CHAPTER-II

ON MODIFICATIONS OF GTS VIA HEREDITARY CLASSES

In this chapter, we discuss the properties of hereditary classes, characterize \( \mu \)– codense hereditary classes and generalize some of the results established in [44].

The following Theorem 2.1 gives a characterization of \( \mu \)– codense hereditary classes.

**Theorem 2.1.** If \((X, \mu)\) is a GTS with a hereditary class \( \mathcal{H} \), then \( \mathcal{H} \) is \( \mu \)– codense if and only if \( i_{\mu}(I) = \emptyset \) for every \( I \in \mathcal{H} \).

**Proof.** Suppose \( \mathcal{H} \) is \( \mu \)– codense. If \( I \in \mathcal{H} \), then \( i_{\mu}(I) \in \mu \cap \mathcal{H} \) and so \( i_{\mu}(I) = \emptyset \). Conversely, suppose the condition holds. If \( A \in \mu \cap \mathcal{H} \), then \( A \in \mathcal{H} \) and so \( i_{\mu}(A) = \emptyset \) which implies that \( A = \emptyset \). Therefore, \( \mathcal{H} \) is \( \mu \)– codense.

**Corollary 2.2.** If \((X, \mu)\) is a GTS with a hereditary class \( \mathcal{H} \), then \( \mathcal{H}_\mu \) is \( \mu \)– codense.

**Theorem 2.3.** If \((X, \mu)\) is a GTS with a hereditary class \( \mathcal{H} \) and \( A \in \mathcal{H} \), then \( i_{\mu}(A^*) = \emptyset \) and so \( c_{\mu}(X - A^*) = X \) for every \( A \in \mathcal{H} \).

**Proof.** Since \( A \in \mathcal{H} \), by Lemma 1.18(a), \( A^* = X - M_\mu \) where \( M_\mu = \cup \{ M \mid M \in \mu \} \). Since \( M_\mu \) is the largest \( \mu \)– open subset of \( X \), it follows that \( i_{\mu}(A^*) = \emptyset \) and so \( c_{\mu}(X - A^*) = X \), since \( c_{\mu}(X - B) = X - i_{\mu}(B) \) for every subset \( B \) of \( X \) [39].

In Lemma 1.18(i), it is established that if \( \mu \) is a topology on \( X \), then every \( \mu \)– codense hereditary class is a strongly \( \mu \)– codense hereditary class. The following
Theorem 2.4 shows that the above result is true for a quasi topology $\mu$. Theorem 2.5 below give characterizations of strongly $\mu$-codense hereditary classes.

**Theorem 2.4.** If $(X,\mu)$ is a quasi topological space with a hereditary class $\mathcal{H}$, then the following statements are equivalent.

(a) $\mathcal{H}$ is $\mu$-codense.

(b) $\mathcal{H}$ is strongly $\mu$-codense.

**Proof.** $(a) \Rightarrow (b)$. Suppose $M,N \in \mu$ and $M \cap N \in \mathcal{H}$. By Lemma 1.15(a), it follows that $M \cap N \in \mu$ and so $M \cap N \in \mu \cap \mathcal{H}$. Since $\mathcal{H}$ is $\mu$-codense, $M \cap N = \emptyset$ and so $\mathcal{H}$ is strongly $\mu$-codense.

$(b) \Rightarrow (a)$ follows from the definition of strongly $\mu$-codense hereditary class.

**Theorem 2.5.** If $(X,\mu)$ is a quasi topological space with a hereditary class $\mathcal{H}$, then the following statements are equivalent.

(a) $\mathcal{H}$ is strongly $\mu$-codense.

(b) $M \subset M^*$ for every $M \in \mu$.

(c) $S \subset S^*$ for every $S \in \sigma$.

(d) $c_\mu(M) = M^*$ for every $M \in \mu$.

(e) $c_\mu(M) = M^*$ for every $M \in \sigma$.

(f) $i_\mu(A) \subset i_\mu(A^*)$ for every subset $A$ of $X$.

