Chapter 7

Effects of Hall current on MHD flow through a uniform channel bounded by porous media - Case B

7.1 Introduction

In the previous chapter, the effect of Hall current on MHD flow through a uniform channel bounded by porous media using BJ slip condition was discussed. As mentioned earlier, BJ slip condition is valid in the case where the width of the porous media is much larger than the width of the flow region. The assumption of large thickness of porous media made required to make use of BJ slip condition is relaxed by using the slip condition proposed by Rudraiah [1985].

Author’s Contribution

To achieve the objective of this chapter namely to know the effect of Hall current on MHD flow through a uniform channel bounded by porous media where the thickness of the porous layer is smaller than the width of the fluid layer, the required basic equations are derived and the slip condition developed by Rudraiah [1985] is used. The combined effect porous parameter,
Hartmann number, width of the porous media and Hall parameter on the primary and secondary velocities, wall shear stress are discussed in detail.

### 7.2 Formulation and Solution of the Problem

Consider an infinitely long channel of uniform width $2h$ through which a laminar, steady and viscous hydromagnetic fluid flows. The channel is bounded externally on either side by densely packed saturated porous layers each of width $H$ at $y = \pm h$. We assume that the fluid behaves like a homogeneous conducting Newtonian fluid with constant density $\rho$ and viscosity $\mu$. It is assumed that the fluid has a small electrical conductivity $\sigma_e$ and the electro-magnetic force produced is also very small.

Let $u$ and $v$ be the velocity components of the fluid at a point $(x, y)$ where $x$ is measured along the channel and $y$ normal to it. It is further assumed that the channel is symmetrical about the $x$-axis. The porous layer is assumed to be homogeneous, isotropic and densely packed. A uniform magnetic field $H_0$ is applied normal to the motion of the fluid. $B_0 = \mu_e H_0$, is the electromagnetic induction where $\mu_e$ is the magnetic permeability. Since the magnetic Reynolds number is very small, the induced magnetic field can be neglected in comparison with the applied magnetic field. Since the channel is infinitely long, all physical quantities (except pressure) depend only on $y$.

The basic equations governing the flow in the presence of magnetic field and Hall current are as follows:

**Continuity equation:**

$$\nabla \cdot \vec{q} = 0 \quad (7.2.1)$$

**Momentum equation:**

$$\rho (\vec{q} \cdot \nabla) \vec{q} = -\nabla p + \rho \vec{F} + \mu \nabla^2 \vec{q} + \vec{j} \times \vec{B} \quad (7.2.2)$$
Generalised Ohm’s Law:

\[ \vec{J} + \frac{\omega_e \tau_e}{H_0} (\vec{J} \times \vec{H}) = \sigma_e \left[ \mu_e (\vec{q} \times \vec{H}) + \nabla P_e \right] \]  

(7.2.3)

where \( \rho \) - density of the fluid, \( \vec{q} \) - velocity vector of the fluid, \( p \) - pressure of the fluid, \( \mu \) - coefficient of viscosity of the fluid, \( F \) - body force per unit mass, \( j \) - total electric current density, \( \vec{B} \) - magnetic field, \( \vec{j} \times \vec{B} \) - Lorentz force (with \( j \) calculated from the generalized ohm’s law for Hall current problem), \( \sigma_e \) - electrical conductivity of the fluid, \( \mu_e \) - the magnetic permeability, \( \omega_e \) - the cyclotron frequency, \( \tau_e \) - the electron collision time, \( e \) - the electric charge, \( n_e \) - the number density of the electron and \( P_e \) - the electron pressure.

The equation of continuity \( \nabla \cdot \vec{q} = 0 \) gives \( v = 0 \) where \( \vec{q} = (u,v,w) \). In the case of a strong applied magnetic field, the Hall effect induces a secondary flow, and so there will be two components of the velocity \( u \) and \( w \). The solenoidal relation \( \nabla \cdot \vec{H} = 0 \) for the magnetic field gives \( H_y = H_0 = \) constant everywhere in the fluid. The conservation of electric charge \( \nabla \cdot \vec{J} = 0 \) gives \( J_y = \) constant. This constant is zero, since \( J_y = 0 \) on the walls which are electrically non-conducting. In the absence of an external electric field, the effect of polarization of the ionized fluid is negligible. It is also assumed that the applied electric field \( \vec{E} = 0 \). In the generalized Ohm’s law, the ion slip and thermo-electric effects are neglected. Further, for weakly ionized gases the electron pressure is also negligible.

