Chapter 6

A New Class of Sets in Čech Closure Space

6.1 Introduction and Preliminaries.
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6.1 Introduction

I. Arockiarani and C. Janaki [8] introduced $\pi g \alpha$-closed sets in topological space. K. Chandrasekhara Rao et al [32] introduced the concepts of Čech weakly $\delta$-Open sets and studied their properties. A thorough discussion on closure functions is due to Hammer [76,77] and later it was studied by Gnilka [73,74] and M.M. Day [37].

6.2 Čech -$\pi g \alpha$-closed sets

**Definition 6.2.1.** The finite union of Čech regular-open sets is called Čech $\pi$-open.

**Remark 6.2.2.** Every Čech regular open set is Čech $\pi$-open but the converse need not be true as seen in the following example.

**Example 6.2.3.** Let $X = \{a, b, c\}$. Define the function $k: P(X) \to P(X)$ as follows. $k(\phi) = \phi$, $k(\{a\}) = k(\{a, c\}) = \{a, c\}$, $k(\{b\}) = k(\{b, c\}) = \{b, c\}$, $k(\{c\}) = c$, $k(\{a, b\}) = k(X) = X$. $\{a, b\}$ is Čech $\pi$-open but not Čech regular open.

**Definition 6.2.4.** A subset $A$ of a Čech closure space $(X, k)$ is called Čech $\pi g \alpha$-closed if $k_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is Čech $\pi$-open in $X$.

**Theorem 6.2.5.** Let $(X, k)$ be a Čech closure space and $A \subset X$. Then the following are true

1. If $A$ is Čech closed then $A$ is Čech $\pi g \alpha$-closed.
2. If $A$ is Čech $g$-closed then $A$ is Čech $\pi g \alpha$-closed.
3. If A is Čech \(\pi\)-open and Čech \(\pi g\alpha\)-closed then A is Čech \(\alpha\)-closed.

Proof.

1. Let A be Čech closed. Let A \(\subset U\) where U is Čech \(\pi\)-open. Since every Čech closed set is Čech \(\alpha\)-closed, \(k_\alpha(A) \subset k(A) = A \subset U\). Hence A is Čech \(\pi g\alpha\)-closed.

2. Let A \(\subset U\) and U be Čech \(\pi\)-open. By assumption, \(k(A) \subset U\). Hence \(k_\alpha(A) \subset U\) implies A is Čech \(\pi g\alpha\)-closed set.

3. Let A be Čech \(\pi g\alpha\)-closed subset of Čech closure space \((X,k)\) and A be Čech \(\pi\)-open in X. Then \(k_\alpha(A) \subset A\). But always A \(\subset k_\alpha(A)\). Hence A = \(k_\alpha(A)\) implies that A is Čech \(\alpha\)-closed.

Converse of the above theorem need not be true as seen in the following examples.

**Example 6.2.6.** Let \(X = \{a,b,c\}\). Define \(k : P(X) \rightarrow P(X)\) as follows

1. \(k(\phi) = \phi, \, k(X) = X, \, k(\{a\}) = \{a\}, k(\{b\}) = k(\{a,b\}) = k(\{a,c\}) = k(\{b,c\}) = k(\{c\}) = X\). \(b,c\) is Čech \(\pi g\alpha\)-closed but not Čech closed set or Čech \(g\)-closed set.

2. \(k(\phi) = \phi, \, k(X) = X, \, k(\{a\}) = k(\{b\}) = k(\{a,b\}) = k(\{a,c\}) = k(\{b,c\}) = k(\{c\}) = X, \, k(\{c\}) = \{c\}\). Then \(\{c\}\) is Čech \(\alpha\)-closed, Čech \(\pi g\alpha\)-closed but not Čech \(\pi\)-open.

**Proposition 6.2.7.** If A and B are Čech \(\pi g\alpha\)-closed sets then so is \(A \cup B\).
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Proof. Let $U$ be Čech $\pi$-open set in $X$. Let $A \cup B \subseteq U$. Then $k_\alpha(A) \cup k_\alpha(B) \subseteq U$. Therefore $k_\alpha(A \cup B) \subseteq U$ implies that $A \cup B$ is Čech $\pi g\alpha$-closed set.

\[ \Box \]

Proposition 6.2.8. If $A$ is Čech $g$-closed, Čech $\pi$-open in $X$ and $F$ is Čech closed in $X$ then $A \cap F$ is Čech closed in $X$.

