CHAPTER 5
MULTI-OBJECTIVE BI-MATRIX GAME USING
EXTREME MEASURES

5.1 Introduction

In this Chapter, fuzzy matrix games formulated by Sakawa and Nishizaki [41] is considered. They incorporate fuzzy goals in single and multiple objective environment and they discuss equilibrium solution in multi-objective bi-matrix games using fuzzy goals and fuzzy payoffs, they formulated maximin problem with respect to degree of attainment of the fuzzy goal using relaxation methods. Bi-matrix game using the relaxation method and maximin solution is formulated as a non-linear programming problem, and this problem is further reduced to linear programming problem by making use of Sakawa method. The variable transformation has done and it is proposed by Charnes and Copper [10].

5.1 Multi-objective bi-matrix game with fuzzy pay-offs

The multi-objective bi-matrix game with fuzzy pay-offs is defined as \((\tilde{A}_1, \tilde{A}_2)\) where

\[
\tilde{A}_1 = (\tilde{A}_1, ..., \tilde{A}_k) \quad k = 1, ..., r \quad \text{and} \quad \tilde{A}_2 = (\tilde{A}_1, ..., \tilde{A}_l) \quad l = 1, ..., s.
\]

Fuzzy multiple pay-offs of players I & II are \((a_{i1}, ..., a_{ir})\) & \((a_{i1}, ..., a_{is})\) respectively.

5.1.1 Equilibrium Solution for bi-matrix game

For bi-matrix game \((\tilde{A}_1, \tilde{A}_2)\) a Nash equilibrium solution is a pair of strategies m-dimensional column vector \(x^*\) and n dimensional row vector \(y^*\)

\[
\begin{align*}
& \left\{ x^T A_i y^* \geq x^T A_j y^* \quad \forall \ x \in S^n \text{ and} \right. \\
& \left. \text{if} \quad x^T A_2 y^* \geq x^T A_3 y \quad \forall \ y \in S^l. \right.
\end{align*}
\]
Let $\mu_{a_{ij}}$ be a membership function of the fuzzy number $a_{ij}$, $k$th fuzzy expected pay-off

$E_k (x, y)$ of player I is represented by

$$\mu_{a_{ij}} (p) = \sup_{p=x^Tpy} \min_{a_{ij}} \mu_{a_{ij}} (p_{ij}) \text{ where } P = [P_{ij}] \text{ be an m by n matrix.}$$

We assume that fuzzy pay-offs $a_{ij}$ are L-R fuzzy numbers

$$(a_{ij}, a_{ij}, a_{ij})_{L_{ij}^{kl} R_{ij}^{kl}} \& (a_{ij}, a_{ij}, a_{ij})_{L_{ij}^{kl} R_{ij}^{kl}}$$

Membership functions of which are

$$\mu_{a_{ij}} (p) = \begin{cases} 
L_{a_{ij}}^{kl} & \left( \frac{a_{ij}^1 - p}{a_{ij}^1} \right) & \text{if } p \leq a_{ij}^1, a_{ij} > 0 \\
R_{a_{ij}}^{kl} & \left( \frac{-a_{ij}^1 + p}{a_{ij}^1} \right) & \text{if } p \geq a_{ij}^1, a_{ij} > 0 
\end{cases}$$

$$\mu_{a_{ij}} (p) = \begin{cases} 
L_{a_{ij}}^{kl} & \left( \frac{a_{ij}^2 - p}{a_{ij}^2} \right) & \text{if } p \leq a_{ij}^2, a_{ij} > 0 \\
R_{a_{ij}}^{kl} & \left( \frac{-a_{ij}^2 + p}{a_{ij}^2} \right) & \text{if } p \geq a_{ij}^2, a_{ij} > 0 
\end{cases}$$

A reference function $L_{a_{ij}}^{kl}$ of the fuzzy number $a_{ij}$ is strictly decreasing on $(0, \infty)$ using parameter $(a_{ij}, a_{ij}, a_{ij})_{L_{ij}^{kl} R_{ij}^{kl}}$ of the fuzzy pay-off $a_{ij}$, the expected fuzzy pay-off can be represented as $(x^T A^k y, x^T \hat{A} k y, x^T \hat{A}^k y) L_{ij}^{kl} R_{ij}^{kl}$ where $\hat{A}^k$ and $\hat{A}^k$ are m x n matrices the ij element of which are $\hat{a}_{ij}$ & $\hat{a}_{ij}$. The fuzzy expected pay-off of player II is represented in the same manner.
5.2 An Equilibrium Solution in single objective incorporating fuzzy goals:

Each of the pay-offs represents an objective or an attribute and it has a different unit of measure in multi-objective games. By incorporating fuzzy goals for an objective and the equilibrium solution to the problems is considered, in terms of maximization of degree of attainment for the aggregated fuzzy goal. Each of the measures for objectives can be transformed to the degree of attainment of the fuzzy goal as a commensurable measure.