(g) $i_\sigma(A) \subset i_\sigma(A^*)$ for every subset $A$ of $X$.

(h) $c_\sigma(A) \subset A^*$ for every $A \in \sigma$.

**Proof.** (a) and (b) are equivalent by Lemma 1.18(b).
(b) ⇒ (c). Suppose $M \subseteq M^*$ for every $M \in \mu$. Let $S \in \sigma$. Then there exists a $\mu$-open set $M$ such that $M \subseteq S \subseteq c_\mu(M)$ [111]. Now $S \subseteq c_\mu(M) = M^*$, by Lemma 1.18(c) and so $S \subseteq M^* \subset S^*$, by Lemma 1.18(d), which proves (c).

(c) implies (b) follows from the fact that $\mu \subset \sigma$.

(c) and (e) are equivalent by Lemma 1.18(c).

(b) and (d) are equivalent by Lemma 1.18(c).

(c) ⇒ (g). For subset $A$ of $X$, $i_\sigma(A) \in \sigma$ and so $i_\sigma(A) \subset (i_\sigma(A))^* \subset A^*$ and so $i_\sigma(A) \subset i_\sigma(A^*)$, since $i_\sigma \in \Gamma_2$.

(g) ⇒ (c). If $A \in \sigma$, then $A = i_\sigma(A) \subset i_\sigma(A^*) \subset A^*$.

(b) ⇒ (f). For subset $A$ of $X$, $i_\mu(A) \in \mu$ and so $i_\mu(A) \subset (i_\mu(A))^* \subset A^*$ and so $i_\mu(A) \subset i_\mu(A^*)$.

(f) ⇒ (b). If $A \in \mu$, then $A = i_\mu(A) \subset i_\mu(A^*) \subset A^*$.

(h) follows from (e), since $\mu \subset \sigma$.

(h) ⇒ (c). Since $A \subset c_\sigma(A) \subset A^*$, (c) follows from (h).

In Lemma 1.18(j), it is established that if $\mu$ is a topological space on $X$ and $\mathcal{H}$ is a hereditary class, then $M \cap A^* \subset (M \cap A)^*$ for every $M \in \mu$ and $A \subseteq X$.

The following Theorem 2.6 shows that the above result is true in a quasi topological space.

**Theorem 2.6.** If $(X, \mu)$ is a quasi topological space with a hereditary class $\mathcal{H}$, then $M \cap A^* \subset (M \cap A)^*$ for every $M \in \mu$ and $A \subseteq X$.

**Proof.** Suppose $x \notin (M \cap A)^*$. Then there exists $N \in \mu$ containing $x$ such that
N \cap (M \cap A) \in \mathcal{H}. If \( x \notin M \), then \( x \notin M \cap A^* \). If \( x \in M \), then \( x \in M \cap N \) and \( M \cap N \in \mu \) by Lemma 1.15(a). Therefore, \( N \cap (M \cap A) \in \mathcal{H} \) implies that \( x \notin A^* \) and so \( x \notin M \cap A^* \). Hence \( M \cup A^* \in (M \cap A)^* \) for every \( M \in \mu \) and \( A \subset X \).

The following Lemma 2.7 shows that \( \mathcal{H}_\mu \) is an ideal in a quasi topological space. In Lemma 1.22, it is established that for the hereditary class \( \mathcal{H}_\mu \), \( A^* \subset c_\mu i_\mu c_\mu(A) \) for every subset \( A \) of \( X \). The following Theorem 2.8 shows that, in a quasi topological space, equality holds in the above relation and \( \mathcal{H}_\mu \) is strongly \( \mu \)-codense.

**Lemma 2.7.** If \((X, \mu)\) is a quasi topological space with a hereditary class \( \mathcal{H} \), then \( \mathcal{H}_\mu \) is an ideal.

