Chapter 2

Global Transversals sets and Global Transversal Irredundant sets

This chapter is devoted to a study of the new graph invariant namely global transversal number.

2.1 Introduction

In the study of transversals, usually the complement of the graph is not considered. A transversal with a particular property may not be a transversal in that property for the complement. This motivated us to study transversals which are clique transversals in $G$ and $\overline{G}$. Such transversals are called global transversals. The property of being a global transversal is a super-hereditary property. The condition for minimality led to the definition of global transver-
2.2 Definition and properties of Global Transversal sets

Definition 2.2.1 A clique in a graph is a maximal induced subgraph of the graph and a maximum clique is a clique of maximum cardinality.

Definition 2.2.2 A Transversal which meets all maximum cliques as well as maximum independent sets is called a global transversal set of $G$. The minimum cardinality of global transversal set of $G$ is called the global transversal number of $G$ and it is denoted by $\tau_g(G)$.

Note 2.2.3 $V(G)$ is always a global transversal set.

2.2.1 $\tau_g(G)$ for Standard Graphs

In the following, we obtain $\tau_g(G)$ for standard graphs.

1. $\tau_g(K_n) = n, \forall n.$
2. $\tau_g(K_n) = 1, \forall n.$
3. $\tau_g(C_n) = \lceil \frac{n}{2} \rceil + 1.$
4. $\tau_g(P_n) = \lceil \frac{n}{2} \rceil$.

5. $\tau_g(W_{n+1}) = \lceil \frac{n}{2} \rceil + 1$.

6. $\tau_g(D_{r,s}) = 3, r, s \geq 2$.

**Proposition 2.2.4** $\tau_g(K_{1,n}) = 2 \forall n \geq 2$.

**Proof** Let $V(K_{1,n}) = \{u, v_1, v_2, ..., v_n\}$ where $u$ is the centre. Then $\{u, v_1\}$ is a global transversal and clearly a single vertex cannot constitute a global transversal. Hence $\tau_g(K_{1,n}) = 2, \forall n \geq 2$.

**Proposition 2.2.5** $\tau_g(K_{m,n}) = \min\{m, n\} + 1$.

**Proof** Let $m, n \geq 2$. Since $K_{m,n}$ cannot contain any complete subgraph of order $\geq 3$, $\omega(K_{m,n}) = 2$. Any clique joins a vertex of one bipartite set with a vertex of another bipartite set. Maximum independent set of $K_{m,n}$ is unique if $m \neq n$ and both partite sets are maximum independent sets if $m = n$. Hence $\tau_g(K_{m,n}) = \min\{m, n\} + 1$.

**Illustration 2.2.6**

To illustrate this, consider the two graphs $K_{2,4}, K_{4,4}$ given in Figure 1, Figure 2.
\[ \tau_g(K_{2,4}) = \min\{2, 4\} + 1 = 2 + 1 = 3. \] 
\( \{u_1, u_2, v_1\} \) is a minimum global transversal set. Thus \( \tau_g(K_{2,4}) = 3. \)

\[ \tau_g(K_{4,4}) = \min\{4, 4\} + 1 = 4 + 1 = 5. \] 
\( \{u_1, u_2, u_3, u_4, v_1\} \) is a minimum global transversal set.
Thus $\tau_g(K_{4,4}) = 5$.

**Theorem 2.2.7** For any bipartite graph $G, \tau_g(G) = \alpha_0(G) + 1$.

**Proof**
Let $S$ be a $\alpha_0$-set of $G$. Then $V - S$ is a $\beta_0$-set of $G$. Let $u \in V - S.S \cup \{u\}$ meets all $\beta_0$ sets as well as all cliques of $G$. Therefore $\tau_g(G) \leq \alpha_0(G) + 1$. Let $S_1$ be a $\tau_g$-set of $G$. Suppose $|S_1| < \alpha_0(G) + 1$. If $V - S_1$ is independent, then $|V - S_1| > n - (\alpha_0(G) + 1) = \beta_0(G) + 1$, a contradiction. Therefore $V - S_1$ is not independent. Therefore $V - S_1$ contains a $K_2$. Since $G$ is bipartite, $\omega(G) = 2$. Therefore $S_1$ does not meet a maximum clique in $V - S_1$, a contradiction. Therefore $\tau_g(G) = \alpha_0(G) + 1$.