Remark 6.2.9. Intersection of two Čech $\pi g\alpha$-closed sets need not be Čech $\pi g\alpha$-closed.

Example 6.2.10. Let $X = \{a, b, c, d\}$. Define $k : P(X) \to P(X)$ as follows $k(\emptyset) = \emptyset$, $k(X) = X$, $k(\{c\}) = c$, $k(\{d\}) = d$, $k(\{c, d\}) = \{c, d\}$, $k(\{a, c, d\}) = (a, c, d)$, $k(\{b, c, d\}) = \{b, c, d\}$ and $k(\{a\}) = k(\{b\}) = k(\{a, c\}) = k(\{a, d\}) = k(\{b, c\}) = k(b, d) = k(\{a, b, c\}) = (\{a, b, d\}) = k(\{x\}) = X$. Here $\{a, b, d\}$ and $\{a, b, c\}$ are Čech $\pi g\alpha$-closed sets but $\{a, b\}$ is not Čech $\pi g\alpha$-closed.

Proposition 6.2.11. If $A$ is Čech $\pi g\alpha$-closed set, then $k_\alpha(A) - A$ contains no non-empty Čech $\pi$-closed set.

Proof. Let $A$ be a Čech $\pi g\alpha$-closed set and $F$ be a non-empty Čech $\pi$-closed set contained in $k_\alpha(A) - A$. Now $F \subseteq k_\alpha(A)$ and $F \subseteq A^c$.

$(F \subseteq A^c) \Rightarrow A \subseteq F^c$. Since $F$ is Čech $\pi$-closed, $F^c$ is Čech $\pi$-open. Thus we have $k_\alpha(A) \subseteq F^c$. Consequently $F \subseteq [k_\alpha(A)]^c$. Hence we get $F \subseteq k_\alpha(A) \cap [k_\alpha(A)]^c = \emptyset$. Hence $k_\alpha(A) - A$ contains no non-empty Čech $\pi$-closed set.

\[ \Box \]

Corollary 6.2.12. Let $A$ be Čech $\pi g\alpha$-closed set. Then $A$ is Čech $\alpha$-closed set iff $k_\alpha(A) - A$ is Čech $\pi$-closed set.

Proof. Suppose that $A$ is Čech $\pi g\alpha$-closed set and Čech $\alpha$-closed set. Since $A = k_\alpha(A)$, we have $k_\alpha(A) - A = \emptyset$ which is Čech $\pi$-closed.
Conversely, suppose that $A$ is Čech $\pi g\alpha$-closed and $k_{\alpha}(A) - A$ is Čech $\pi$-closed. Then $k_{\alpha}(A) - A$ contains no non-empty Čech $\pi$-closed set. Since $k_{\alpha}(A) - A$ is itself Čech $\pi$-closed, $k_{\alpha}(A) - A = \emptyset$ and hence $A$ is Čech $\alpha$-closed.

\[ \square \]

**Theorem 6.2.13.** Let $A \subset Y \subset X$ be Čech open in $X$. Suppose that $A$ is Čech $\pi g\alpha$-closed in $(X, k)$. Then $A$ is Čech $\pi g\alpha$-closed relative to $Y$.

**Proof.** Let $S$ be any Čech $\pi$-open set in $Y$ such that $A \subset S$. Then $S = U \cap Y$ for some Čech $\pi$-open set $U$ in $X$. Therefore $A \subset U \cap Y \subset U$. Since $A$ is Čech $\pi g\alpha$-closed set in $X$, we have $k_{\alpha}(A) \subset U$. Hence $k_{\alpha}(A) \cap Y \subset U \cap Y = S$. Thus $A$ is Čech $\pi g\alpha$-closed relative to $Y$.

\[ \square \]

**Theorem 6.2.14.** Let $A \subset Y \subset X$ and $Y$ be Čech regular open and Čech $\pi g\alpha$-closed in $X$. If $A$ is Čech $\pi g\alpha$-closed relative to $Y$ then $A$ is Čech $\pi g\alpha$-closed relative to $X$.