**Proof.** If \( A \) and \( B \in \mathcal{H}_\mu \), then \( i_\mu(c_\mu(A \cup B)) = i_\mu(c_\mu(A) \cup c_\mu(B)) \), by Lemma 1.15(c) and so \( i_\mu(c_\mu(A \cup B)) \subset i_\mu(c_\mu(A)) \cup c_\mu(B) = \emptyset \cup c_\mu(B) \), by Lemma 1.19(b). Therefore, \( i_\mu(c_\mu(A \cup B)) \subset i_\mu(c_\mu(B)) = \emptyset \) and so \( A \cup B \in \mathcal{H}_\mu \). Hence \( \mathcal{H}_\mu \) is an ideal.

**Theorem 2.8.** If \((X, \mu)\) is a quasi topological space with a hereditary class \( \mathcal{H}_\mu \), then the following hold.

(a) \( A^* = c_\mu i_\mu c_\mu(A) \) for every subset \( A \) of \( X \).

(b) \( X = X^* \) and hence \( \mathcal{H}_\mu \) is strongly \( \mu \)-codense.

(c) \( \mu^* = \{ A \mid A \subset i_\mu c_\mu i_\mu(A) \} = \alpha_\mu \).

**Proof.** If \( A \subset X \), then \( A^* \subset c_\mu i_\mu c_\mu(A) \) by Lemma 1.22. Let \( x \notin A^* \). Then there exists a \( \mu \)-open set \( M \) containing \( x \) such that \( M \cap A \in \mathcal{H}_\mu \), and so \( i_\mu c_\mu(M \cap \)
A) = \emptyset. By Lemma 1.19(a), \( M \cap c_\mu(A) \subset c_\mu(M \cap A) \) and so \( i_\mu c_\mu(M \cap c_\mu(A) \subset i_\mu c_\mu(M \cap A) = \emptyset \). Therefore, \( i_\mu c_\mu(M \cap c_\mu(A)) = \emptyset \). If \( y \in M \cap i_\mu c_\mu(A) \), then \( y \in M \) and \( y \in i_\mu c_\mu(A) \). Therefore, there exists a \( \mu \)-open set \( V \) containing \( y \) such that \( V \subset c_\mu(A) \) and so \( y \in M \cap V \subset M \cap c_\mu(A) \subset c_\mu(M \cap c_\mu(A)) \) which implies that \( y \in i_\mu c_\mu(M \cap c_\mu(A)) \), a contradiction to the fact that \( i_\mu c_\mu(M \cap c_\mu(A)) = \emptyset \). Therefore, \( M \cap i_\mu c_\mu(A) = \emptyset \) which implies that \( x \notin c_\mu i_\mu c_\mu(A) \). Hence \( A^* = c_\mu i_\mu c_\mu(A) \) for every subset \( A \) of \( X \).

(b) Now \( X^* = c_\mu i_\mu c_\mu(X) = c_\mu i_\mu(X) \), since \( c_\mu(X) = X \). Since \( i_\mu(X) = M_\mu \), it follows that \( c_\mu i_\mu(X) = X \). Therefore \( X = X^* \). By Lemma 1.18(e), \( H_\mu \) is \( \mu \)-codense. By Theorem 2.4, \( H_\mu \) is strongly \( \mu \)-codense.

(c) \( \mu^* = \{ A \subset X \mid c^*(X - A) = X - A \} = \{ A \subset X \mid (X - A)^* \subset X - A \} = \{ A \subset X \mid c_\mu i_\mu c_\mu(X - A) \subset X - A \} = \{ A \subset X \mid A \subset i_\mu c_\mu i_\mu(A) \} = \alpha_\mu \).

Lemma 1.24 characterizes the subsets \( A \) of \( X \) satisfying the condition \( A \subset A^* \) in which the condition (c) is redundant by Lemma 1.23. The following Theorem 2.9 gives more characterizations of such sets in a quasi topological space.

**Theorem 2.9.** If \( (X, \mu) \) is a quasi topological space with a hereditary class \( H \), then the following following statements are equivalent.

(a) \( A \subset A^* \).