**Corollary 2.2.8** For any Tree $T$, $\tau_g(T) = \alpha_0(T) + 1$.

**Remark 2.2.9**

The above property is not true for graphs which are not bipartite. To illustrate this consider the following graph $G$ in figure 3.
Figure 3

\{1,2\},\{1,4\} are \(\tau_g\)-sets of \(G\). \(\alpha_0(G) = 2\). Therefore \(\tau_g(G) < \alpha_0(G) + 1\).

**Remark 2.2.10** There are graphs which are not bipartite in which \(\tau_g(G) = \alpha_0(G) + 1\) holds.

Consider \(C_n\), \(n\) odd. \(\tau_g(C_n) = \lceil \frac{n}{2} \rceil + 1\),
\(\alpha_0(C_n) = \lceil \frac{n}{2} \rceil\),
\(\tau_g(C_n) = \alpha_0(C_n) + 1\).

**Remark 2.2.11** For any graph \(G\), \(\tau_g(G) \leq \alpha_0(G) + 1\). (Since any vertex cover \(C\) together with a vertex from \(V - C\) is a global transversal of \(G\)).

**Observation 2.2.12** If \(G\) is any graph, \(\tau_g(G.K_1) = |V(G)|+1 = \alpha_0(G.K_1) + 1\).
Observation 2.2.13

Given a graph $G$, there are induced subgraphs $S$ whose $\tau_g$ may be equal to greater or less than that of $G$.

For:

Let $G$:

The $\beta_0$ sets are $\{v_1, v_6, v_3, v_5\}, \{v_1, v_7, v_3, v_5\}$. The only clique is $\{v_2, v_6, v_7\}$. $\{v_1, v_2\}$ is a $\tau_g$ set of $G$. Therefore $\tau_g(G) = 2$.

Let $H$:

$\tau_g(H) = 3 > \tau_g(G)$,

Let $H_1$:
τ_g(H_1) = 1 < τ_g(G).

Let H_2

\[ \tau_g(H_2) = 2 = \tau_g(G). \]

**Observation 2.2.14** Let T be a global transversal set of G. Then any super set of T is also global transversal set of G.

**Definition 2.2.15** T is called a minimal global transversal set of G if no proper subset of T is a global transversal set of G. Since global transversal property is super hereditary, a global transversal set is minimal if and only if it is 1-minimal. The minimum cardinality of a minimal global transversal
set of $G$ is the global transversal number of $G$ and is denoted by $\tau_g(G)$ and the maximum cardinality of a minimal global transversal set of $G$ is called the upper global transversal number of $G$ and is denoted by $\tau_G(G)$.

**Example 2.2.16**

Consider the graph $G$ given in figure 4.

The maximum independent sets of the graph $G$ are

\{
v_1, v_8, v_{10}, v_4, v_6\}, \{v_1, v_8, v_{10}, v_5, v_7\}, \{v_1, v_8, v_{10}, v_4, v_7\}, \{v_1, v_8, v_{11}, v_4, v_6\}, \{v_1, v_9, v_{10}, v_4, v_7\}, \\
\{v_1, v_9, v_{11}, v_4, v_7\}, \{v_1, v_9, v_{11}, v_4, v_6\}, \{v_1, v_9, v_{11}, v_5, v_7\}.

Clique of $G$ are $\{v_2, v_8, v_9\}$, and $\{v_{11}, v_{10}\}$.
The minimal global transversal sets of $G$ are
\[
\{v_1, v_2, v_3\}, \{v_2, v_3, v_4, v_5\}, \{v_2, v_3, v_4, v_6\}, \\
\{v_2, v_3, v_5, v_7\}, \{v_2, v_3, v_4, v_6\}, \\
\{v_2, v_3, v_4, v_6\}, \{v_2, v_3, v_4, v_6\}, \\
\{v_2, v_3, v_4, v_6\}, \{v_2, v_3, v_4, v_6\}, \\
\{v_2, v_3, v_4, v_6\}, \{v_2, v_3, v_4, v_6\},
\]
Therefore $\tau_g(G) = 3$, $\tau_G(G) = 4$.