The governing equations of the flow(after neglecting inertia effects) in the dimensionless form are:

\[
R \frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial y^2} - \frac{M^2}{1 + m^2} (u + mw) \]  

(7.2.4)

\[ v = 0 \]  

(7.2.5)

\[
R \frac{\partial p}{\partial y} = 0 \]  

(7.2.6)

\[
R \frac{\partial p}{\partial z} = \frac{\partial^2 w}{\partial y^2} + \frac{M^2}{1 + m^2} (mu - w) \]  

(7.2.7)

where

\[ M^2 = \frac{\sigma_e \mu^2 H_0^2 h^2}{\mu_f^2}, \quad R = \frac{U h}{\nu} \]  

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These equations are solved using the BJR slip boundary conditions, which, in the dimensionless form (Rudraiah \(1985\)) are given by,

\[
\begin{align*}
\frac{du}{dy} &= \delta_1 Q_1 + \delta_2 (u_{B_1} - Q_1) \quad \text{at} \quad y = -1 \\
\frac{du}{dy} &= -\delta_1 Q_1 - \delta_2 (u_{B_2} - Q_1) \quad \text{at} \quad y = 1 \\
u &= u_{B_1} \quad \text{at} \quad y = 1 \\
u &= u_{B_2} \quad \text{at} \quad y = -1 \\
\int_{-1}^{1} u(y)dy &= n_f \\
v &= \epsilon Q_y \\
Q_1 &= -\frac{\delta_3^2 R}{\sigma_p^2} \frac{\partial p}{\partial x} \\
w &= 0 \quad \text{at} \quad y = \pm 1
\end{align*}
\]

where

\[
\delta_1 = \frac{\lambda \sqrt{\lambda} \sigma_p \epsilon}{\delta_0 \sinh \left( \frac{\sigma_p}{\sqrt{\lambda}} \right)}, \quad \delta_2 = \frac{\sqrt{\lambda}}{\delta_0} \sigma_p \coth \left( \frac{\sigma_p}{\sqrt{\lambda}} \right) \\
\sigma_p = \frac{H}{\sqrt{k}}, \quad \delta_0 = \frac{H}{h}
\]

Here, \(\lambda\) is a positive constant called viscosity factor, \(n_f\) is the net flux through the channel, \(u_{B_1}\) and \(u_{B_2}\) are the slip velocities, \(Q_1\) is the Darcy velocity, \(k\) is the permeability of the porous material and \(\sigma_p\) is called the porous parameter.

Since the channel is of uniform width, we use the approximation that the wall slope is everywhere negligible (Chandrasekhara and Rudraiah [1980]). Eliminating the pressure terms between (7.2.4), (7.2.6) and (7.2.7), the basic equations in non-dimensional form are as follows :

\[
\begin{align*}
\frac{d^3 u}{dy^3} - F \frac{du}{dy} - mF \frac{dw}{dy} &= 0 \\
\frac{d^3 w}{dy^3} + mF \frac{du}{dy} - F \frac{dw}{dy} &= 0
\end{align*}
\]
since $u$ and $w$ are function of $y$ alone. Here,

$$F = \frac{M^2}{1 + m^2}$$

Solving the simultaneous equations (7.2.18) and (7.2.19) subject to the conditions from (7.2.8) to (7.2.15), the expressions for primary and secondary velocity components are given by,

$$u(y) = B_1 + 2B_2 \cosh(\psi_1 y) \cos(\psi_2 y) + 2B_3 \sinh(\psi_1 y) \sin(\psi_2 y)$$

$$w(y) = \frac{1}{mF(\psi_1^2 + \psi_2^2)} \left[ (\eta_{17} B_2 + \eta_{18} B_3) (\psi_1 \cosh(\psi_1 y) \cos(\psi_2 y) + \psi_2 \sinh(\psi_1 y) \sin(\psi_2 y)) \right]$$

$$+ \frac{1}{mF(\psi_1^2 + \psi_2^2)} \left[ (\eta_{17} B_3 - \eta_{18} B_2) (\psi_1 \sinh(\psi_1 y) \sin(\psi_2 y) - \psi_2 \cosh(\psi_1 y) \cos(\psi_2 y)) \right]$$

$$- \frac{1}{mF(\psi_1^2 + \psi_2^2)} \left[ \eta_{15} (\eta_{17} B_2 + \eta_{18} B_3) + \eta_{16} (\eta_{17} B_3 - \eta_{18} B_2) \right]$$