(b) \( A = D \cap A^* \) for some \( \mu \)-dense set \( D \).

(c) \( A = B \cap A^* \) for some \( \beta \)-open set \( B \).

(d) \( A = G \cap A^* \) for some \( b \)-open set \( G \).
Proof. (a) ⇒ (b). Let $D = A \cup (X - A^*)$. Then $D \cap A^* = (A \cup (X - A^*)) \cap A^* = A$. Also $c_\mu(D) = c_\mu(A \cup (X - A^*)) = c_\mu(A) \cup c_\mu(X - A^*)$, by Lemma 1.15(c) and so $c_\mu(D) = c_\mu(A) \cup (X - i_\mu(A^*)) = c_\mu(A) \cup (X - i_\mu c_\mu(A))$, by Lemma 1.18(c). Therefore, $c_\mu(D) \supset c_\mu(A) \cup (X - c_\mu(A)) = X$ and so $D$ is $\mu -$ dense.

(b) ⇒ (c). We prove that every $\mu -$ dense set is a $\beta -$ open set. If $D$ is $\mu -$ dense, then $c_\mu(i_\mu c_\mu(D)) = c_\mu(i_\mu c_\mu(X)) = c_\mu(M_\mu) = X$ and so $D$ is a $\beta -$ open set.

(c) implies (d) and (d) implies (a) are clear.

In Lemma 1.25, it is established that if $\mathcal{H}$ is strongly $\mu -$ codense, then $\delta \subset \delta_\mathcal{H}$. Since the concepts strongly $\mu -$ codense and $\mu -$ codense are equivalent in quasi topological spaces, by Theorem 2.4, we have the following Theorem 2.10. Theorem 2.11 below shows that the converse is true, if $\mathcal{H} \subset \mathcal{H}_\mu$. Also, it shows that Lemma 1.26 is true for quasi topological space.

**Theorem 2.10.** If $(X, \mu)$ is a quasi topological space with a hereditary class $\mathcal{H}$ and $\mathcal{H}$ is $\mu -$ codense, then $\delta \subset \delta_\mathcal{H}$.

**Theorem 2.11.** If $(X, \mu)$ is a quasi topological space with a hereditary class $\mathcal{H}$ and $\mathcal{H} \subset \mathcal{H}_\mu$, then $\delta_\mathcal{H} \subset \delta$.

**Proof.** Suppose $A \in \delta_\mathcal{H}$. Then $i_\mu c_\mu^*(A) \subset c_\mu^* i_\mu(A)$. Since $\mathcal{H} \subset \mathcal{H}_\mu$, and $\mathcal{H}_\mu$ is $\mu -$ codense by Theorem 2.8(b), it is easy to verify that $\mathcal{H}$ is a $\mu -$ codense. clearly, $A^*(\mathcal{H}_\mu) \subset A^*(\mathcal{H})$. $A^*(\mathcal{H}_\mu) \subset A^*(\mathcal{H}) \Rightarrow c_\mu i_\mu c_\mu(A) \subset A^*(\mathcal{H}) \Rightarrow i_\mu c_\mu i_\mu c_\mu(A) \subset i_\mu(A^*) \Rightarrow i_\mu c_\mu(A) \subset i_\mu(A^*) \subset i_\mu c_\mu(A) \subset c_\mu^* i_\mu(A)$. By Theorem 2.4, $\mathcal{H}$ is strongly $\mu -$ codense. By Theorem 2.5(d), $c_\mu^* i_\mu(A) = (i_\mu(A))^*$. By Theorem 2.5(b), $i_\mu(A) \subset
\[(i_\mu(A))^* \text{ and so } (i_\mu(A))^* = c_\mu(i_\mu(A)). \text{ Therefore } i_\mu c_\mu(A) \subset c_\mu i_\mu(A) \text{ and so } A \in \delta.\]

In Lemma 1.27, it is established that if \( \mu \) is a topology on \( X \), \( \mathcal{H} \) is a hereditary class and \( A \in \delta_\mathcal{H} \), then \( i_\mu(A^*) \subset (i_\mu(A))^* \). The following Theorem 2.12 shows that the above result is true in quasi topological spaces.