**Example 2.2.17**

![Graph Image]

The unique maximum independent set of the above given graph is $\{v_1, v_2, v_3, v_6, v_7, v_8, v_9\}$.

Clique are $\{v_4, v_1\}, \{v_4, v_2\}, \{v_4, v_3\}, \{v_4, v_5\}, \{v_5, v_9\}, \{v_5, v_7\}, \{v_5, v_8\}, \{v_5, v_9\}$.

The minimal global transversal sets are $\{v_1, v_4, v_5\}, \{v_6, v_7, v_8, v_9, v_4\}$ and $\{v_1, v_2, v_3, v_5\}$.

Therefore $\tau_g(G) = 3$, $\tau_G(G) = 5$.

**Remark 2.2.18** Given any positive integer $k$, there exists a graph $G$ such
that $\tau_G(G) - \tau_g(G) = k$. 

Proof Consider $D_{r,s}$ where $s = k + 2$ and $r \leq s, \tau_g(D_{r,s}) = 3$ and $\tau_G(D_{r,s}) = k + 3$. Therefore $\tau_G(D_{r,s}) - \tau_g(D_{r,s}) = k$.

Theorem 2.2.19 Let $T$ be a global transversal set of $G$. Then $T$ is minimal if and only if for every $u \in T$, either there exists a maximum independent set $S$ such that $S \cap T = \{u\}$ or there exists a maximum clique $C$ such that $C \cap T = \{u\}$.

Proof Obvious. \hfill \blacksquare

Theorem 2.2.20 Let $G$ and $\overline{G}$ have no isolates. Then any global transversal set $T$ is a dominating set for $G$ as well as $\overline{G}$. That is, $T$ is a global dominating set of $G$.

Proof
Let $T$ be a global transversal set of $G$. Let $u \in V - T$. Then $u$ is a vertex belonging to a clique say $C$ of $G$ (Since $u$ is not an isolate). Therefore $T \cap C \neq \emptyset$. Let $v \in T \cap C$. Since $v \in C$, $u$ and $v$ are adjacent. Therefore $T$ is a dominating set of $G$. Since $T$ is a clique transversal set for $\overline{G}$, $T$ is also a dominating set of $\overline{G}$. Therefore $T$ is a global dominating set of $G$. \hfill \blacksquare
Corollary 2.2.21 \( \gamma_g(G) \leq \tau_G(G) \)

2.3 Private maximum independant set neighbour and private maximum clique neighbour

Definition 2.3.1 Let \( u \in V(G) \)

(i) The maximum independent set neighbourhood of \( u \) in \( G \) is defined as \( \{ S \subseteq V(G) : S \text{ is a maximum independent set of } G \text{ and } u \in S \} \). This is denoted by \( N_{\beta}(u) \).

(ii) The maximum clique neighbourhood of \( u \) in \( G \) is defined as \( \{ C \subseteq V(G) : C \text{ is a maximum clique and } u \in C \} \). This is denoted by \( N_c(u) \).

Definition 2.3.2 Let \( T \) be a global transversal set of \( G \). Let \( u \in T \). Let \( S \) be a maximum independant set of \( G \). \( S \) is said to be a private maximum independent neighbour of \( u \) with respect to \( T \) if \( S \in N_{\beta}(u) \) and \( S \notin N_{\beta}(v) \) for any \( v \neq u, v \in T \). The set of all private maximum independent set neighbours of \( u \) is denoted by \( pn_{\beta}(u, T) \).