**Proof.** Let $A$ be Čech $\pi g\alpha$-closed relative to $Y$. Let $A \subset U$ where $U$ is Čech $\pi$-open in $X$. Since $A \subset Y$, we have $A \subset U \cap Y$ where $U \cap Y$ is Čech $\pi$-open in $Y$. Therefore $k_{\alpha}(A_Y) \subset U \cap Y$ and $k_{\alpha}(A_Y) \subset U$. Also $k_{\alpha}(A) = k_{\alpha}(A_Y) \subset U$. Hence $A$ is Čech $\pi g\alpha$-closed relative to $X$.

\[ \square \]

**Proposition 6.2.15.** If $A$ is Čech $\pi g\alpha$-closed, then $k_{r}(x) \cap A \neq \emptyset$ holds for each $x \in k_{\alpha}(A)$.

**Proof.** Let $A$ be Čech $\pi g\alpha$-closed set. Suppose that $k_{r}(x) \cap A = \emptyset$ holds for each $x \in k_{\alpha}(A)$. We have $A \subset [k_{r}(x)]^c$. Now $k_{r}(x)$ is Čech regular closed. Hence $[k_{r}(x)]^c$ is Čech regular open and hence Čech $\pi$-open. Since $A$ is Čech $\pi g\alpha$-closed set, $k_{\alpha}(A) \subset [k_{r}(x)]^c \Rightarrow k_{\alpha}(A) \cap k_{r}(x) = \emptyset \Rightarrow x \notin k_{\alpha}(A)$ which is a contradiction. Hence $k_{r}(x) \cap A \neq \emptyset$ holds for each $x \in k_{\alpha}(A)$.

\[ \square \]
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**Proposition 6.2.16.** Let \((X, k)\) be a Čech closure space. For each \(x \in X\), \(\{x\}\) is Čech \(\pi\)-closed (or) \(\{x\}^c\) is Čech \(\pi g \alpha\)-closed set.

**Proof.** Suppose that \(\{x\}\) is not Čech \(\pi\)-closed. Then \(\{x\}^c\) is not Čech \(\pi\)-open. The only Čech \(\pi\)-open sets containing \(\{x\}^c\) is \(X\). Thus \(\{x\}^c \subset X \Rightarrow k_\alpha(\{x\}^c) \subset X \Rightarrow \{x\}^c\) is Čech \(\pi g \alpha\)-closed set. □

**Definition 6.2.17.** A subset \(A\) in Čech closure space \((X, k)\) is called Čech \(\pi g \alpha\)-open if \(A^c\) is Čech \(\pi g \alpha\)-closed in \((X, k)\).

**Theorem 6.2.18.** A subset \(A\) of \((X, k)\) is Čech \(\pi g \alpha\)-open iff \(F \subset \text{int}_k(A)\) whenever \(F\) is Čech \(\pi\)-closed and \(F \subset A\).

**Proof.** Suppose that \(A\) is Čech \(\pi g \alpha\)-open in \((X, k)\). Let \(F\) be a Čech \(\pi\)-closed set and \(F \subset A\). Then \(F^c\) is Čech \(\pi\)-open set and \(A^c \subset F^c\). Since \(A^c\) is Čech \(\pi g \alpha\)-closed, we have \(k_\alpha(A^c) \subset F^c\) and hence \(F \subset [k_\alpha(A^c)]^c \Rightarrow F \subset \text{int}_k(A)\). That is, \(F \subset \text{int}_k(A)\) whenever \(F\) is Čech \(\pi\)-closed set and \(F \subset A\).

Conversely, suppose that \(F \subset \text{int}_k(A)\) whenever \(F\) is Čech \(\pi\)-closed and \(F \subset A\). Let \(V\) be Čech \(\pi\) open set in \(X\) such that \(A^c \subset V\). Then \(V^c \subset A\) and \(V^c\) is Čech \(\pi\)-closed in \(X\).

