\)-open and \(π^∗g\)-closed. Then \(X = U \cup (X - U)\), a disjoint union of non-empty sets which contradicts the fact that \(X\) is \(π^∗g\)-connected. Hence \(U = φ\) or \(U = X\).

2 \(\Rightarrow\) 1. Suppose that \(X = A \cup B\) where \(A\) and \(B\) are disjoint non-empty \(π^∗g\)-open subsets of \(X\). Since \(A = X - B\) and \(B = X - A\), \(A\) and \(B\) are both \(π^∗g\)-open
and $\pi^g$-closed. By assumption, $A = \emptyset$ or $X$ which is a contradiction. As a result $X$ is $\pi^g$-connected.

$2 \Rightarrow 3$. Let $f : (X,\tau) \to (Y,\sigma)$ be a $\pi^g$-continuous map where $Y$ is a discrete space with at least two points. Then $\{f^{-1}(y) : y \in Y\}$ is $\pi^g$-open and $\pi^g$-closed covering. By assumption, $f^{-1}(\{y\}) = \emptyset$ or $X$ for each $y \in Y$. If $f^{-1}(\{y\}) = \emptyset$ for all $y \in Y$ then $f$ fails to be a map. Then, there exist only one point $y \in Y$ such that $f^{-1}(\{y\}) \neq \emptyset$ and hence $f^{-1}(\{y\}) = X$ which shows that $f$ is a constant map.

$3 \Rightarrow 2$. Let $U$ be both $\pi^g$-open and $\pi^g$-closed in $X$. Suppose $U \neq \emptyset$. Let $f : (X,\tau) \to (Y,\sigma)$ be $\pi^g$-continuous map defined by $f(U) = \{y\}$ and $f(X-Y) = \{w\}$ for some distinct points $y$ and $w$ in $Y$. By assumption, $f$ is constant. Therefore $U = X$ and (2) holds.

Remark 2.4.18. It is obvious that every $\pi^g$-connected space is connected but the converse is not true.

Example 2.4.19. Let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Since the only clopen sets of $(X,\tau)$ are $X$ and $\emptyset$, $(X,\tau)$ is a connected space. $(X,\tau)$ is not $\pi^g$-connected since $\{a,c\}$ is both $\pi^g$-open and $\pi^g$-closed.