Definition 2.3.3 Let \( T \) be a global transversal set of \( G \). Let \( u \in T \). Let \( C \) be a maximum clique of \( G \). \( C \) is said to be a private maximum clique neighbour
of $u$ with respect to $T$ if $C \in N_c(u)$ and $C \notin N_c(v)$ for all $v \neq u$, $v \in T$. The set of all private maximum clique neighbours of $u$ is denoted by $p\text{nc}(u, T)$

**Remark 2.3.4** Let $T$ be a global transversal set of $G$. $T$ is minimal if and only if for any $u \in T$, $p\text{b}0(u, T) \neq \phi$ or $p\text{nc}(u, T) \neq \phi$.

### 2.4 Definition and properties of global transversal irredundant sets

**Definition 2.4.1** Let $T$ be a non empty subset of $V(G)$ satisfying the following:

For all $u \in T$, $p\text{b}0(u, T) \neq \emptyset$ or $p\text{nc}(u, T) \neq \emptyset$. Then $T$ is called a global transversal irredundant set (in short g.t irredundant set) of $G$.

**Observation 2.4.2** If $T$ is a g.t irredundant set of $G$, then any subset of $T$ is also a g.t irredundant set of $G$. That is, g.t irredundance is a hereditary property.

**Proof**

Let $T$ be a g.t irredundant set of $G$. Let $S \subseteq T$. Let $u \in S$. Therefore $u \in T$. Therefore $p\text{b}0(u, T) \neq \phi$ or $p\text{nc}(u, T) \neq \phi$. That is there exists a maximum
independent set $A$ of $G$ such that $A \cap T = \{u\}$ or there exists a maximum clique $C$ of $G$ such that $T \cap C = \{u\}$. Therefore $A \cap S = \{u\}$ or $C \cap S = \{u\}$. Therefore $pn_{g,t}(u, S) \neq \phi$ or $pn_c(u, S) \neq \phi$. Therefore $S$ is a $g.t$ irredundant set of $G$.

**Definition 2.4.3** The minimum (maximum) cardinality of a maximal $g.t$ irredundant set of a graph $G$ is called (upper) $g.t$ irredundance number of $G$ and is denoted by $ir_{g,t}(G)(IR_{g,t}(G))$.

**Example 2.4.4**

(i) $ir_{g,t}(K_n) = IR_{g,t}(K_n) = n$.

(ii) $ir_{g,t}(P_{2n}) = n$, $IR_{g,t}(P_{2n}) = n + 1$.

(iii) $ir_{g,t}(P_{2n+1}) = n$, $IR_{g,t}(P_{2n+1}) = n + 1$, $n \geq 1$.

**Illustration 2.4.5**
The maximum independent sets of $C_6$ are $\{v_1, v_3, v_5\}, \{v_2, v_4, v_6\}$.

Cliques of $C_6$ are , $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_1\}$.

$\{v_1, v_2, v_4, v_5\}, \{v_1, v_2, v_3, v_5\}, \{v_2, v_3, v_4, v_6\}, \{v_1, v_3, v_4, v_5\}$ are maximal g.t irredundant sets. $\{v_2, v_4, v_5, v_6\}, \{v_5, v_6, v_1, v_3\}, \{v_6, v_1, v_2, v_4\}$ are also maximal g.t irredundant sets.

For: Consider $S = \{v_1, v_2, v_4, v_5\}$. Clearly $S$ is a g.t irredundant set of $C_6$. Consider $\{v_1, v_2, v_3, v_4, v_5\}$. $v_2$ belongs to $\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4, v_6\}$ and none of them is a private neighbour of $v_2$. Therefore $\{v_1, v_2, v_3, v_4, v_5\}$ is not a g.t irredundant set. Consider $\{v_1, v_2, v_4, v_5\}$. $v_5$ belongs to $\{v_4, v_5\}, \{v_5, v_6\}, \{v_1, v_3, v_5\}$ and none of them is a private neighbour of $v_5$. Therefore $\{v_1, v_2, v_4, v_5\}$ is not a g.t irredundant set of $C_6$. $\{v_1, v_2, v_4, v_5\}$ is a maximal g.t irredundant set of $C_6$. Consider 3 element set of $C_6$. There are twenty,three-element sets.