Therefore \(V^c \subset \text{int}_k(A) \Rightarrow [\text{int}_k(A)]^c \subset V \Rightarrow k_\alpha(A^c) \subset V \Rightarrow A^c\) is Čech \(\pi g \alpha\)-closed in \(X\) and hence \(A\) is Čech \(\pi g \alpha\)-open in \(X\). □

**Proposition 6.2.19.** If \(A\) and \(B\) are Čech \(\pi g \alpha\)-open sets so is \(A \cap B\).
Proof. Let \(A^c \cup B^c \subset U\) where \(U\) is Čech \(\pi\)-open. This implies \(A^c \subset U\) and \(B^c \subset U\). Hence \(k_\alpha(A^c) \subset U\) and \(k_\alpha(B^c) \subset U\). Thus \(k_\alpha(A^c \cup B^c) \subset U \Rightarrow (A \cap B)^c\) is a Čech \(\pi g \alpha\)-closed set. Hence \(A \cap B\) is Čech \(\pi g \alpha\)-open set.

Proposition 6.2.20. If \(A\) is Čech \(\pi g \alpha\)-open set in \(X\) and \(\text{int}_{k_\alpha}(A) \subset B \subset A\) then \(B\) is Čech \(\pi g \alpha\)-open set.

Proof. Let \(F\) be Čech \(\pi\)-closed set such that \(F \subset B\). Since \(B \subset A\), we have \(F \subset A\) and \(A\) is Čech \(\pi g \alpha\)-open implies \(F \subset \text{int}_{k_\alpha}(A)\). By assumption, \(F \subset \text{int}_{k_\alpha}(B)\). Hence \(B\) is Čech \(\pi g \alpha\)-open set.

Corollary 6.2.21. A subset \(A\) is Čech \(\pi g \alpha\)-closed set if \(k_\alpha(A) - A\) is Čech \(\pi g \alpha\)-open set.

Proof. Let \(F\) be Čech \(\pi\)-closed set such that \(F \subset k_\alpha(A) - A\). By proposition 6.2.11, \(F = \emptyset\). Therefore \(F \subset \text{int}_{k_\alpha}(k_\alpha(A) - A)\). Hence \(k_\alpha(A) - A\) is Čech \(\pi g \alpha\)-open set.

Proposition 6.2.22. The intersection of a Čech \(\pi g \alpha\)-open set and a Čech \(\pi\)-open set is always Čech \(\pi g \alpha\)-open set.

Definition 6.2.23.
1. A space \((X, k)\) is said to be Čech \(T_\pi\)-space, if every Čech closed set is Čech \(\pi\)-closed.
2. Two sets \(A\) and \(B\) are said to be separated if \(k(A) \cap B = A \cap k(B) = \emptyset\).

Theorem 6.2.24. If \(A\) and \(B\) are separated Čech \(\pi g \alpha\)-open sets in a Čech \(T_\pi\)-space, then \(A \cup B\) is Čech \(\pi g \alpha\)-open set.

Proof. Since \(A\) and \(B\) are Čech \(\pi g \alpha\)-open separated sets in \(X\), \(A \cap k(B) = k(A) \cap B = \emptyset\). Let \(F\) be a Čech \(\pi\)-closed set such that \(F \subset A \cup B\).
Therefore $F \cap k(A) \subset (A \cup B) \cap K(A) \subset (A \cap k(A)) \cup (B \cap k(A)) \subset A$. Similarly we can show that $F \cap k(B) \subset B$. Since $F \cap k(A)$ and $F \cap k(B)$ are Čech closed and since $X$ is Čech $T_\pi$-space, $F \cap k(A)$ and $F \cap k(B)$ are Čech $\pi$-closed.

$F \cap k(A) \subset A$ and $A$ is Čech $\pi g_\alpha$-open $\Rightarrow F \cap k(A) \subset \text{int } k_\alpha(A)$. Similarly, $F \cap k(B) \subset B$ and $B$ is Čech $\pi g_\alpha$-open $\Rightarrow F \cap k(B) \subset \text{int } k_\alpha(B)$. Hence $A \cup B$ is Čech $\pi g_\alpha$-open. $lacksquare$

6.3 $(k_1, k_2)$ — $\pi g_\alpha$ closed sets

In this section we extend the concepts of Čech $\pi g_\alpha$ closed sets to Bi-Čech closure spaces and study some of the separation axioms.

Definition 6.3.1. A subset $A$ in a Čech closure space $(X, k_1, k_2)$ is said to be $(k_1, k_2)$- $\pi g_\alpha$ closed if $k_2 \alpha(A) \subset U$, whenever $A \subset U$ and $U$ is $k_1$- $\pi$ open set in $X$.