(7.2.21)

where

$$r = \sqrt{1 + m^2}$$

$$\theta = \arctan(m)$$

$$\psi_1 = \sqrt{Fr} \cos \left( \frac{\theta}{2} \right)$$

$$\psi_2 = \sqrt{Fr} \sin \left( \frac{\theta}{2} \right)$$

$$\eta_1 = \psi_1 \cosh(\psi_1) \sin(\psi_2) - \psi_2 \sinh(\psi_1) \cos(\psi_2)$$

$$\eta_2 = \psi_1 \cosh(\psi_1) \sin(\psi_2) + \psi_2 \sinh(\psi_1) \cos(\psi_2)$$

$$\eta_3 = \psi_1 \sinh(\psi_1) \cos(\psi_2) - \psi_2 \cosh(\psi_1) \sin(\psi_2)$$

$$\eta_4 = \psi_1 \sinh(\psi_1) \cos(\psi_2) + \psi_2 \cosh(\psi_1) \sin(\psi_2)$$

$$\eta_5 = \psi_1 \sin(\psi_1) \cos(\psi_2) + \psi_2 \sinh(\psi_1) \cosh(\psi_2)$$

$$\eta_6 = \cosh^2(\psi_1) - \cos^2(\psi_2)$$

$$\eta_7 = \sinh(\psi_1) \sin(\psi_2)$$

$$\eta_8 = \cosh(\psi_1) \cos(\psi_2)$$
\[
\eta_9 = \frac{\eta_3}{(\psi_1^2 + \psi_2^2)\eta_5 - 2\psi_1\psi_2\eta_6}
\]
\[
\eta_{10} = \frac{\eta_3}{(\psi_1^2 + \psi_2^2)\eta_8 - \eta_4}
\]
\[
\eta_{11} = \frac{\eta_3}{(\psi_1^2 + \psi_2^2)\eta_5 - 2\psi_1\psi_2\eta_6}
\]
\[
\eta_{12} = \frac{(\psi_1^2 + \psi_2^2)\eta_7 - \eta_2}{(\psi_1^2 + \psi_2^2)\eta_5 - 2\psi_1\psi_2\eta_6}
\]
\[
\eta_{13} = \frac{(\psi_1^2 + \psi_2^2)[\delta_2\eta_4 + \eta_3] - 2\eta_2\eta_1}{(\psi_1^2 + \psi_2^2)\eta_5 - 2\psi_1\psi_2\eta_6}
\]
\[
\eta_{14} = \frac{(\psi_1^2 + \psi_2^2)[\delta_2\eta_7 + \eta_1] - 2\eta_2\eta_1}{(\psi_1^2 + \psi_2^2)\eta_5 - 2\psi_1\psi_2\eta_6}
\]
\[
\eta_{15} = \psi_1\cosh(\psi_1)\cos(\psi_2) + \psi_2\sinh(\psi_1)\sin(\psi_2)
\]
\[
\eta_{16} = \psi_1\sinh(\psi_1)\sin(\psi_2) - \psi_2\cosh(\psi_1)\cos(\psi_2)
\]
\[
\eta_{17} = 2\psi_1(\psi_1^2 - 3\psi_2^2 - F)
\]
\[
\eta_{18} = 2\psi_2(3\psi_1^2 - \psi_2^2 - F)
\]
\[
\eta_{19} = 1 + \eta_1\eta_{12} - \eta_2\eta_9
\]
\[
\eta_{20} = \eta_5\eta_{17} - \eta_6\eta_{18}
\]
\[
\eta_{21} = \eta_5\eta_{18} + \eta_6\eta_{17}
\]
\[
\eta_{22} = \eta_2\eta_{11} - \eta_4\eta_{14}
\]
\[
\eta_{23} = \eta_4\eta_{13} - \eta_2\eta_{10}
\]
\[
\eta_{24} = \frac{\eta_3\eta_{12} - \eta_1\eta_9}{\eta_3\eta_{14} - \eta_1\eta_{11} + \Delta\sigma}
\]
\[
\eta_{25} = \frac{1 + \eta_3\eta_{13} - \eta_1\eta_{10}}{\eta_3\eta_{14} - \eta_1\eta_{11} + \Delta\sigma}
\]
\[
u_{B_1} = \frac{1}{2} (\eta_3\eta_{12} - \eta_1\eta_9) n_f + 2 (1 + \eta_3\eta_{13} - \eta_1\eta_{10}) (\delta_2 - \delta_1) Q_1\]
\[
F = \frac{4\sigma_0(\psi_1^2 + \psi_2^2)\delta_0^{-2} + 2Q_1(\delta_2 - \delta_1)[\eta_{20}(\eta_{14}\eta_{25} - \eta_{13}) + \eta_{21}(\eta_{10} - \eta_1\eta_{25}) - 2F(\eta_{23} + \eta_{22}\eta_{25})]}{\eta_{20}(\eta_{14}\eta_{24} - \eta_{12}) + \eta_{21}(\eta_9 - \eta_1\eta_{24}) - 2F(\eta_{19} + \eta_{22}\eta_{24})}
\]