**Theorem 2.12.** If \( (X, \mu) \) is a quasi topological space with a hereditary class \( \mathcal{H} \) and \( A \in \delta_\mathcal{H} \), then \( i_\mu(A^*) \subset (i_\mu(A))^* \).

**Proof.** If \( A \in \delta_\mathcal{H} \), then \( i_\mu c_\mu^*(A) \subset c_\mu^* i_\mu(A) \). Now \( i_\mu(A^*) \subset i_\mu c_\mu^*(A) \subset c_\mu^* i_\mu(A) = c_\mu(i_\mu(A)) \cup (i_\mu(A))^* \). Suppose \( x \in i_\mu(A^*) \). Then \( x \in i_\mu(A) \cup (i_\mu(A))^* \). Since \( x \in i_\mu(A^*) \), there exists a \( \mu \)-open set \( G \) containing \( x \) such that \( G \subset A^* \). If \( x \notin (i_\mu(A))^* \), then there exists a \( \mu \)-open set \( H \) containing \( x \) such that \( H \cap i_\mu(A) \in \mathcal{H} \). Now \( H \cap i_\mu(A) = H \cap (i_\mu(A) \cap A) = (H \cap i_\mu(A)) \cap A \in \mathcal{H} \). By Lemma 1.18(f), \( H \cap i_\mu(A) \cap A^* = \emptyset \). Since \( G \subset A^* \), \( H \cap i_\mu(A) \cap G = \emptyset \) which implies that \( (H \cap G) \cap i_\mu(A) = \emptyset \) and so \( x \notin i_\mu(A) \) which is a contradiction to the fact \( x \in i_\mu(A) \cup (i_\mu(A))^* \). Therefore \( x \in (i_\mu(A))^* \) and so \( i_\mu(A^*) \subset (i_\mu(A))^* \).

The following Theorem 2.13 shows that the converse of Theorem 2.12 is true if \( \mathcal{H} \) is \( \mu \)-codense. Also, it shows that Lemma 1.28, is true for a quasi topological space.

**Theorem 2.13.** If \( (X, \mu) \) is a quasi topological space, \( \mathcal{H} \) is \( \mu \)-codense hereditary class and \( i_\mu(A^*) \subset (i_\mu(A))^* \), then \( A \in \delta_\mathcal{H} \).

**Proof.** Since \( A^* \) is \( \mu \)-closed, by Lemma 1.18(g), \( i_\mu c_\mu^*(A) = i_\mu(A \cup A^*) \subset \emptyset \). Therefore \( i_\mu(A^*) \subset (i_\mu(A))^* \) and so \( A \in \delta_\mathcal{H} \).
\( i_\mu(A) \cup A^* \), by Lemma 1.19(b). Since \( \mathcal{H} \) is \( \mu \)-codense, by Theorem 2.4 and 2.5(b), \( i_\mu(A) \subset (i_\mu(A))^* \). Now \( i_\mu c_\mu^*(A) = (i_\mu(A))^* \cup A^* \subset (i_\mu(A) \cup A)^* = A^* \) by Lemma 1.18(d) and so \( i_\mu c_\mu^*(A) \subset i_\mu(A^*) \subset (i_\mu(A))^* \subset i_\mu(A) \cup (i_\mu(A))^* = c_\mu^* i_\mu(A) \).

Therefore, \( A \in \delta_{\mathcal{H}} \).

**Corollary 2.14.** If \((X, \mu)\) is a quasi topological space, \( \mathcal{H} \) is a \( \mu \)-codense hereditary class of subsets of \( X \) and \( A \subset X \), then \( A \in \delta_{\mathcal{H}} \) if and only if \( i_\mu(A^*) \subset (i_\mu(A))^* \).

**Proof.** The proof follows from Theorems 2.12 and 2.13.