Proposition 6.3.2. If $A$ and $B$ are $(k_1, k_2)$- $\pi g_\alpha$ closed sets then so is $A \cup B$.

Proof. Let $A$ and $B$ be two $(k_1, k_2)$- $\pi g_\alpha$ closed sets. Let $U$ be $k_1$- $\pi$ open set in $X$. Let $(A \cup B) \subset U$. Since $A$ and $B$ are $(k_1, k_2)$-$\pi g_\alpha$-closed sets, $k_2 \alpha(A) \subset U$ and $k_2 \alpha(B) \subset U$. $k_2 \alpha(A) \cup k_2 \alpha(B) \subset U$. Hence $k_2 \alpha(A \cup B) \subset U$. Thus $A \cup B$ is $(k_1, k_2)$-$\pi g_\alpha$ closed set. $lacksquare$

Proposition 6.3.3. If $A$ is $(k_1, k_2)$- $\pi g_\alpha$ closed set then $k_2 \alpha(A)$ - $A$ contains no non-empty $k_1$-$\pi$ closed set.

Proof. Let $A$ be $(k_1, k_2)$-$\pi g_\alpha$ closed set. Let $U$ be a non-empty $k_1$-$\pi$ closed contained in $k_2 \alpha(A)$ - $A$. Now, $U \subset k_2 \alpha(A)$ and $U \subset A^c$ and $A \subset U^c$. Since $U$ is $k_1$-$\pi$ closed, $U^c$ is $k_1$-$\pi$ open. Thus $k_2 \alpha(A) \subset U^c$. Consequently, $U \subset [k_2 \alpha(A)]^c$ and
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U ⊂ k₂α(A) ∩ [k₂α(A)] c = φ. Therefore, U = φ and k₂α(A) − A contains no non-empty k₁-π closed set.

\[\text{Proposition 6.3.4. Let } (X, k₁, k₂) \text{ be a biČech closure space. For each } x \in X, \{x\} \text{ is } k₁-\pi \text{ closed or } \{x\}^c \text{ is } (k₁,k₂) -\pi gα \text{ closed set.} \]

\[\text{Proof. Let } (X, k₁, k₂) \text{ be biČech closure space. Suppose that } \{x\} \text{ is not } k₁-\pi \text{ closed set, then } \{x\}^c \text{ is not } k₁-\pi \text{ open set. Therefore the only } k₁-\pi \text{ open set containing } \{x\}^c \text{ is } X. \text{ Thus } \{x\}^c \subset X. \text{ Also } k₂α[\{x\}^c] \subset k₂α(X) = X. \text{ Hence } \{x\}^c \text{ is a } (k₁,k₂) -\pi gα \text{ closed set.} \]

\[\text{Proposition 6.3.5. Let } A \text{ be } (k₁,k₂)-\pi gα \text{ closed set and if } A \text{ is } k₁-\pi \text{ open set then } A = k₂α(A). \]

\[\text{Proof. Let } A \text{ be } (k₁,k₂)-\pi gα \text{ closed subset of a biČech closure space } (X, k₁, k₂) \text{ and let } A \text{ be } k₁-\pi \text{ open set. Then } k₂α(A) \subset U \text{ whenever } A \subset U \text{ and } U \text{ is } k₁-\pi \text{ open set in } X. \text{ Since } A \text{ is } k₁-\pi \text{ open and } A \subset A, \text{ we have } k₂α(A) \subset A. \text{ But always } A \subset k₂α(A). \text{ Thus, } A = k₂α(A). \]

\[\text{Proposition 6.3.6. Let } A \subseteq Y \subseteq X \text{ and suppose that } A \text{ is } (k₁,k₂)-\pi gα \text{ closed set in } (X, k₁, k₂). \text{ Then } A \text{ is } (k₁,k₂)-\pi gα \text{ closed relative to } Y. \]

\[\text{Proof. Let } S \text{ be any } k₁-\pi \text{ open set in } Y \text{ such that } A \subset S. \text{ Then } S = U \cap Y \text{ for some } U \text{ which is } k₁-\pi \text{ open set in } X. \text{ Therefore } A \subset U \cap Y \text{ implies } A \subset U. \text{ Since } A \text{ is } (k₁,k₂)-\pi gα \text{ closed set in } X, \text{ we have } k₂α(A) \subset U. \text{ Hence } Y \cap k₂α(A) \subset Y \cap U = S. \text{ Thus } A \text{ is } πgα -\text{closed relative to } Y. \]