(7.22)

(7.23)
The shear stress components in the lower wall are given by
\[
\begin{align*}
\tau_x &= 2 \left( \psi_2 B_2 - \psi_1 B_3 \right) \cosh(\psi_1) \sin(\psi_2) - 2 \left( \psi_1 B_2 + \psi_2 B_3 \right) \sinh(\psi_1) \cos(\psi_2) \\
\tau_z &= -\frac{1}{m_f} \left[ (\eta_{17} B_2 + \eta_{18} B_3) \sinh(\psi_1) \cos(\psi_2) + (\eta_{17} B_3 - \eta_{18} B_2) \cosh(\psi_1) \sin(\psi_2) \right]
\end{align*}
\]

7.3 Results and Discussion

The purpose of the present discussion is to analyze and compare the effects of Hartmann number \(M\), porous parameter \(\sigma_p\), Hall parameter \(m\) and width of the porous layer \(H\) on the primary velocity \(u\), secondary velocity \(w\) and shear stress in the case where the thickness of the porous layer is much smaller than the width of the fluid layer.

To observe the quantitative effects of \(\sigma_p\), \(M\), \(m\) and \(H\), numerical evaluations of the analytic results obtained for \(u\), \(w\), \(\tau_x\) and \(\tau_z\) for the values of the parameters \(\lambda = 6.0\), \(M = 2, 3, 4, 5\), \(\sigma_p = 10^2, 10^3\), \(Q_1 = 0\) to \(0.0001\), \(H = h/4\) and \(m = 0.5, 1.0, 1.5, 2.0\) are carried out.

The effect of Hartmann number \(M\) on the primary velocity \(u\) is depicted in figure 7.1 and 7.2. A parabolic primary velocity profile is observed with maximum value at the channel centerline and minimum value at the walls. However, a general decrease in the magnitude of primary velocity profile is noticed with an increase in Hartmann number \(M\). Figs. 7.3 and 7.4 are drawn to find the effect of the Hall parameter \(m\) on the primary velocity. It is observed that the primary velocity increases with increasing \(m\) for a given value of \(M\).

From figures 7.1 to 7.4, it is seen that the primary velocity decreases as the porous parameter increases. However when the porous parameter \(\sigma_p\) is further increased beyond \(10^5\), there is no significant difference in the primary velocity (graphs are not shown), since in this case the walls
Figure 7.1: Effect of $M$ on primary velocity $u$ Vs. channel width ($m = 0.5$)

behave like impermeable walls.

The secondary velocity($w$) profile are plotted in Figs. 7.5 and 7.6 for different values of $M$ and for a given value of $m$. It is found that the secondary flow velocity decreases in magnitude with the increase in Hartmann number($M$).

In Figs. 7.7 and 7.8, the effect of Hall parameter($m$) on the secondary flow velocity for a given value of $M$ are shown. It is observed that the secondary flow velocity increases in magnitude with the increase in Hall parameter $m$. This confirms the fact that the secondary velocity component is a result of the Hall effect responding positively to ascending values of $m$.