They are $\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_3, v_4, v_5\}, \{v_4, v_5, v_6\}, \{v_5, v_6, v_1\}$,

$\{v_6, v_1, v_2\}, \{v_1, v_2, v_4\}, \{v_2, v_3, v_5\}, \{v_3, v_4, v_6\}, \{v_4, v_5, v_1\}, \{v_5, v_6, v_2\}$,

$\{v_6, v_1, v_3\}, \{v_1, v_2, v_5\}, \{v_2, v_3, v_6\}, \{v_3, v_4, v_1\}, \{v_4, v_5, v_2\}, \{v_5, v_6, v_3\}$,

$\{v_6, v_1, v_4\}, \{v_1, v_3, v_5\}$ and $\{v_2, v_4, v_6\}$. None of them is maximal.Any 5 element subset of $V(C_6)$ is not a g.t irredundant set. Therefore $ir_{g,t}(C_6) = IR_{g,t}(C_6) = 4$.

**Theorem 2.4.6** $ir_{g,t}(C_n) = IR_{g,t}(C_n) = \left\lceil \frac{n}{2} \right\rceil + 1$. 

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Proof

Case(i): $^\prime n^\prime = 2n$

Let $V(C_{2n})=\{u_1, u_2, ..., u_{2n}\}$. Let $S=\{u_1, u_2, u_3, u_5, u_7, ..., u_{2n}\}$. Clearly $S$ is a maximal $g.t$ irredundant set of $C_{2n}$ of cardinality $n + 1$. Let $T$ be any subset of $V(C_{2n})$ of cardinality $n + 2$. Then either $T$ contains five consecutive terms of $V(C_{2n})$ or three consecutive terms occurring in two different places. In either case $T$ is not a $g.t$ irredundant set of $C_{2n}$. Let $T$ be any $g.t$ irredundant subset of $V(C_{2n})$ of cardinality $n$. Then either $T$ contains all odd suffixed terms or all even suffixed terms or three consecutive terms and other $n - 3$ terms having suffixes with the opposite parity of the suffix of the middle term. In any case $T$ is contained properly in a $g.t$ irredundant set of $C_{2n}$. Also any $g.t$ irredundant subset of $V(C_{2n})$ of cardinality less than or equal to $n - 1$ can be proved to be contained in a maximal $g.t$ irredundant set of $C_{2n}$. Hence $ir_{g.t}(C_{2n}) = n + 1$.

Also $IR_{g.t}(C_n) = n + 1$.

Case(ii): $^\prime n^\prime = 2n + 1$

Let $V(C_{2n+1})=\{u_1, u_2, ..., u_{2n+1}\}$. Let $S=\{u_1, u_2, u_3, u_4, u_6, u_8, ..., u_{2n}\}$. Clearly $S$ is a maximal $g.t$ irredundant set of $C_{2n+1}$ of cardinality $n + 2$. Let $T$ be any subset of $V(C_{2n+1})$ of cardinality $n + 3$. Then either $T$ contains six consecutive terms of $V(C_{2n+1})$ or three consecutive terms occurring in three
different places. In either case \( T \) is not a \( g.t \) irredundant set of \( C_{2n+1} \). Let \( T \) be any \( g.t \) irredundant subset of \( V(C_{2n+1}) \) of cardinality \( n+1 \). Then either \( T \) contains all odd suffixed terms with exactly one even suffixed term or all even suffixed terms with exactly one odd suffixed term or three consecutive terms and other \( n - 3 \) terms having suffixes with the opposite parity of the suffix of the middle term. In any case \( T \) is contained properly in a \( g.t \) irredundant set of \( C_{2n+1} \). Also any \( g.t \) irredundant subset of \( V(C_{2n+1}) \) of cardinality less than or equal to \( n \) can be proved to be contained in a maximal set of \( C_{2n+1} \).

Hence \( \text{ir}_g.t(C_{2n+1}) = \text{IR}_g.t(C_{2n+1}) = n + 2 = \lceil \frac{2n+1}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil + 1. \) 

\[ \square \]

**Theorem 2.4.7** Let \( G \) be a simple graph. Any minimal global transversal set of \( G \) is a maximal \( g.t \) irredundant set of \( G \).