\[\text{Definition 6.3.7. A subset } A \text{ in biČech closure space } (X, k₁, k₂) \text{ is called } (k₁,k₂)-\pi gα \text{ open set if } A^c \text{ is } (k₁,k₂)-\pi gα \text{ closed set in } (X, k₁, k₂). \]
Proposition 6.3.8. A subset $A$ of $(X, k_1, k_2)$ is $(k_1, k_2)$-$\pi g\alpha$ open set if and only if $F \subset (\text{int}_{k_2}(A))$ whenever $F$ is $k_1$-$\pi$ closed set and $F \subset A$.

Proof. Suppose $A$ is $(k_1, k_2)$-$\pi g\alpha$ open set in $(X, k_1, k_2)$. Let $F$ be $k_1$-$\pi$ closed set and $F \subset A$. Then $F^c$ is $k_1$-$\pi$ open set and $A^c \subset F^c$. Since $A^c$ is $(k_1, k_2)$-$\pi g\alpha$ closed set, we have $k_2\alpha(A^c) \subset F^c$. This implies $F \subset \text{int}_{k_2\alpha}(A)$. That is $F \subset \text{int}_{k_2\alpha}(A)$ whenever $F$ is $k_1$-$\pi$ closed set and $F \subset A$. Let $V$ be any $k_1$-$\pi$ open set in $X$ such that $A^c \subset V$. Thus $V^c \subset A$ and $V^c$ is $k_1$-$\pi$ closed. Therefore, $V^c \subset \text{int}_{k_2\alpha}(A)$ and $[\text{int}_{k_2\alpha}(A)]^c \subset V \Rightarrow k_2\alpha(A^c) \subset V$. Hence $A^c$ is $(k_1, k_2)$-$\pi g\alpha$ closed set $A$ is $(k_1, k_2)$-$\pi g\alpha$ open set.

Corollary 6.3.9. A subset $A$ of $(X, k_1, k_2)$ is $(k_1, k_2)$-$\pi g\alpha$ closed set, then $k_2\alpha(A) - A$ is $(k_1, k_2)$-$\pi g\alpha$ open set.

Proof. Let $F$ be $k_1$-$\pi$ closed set such that $F \subset k_2\alpha(A) - A$. Then using proposition 6.3.3, $F = \emptyset$. Therefore $F \subset \text{int}_{k_2\alpha}(k_2\alpha(A)) - A$ and $k_2\alpha(A) - A$ is $(k_1, k_2)$-$\pi g\alpha$ open set.

Proposition 6.3.10. If $A$ and $B$ be $(k_1, k_2)$-$\pi g\alpha$ open set, then so is $A \cap B$.

Proof. Let $A^c \cup B^c \subset U$ where $U$ is $k_1$-$\pi$ open. This implies $A^c \subset U$ and $B^c \subset U$, gives $k_2\alpha(A^c) \subset U$ and $k_2\alpha(B^c) \subset U$. Thus $k_2\alpha(A^c) \cup k_2\alpha(B^c) \subset U$. Therefore $k_2\alpha(A^c \cup B^c) \subset U$. Therefore $A \cap B$ is $(k_1, k_2)$-$\pi g\alpha$ open set.

Definition 6.3.11. A biclosure space $(X, k_1, k_2)$ is called a $\pi g\alpha$ -$T_{\frac{1}{2}}$ biclosure space if every $\pi g\alpha$- closed subset of $(X, k_1, k_2)$ is a $k_2$-$\alpha$ closed.

Proposition 6.3.12. The biclosure space $(X, k_1, k_2)$ is a $\pi g\alpha$ -$T_{\frac{1}{2}}$ space iff every $\{x\}$ of $X$ is either $k_2$-$\alpha$ open or $k_1$-$\pi$ closed.
Proof. Let $x \in X$ and suppose that $\{x\}$ is not a $k_1$-$\pi$ closed subset of $X$. Then $X - \{x\}$ is not a $k_1$-$\pi$ open subset of $X$. The only $\pi$-open subsets of $(X$ containing $X - \{x\}$ is $X$. Hence $X - \{x\}$ is a $(k_1,k_2)$-$\pig\alpha$ closed subset of $X$. Since $(X, k_1, k_2)$ is a $\pig\alpha$-$T_1$ biclosure space, $X - \{x\}$ is $k_2$-$\alpha$ closed subset of $X$. Consequently $\{x\}$ is $k_1$-$\pi$ open subset of $X$.