Figure 7.9 and 7.10 indicate the development of the primary velocity for various width of the porous layer $H$ for fixed values of $M$ and $m$. It is clear from these figures that the primary velocity increases in the central region of the channel for increasing $H$ and it exhibits no significant changes near the boundary of the channel. In general, primary velocity distribution decreases as $\sigma_p$ increases.

Figure 7.11 and 7.12 show the effect of the width of the channel($H$) on the secondary flow velocity at fixed values of $M$ and $m$. It is observed that the secondary flow velocity increases in
Figure 7.2: Effect of $M$ on primary velocity $u$ Vs. channel width ($m = 2.0$)

Figure 7.3: Effect of $m$ on primary velocity $u$ Vs. channel width ($M = 2.0$)
Figure 7.4: Effect of $m$ on primary velocity $u$ Vs. channel width ($M = 4.0$)

Figure 7.5: Effect of $M$ on secondary velocity $w$ Vs. channel width ($m = 0.5$)
Figure 7.6: Effect of $M$ on secondary velocity $w$ Vs. channel width ($m = 2.0$)

Figure 7.7: Effect of $m$ on secondary velocity $w$ Vs. channel width ($M = 2.0$)
Figure 7.8: Effect of $m$ on secondary velocity $w$ Vs. channel width ($M = 4.0$)

Figure 7.9: Effect of $H$ on primary velocity $u$ Vs. channel width ($M = 2.0, m = 1.0$)
Figure 7.10: Effect of $H$ on primary velocity $u$ Vs. channel width ($M = 3.0, m = 2.0$)

Figure 7.11: Effect of $H$ on secondary velocity $w$ Vs. channel width ($M = 2.0, m = 1.0$)
Figure 7.12: Effect of $H$ on secondary velocity $w$ Vs. channel width ($M = 3.0, m = 2.0$)

Table 7.1: Shear Stress($\tau_x$) for primary velocity at different values of $M$ and $m$ ($H = h/4$ and $\sigma = 10^2$)

<table>
<thead>
<tr>
<th>$M$ ↓</th>
<th>$m$ →</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td></td>
<td>-0.9798</td>
<td>-0.2242</td>
</tr>
<tr>
<td>3.0</td>
<td></td>
<td>-0.4525</td>
<td>-0.0798</td>
</tr>
<tr>
<td>4.0</td>
<td></td>
<td>-0.1110</td>
<td>-0.0422</td>
</tr>
</tbody>
</table>

magnitude as $H$ increases. It is also noted that the secondary velocity has larger values in the central core region of the channel.

Finally, the values of shear stress at the lower wall for the primary and secondary flows are given in Table 7.1 and 7.2 for different values of $m$ and $M$. The value of shear stress at the lower wall for the primary and secondary flows decreases numerically as both $M$ and $m$ increases. It is also observed that the values of shear stress at the upper wall are equal and opposite to those at the lower wall.
Table 7.2: Shear Stress($\tau_z$) for secondary velocity at different values of $M$ and $m$ ($H = h/4$ and $\sigma = 10^2$)

<table>
<thead>
<tr>
<th>$M$ ↓ $m$ →</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>-5.4384</td>
<td>-1.7311</td>
</tr>
<tr>
<td>3.0</td>
<td>-2.8101</td>
<td>-1.4720</td>
</tr>
<tr>
<td>4.0</td>
<td>-2.7125</td>
<td>-0.9436</td>
</tr>
</tbody>
</table>

7.4 Conclusion

The effects $\sigma_p$, $M$, $H$ and $m$ on the steady flow of conducting viscous incompressible fluid in a uniform channel covered by porous media using BJR slip condition were investigated. Our results reveal that $M$ produce a retarding effect on both the primary and secondary velocities where as the other parameter i.e., $m$ has an accelerating effect on the primary and secondary velocities. The porous parameter $\sigma_p$ accelerats primary and secondary velocities for large value of $m$ whereas it decreases both the velocities when $m$ is small. It is observed that when $H$ increases, both the primary and secondary velocities decrease. The shear stress for the primary and secondary flows increases numerically as in the case of BJ slip condition for ascending Hall parameter($m$).