**Proof**

Let \( S \) be a minimal global transversal set of \( G \). Then \( S \) is a \( g.t \) irredundant set of \( G \). Suppose \( S \) is not a maximal \( g.t \) irredundant set of \( G \), then \( S \) is contained in a maximal \( g.t \) irredundant set of \( G \) say \( T \). Let \( u \in T - S \). Then \( S \cup \{u\} \) is a \( g.t \) irredundant set of \( G \). Therefore \( pn_c\{u, S \cup \{u\}\} \neq \phi \) or \( pn_{\beta_b}\{u, S \cup \{u\}\} \neq \phi \). Therefore there exists a maximum clique \( C \) such that \( u \in C \) and no vertex of \( S \) belongs to \( C \) or there exists a maximum independant set \( I \) such that \( u \in I \) and no vertex of \( S \) belongs to \( I \). Therefore
$S \cap C = \phi$ or $S \cap I = \phi$, a contradiction, since $S$ is a global transversal set of $G$. Therefore $S$ is a g.t irredundant set of $G$.

\textbf{Remark 2.4.8} \quad ir_{g.t}(G) \leq \tau_g(G) \leq \tau_G(G) \leq IR_{g.t}(G)

\textbf{Illustration 2.4.9}

In the above graph $G$, the only maximum clique is $\{v_3, v_4, v_6\}$ and the only maximum independent set is $\{v_1, v_3, v_5, v_7\}$. $\{v_3\}$ is a minimal global transversal set of minimum cardinality. Therefore $\tau_g(G) = 1$.

Since $ir_{g.t}(G) \leq t_g(G)$, $ir_{g.t}(G) = 1$. $\{v_4, v_5\}$ is a minimal global transversal set of maximum cardinality, $\tau_G(G) = 2$. $\{v_4, v_5\}$ is also a maximal g.t irredundant set of maximum cardinality. Therefore $IR_{g.t}(G) = 2$.

\textbf{Illustration 2.4.10}
In the above graph $G$, each edge is a clique and $\{v_1, v_3, v_5, v_7, v_9, v_{11}\}$ is the unique maximum independent set. Any global transversal set must intersect each edge and hence contains at least five points. If it contains exactly five points, then the transversal contains $v_2, v_4, v_6, v_8$ and $v_{10}$. The transversal also intersects the independent set and none of the points $v_2, v_4, v_6, v_8$ and $v_{10}$ belongs to the unique independent set. Therefore any global transversal set contains at least six vertices. $\{v_1, v_2, v_4, v_6, v_8, v_{10}\}$ is a global transversal set of cardinality 6. Therefore $\tau_G(G) = 6$. $\{v_1, v_3, v_5, v_7, v_9, v_{11}\}$ is also a global transversal set. It can be seen that $\{v_2, v_3, v_4, v_8, v_9, v_{10}\}$ is a maximal g.t irredundant set of $G$. It can be verified that $ir_{g.t}(G) = 6$.

Illustration 2.4.11
In the above graph $G$, $\{v_3, v_4, v_5\}$ is the unique maximum clique. $\{v_1, v_3, v_6\}$, $\{v_1, v_4, v_6\}$, $\{v_2, v_4, v_6\}$, $\{v_2, v_4, v_7\}$, $\{v_1, v_5, v_7\}$, $\{v_1, v_4, v_7\}$, $\{v_1, v_3, v_7\}$ are the maximum independent sets. Here $ir_{g,t}(G) = \tau_g(G) = \tau_G(G) = IR_{g,t}(G) = 3$

**Remark 2.4.12**

(i) $ir_{g,t}(K_{1,n}) = 2, IR_{g,t}(K_{1,n}) = n, \forall n$.

(ii) $ir_{g,t}(K_{m,n}) = \begin{cases} \min\{m, n\} + 1 & \text{if } m \neq n. \\ m + 1 & \text{if } m = n \end{cases}$

$IR_{g,t}(K_{m,n}) = \begin{cases} \max\{m, n\} & \text{if } m \neq n. \\ m + 1 & \text{if } m = n \end{cases}$