Conversely. Let $A$ be $(k_1,k_2)$-$\pig\alpha$ closed subset of $(X, k_1, k_2)$. Suppose $x \notin A$, then $\{x\} \subset X - A$ and we have $A \subset X - \{x\}$.

Case 1. If $\{x\}$ is $k_2$-$\alpha$ open, then $X - \{x\}$ is $k_2$-$\alpha$ closed subset of $X$ and we have $k_2\alpha(A) \subset k_2\alpha(X - \{x\}) = X - \{x\}$ and thus $x \notin k_2\alpha(A)$. Hence $k_2\alpha(A) \subseteq A$ and $A = k_2\alpha(A)$ and $A$ is $k_2$-$\alpha$ closed.

Case 2. If $\{x\}$ is $k_1$-$\pi$ closed subset of $X$ then $X - \{x\}$ is $k_1$-$\pi$ open subset of $X$. Since $A$ is $(k_1,k_2)$-$\pig\alpha$ closed, $k_2\alpha(A) \subset X - \{x\}$. Therefore $x \notin k_2\alpha(A)$ and we get $k_2\alpha(A) \subset A$. But $A \subset k_2\alpha(A)$ and thus $k_2\alpha(A) = A$ and $A$ is $k_2$-$\alpha$ closed in $X$ and $X$ is $\pig\alpha$-$T_1$ space.

\[ \square \]

**Definition 6.3.13.** A biclosure space $(X, k_1, k_2)$ is said to be $(k_1,k_2)$-$\pig\alpha$-Hausdorff space whenever $x$ and $y$ are distinct points of $X$ there exist a $k_1$-$\pig\alpha$-open subset $U$ and $k_2$-$\pig\alpha$ open subset $V$ of $X$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

**Definition 6.3.14.** Let $(X, u_1,u_2)$ and $(Y,v_1,v_2)$ be biclosure spaces. A map $f : X \to Y$ is called $\pig\alpha$-irresolute, if $f^{-1}(F)$ is a $\pig\alpha$-closed subset of $X$ for every $\pig\alpha$-closed subset $F$ of $Y$.

**Definition 6.3.15.** Let $(X, u_1,u_2)$ and $(Y,v_1,v_2)$ be biclosure spaces and let $i \in \{1,2\}$. A map $f : (X, u_i) \to (Y,v_i)$ is called $i$-$\pig\alpha$-irresolute if the map $f : (X, u_i) \to (Y,v_i)$ is $\pig\alpha$-irresolute.
Proposition 6.3.16. Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be biclosure spaces. Let \(f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)\) be injective and \(\pi g\alpha\)-irresolute. If \((Y, v_1, v_2)\) is a \((v_1, v_2)\)-\(\pi g\alpha\)-Hausdorff biclosure space, then \((X, u_1, u_2)\) is a \((u_1, u_2)\)-\(\pi g\alpha\)-Hausdorff space.

Proof. Let \(x\) and \(y\) be any two distinct points of \(X\). Then \(f(x)\) and \(f(y)\) are distinct points of \(Y\). Since \((Y, v_1, v_2)\) is a \((v_1, v_2)\)-\(\pi g\alpha\)-Hausdorff biclosure space, there exists a disjoint \(v_1\)-\(\pi g\alpha\) open subset \(U\) of \((Y, v_1)\) and \(v_2\)-\(\pi g\alpha\) open subset \(V\) of \((Y, v_2)\) containing \(f(x) \in U\) and \(f(y) \in V\) respectively. Since \(f\) is \(\pi g\alpha\)-irresolute and \(U \cap V = \emptyset\), \(f^{-1}(U)\) is a \(u_1\)-\(\pi g\alpha\) open subset of \(X\) and \(f^{-1}(V)\) is an \(u_2\)-\(\pi g\alpha\) open subset of \(X\) such that \(f^{-1}(U) \cap f^{-1}(V) = \emptyset\) and \((X, u_1, u_2)\) is a \(\pi g\alpha\)-Hausdorff biclosure space.