For the sake of simplicity, the superscript 1 representing the number of objectives is omitted. Let player-I fuzzy goal $\tilde{G}_1$ for the pay-off be a fuzzy set on the set of real number $\mathbb{R}$ characterized by the membership function $\mu_{G_1} : \mathbb{R} \rightarrow [0,1]$. The following player-I specifies the finite value $a_1$ of the payoff for which the degree of satisfaction is 0 and the finite value $\tilde{a}_1$, it is define that $\mu_{G_1}(p) = 0$ For the value $p$ undesired than $\tilde{a}_1$, it is defined that $\mu_{G_1}(p) = 1$ and for $a_1 \leq p \leq \tilde{a}_1$, $\mu_{G_1}(p)$ is continuous and strictly increasing. The fuzzy goal $\tilde{G}_2$ of player II is a similar fuzzy set which is characterized by the membership function $\mu_{G_2}(p)$. The membership function value of the fuzzy goal can be interpreted as a degree of attainment of the fuzzy goal. Then we assume that, for any pair of pay-offs, a player prefers the payoff having the larger degree of attainment of the fuzzy goal. We define the degree of attainment of the fuzzy goal by applying the concept of the fuzzy decision by Bellman and Zadeh [3]. For any pair of mixed strategies $(x, y)$ the fuzzy expected payoff for player-I be denoted by $\tilde{E}_i(x, y)$ and let the fuzzy goal for player I be denoted by $\tilde{G}_i$. The degree of attainment of the fuzzy goal is defined as the maximum of the intersection of the fuzzy expected payoff $\tilde{E}_i(x, y)$ and the fuzzy goal $\tilde{G}_i$. 
The degree of attainment of the fuzzy goal for player II can be defined in a similar way. We now consider equilibrium solution with respect to the degree of attainment of the fuzzy goal. A pair of strategies $x^*$ and $y^*$ is said to be an equilibrium solution with respect to the degree of attainment of the fuzzy goal if for any other mixed strategies $x$ and $y$,

$$d_1(x^*, y^*) \geq d_1(x, y^*) \quad \text{and} \quad d_2(x^*, y^*) \geq d_2(x^*, y).$$

Let the membership functions $\mu_{G_i}(p)$, $i=1, 2$ and $\mu_{a_i}$, $\mu_{b_j}$, $i=1, 2, \ldots, m$, $j=1, 2, \ldots, n$ are determined such that $d_i(x, y)$ where $i = 1, 2$ for every $x$ in $X$ and $y$ in $Y$ are continuous.

**Theorem: 5.2.1**

For a bi-matrix game $(A_1, A_2)$ with fuzzy payoffs let $d_i(x, y)$, $i=1, 2$ be continuous. Then there exist equilibrium solution with respect to the degree of attainment of the fuzzy goal.

**Proof:**

Let $x$ and $y$ be any pair of mixed strategies for the game $(A_1, A_2)$ we define
\[ c_i(x^\hat{i}) = \max \left\{ d_i(x^\hat{i}, y) - d_i(x, y), 0 \right\} \]

\[ e_j(y^\hat{j}) = \max \left\{ d_j(x, y^\hat{j}) - d_j(x, y), 0 \right\} \]

For some \( x^\hat{i} \) in \( X \), \( y^\hat{j} \) in \( Y \) and \( x' = (x^\hat{i}_1, x^\hat{i}_2, ..., x^\hat{i}_m)^T \)

\[ x'_i = \frac{x_i + c_i(x^\hat{i})}{1 + \sum_{k=1}^m c_k(x^\hat{k})} \]

\[ y'_j = \frac{y_j + e_j(y^\hat{j})}{1 + \sum_{k=1}^m e_k(y^\hat{k})} \]

Because \( \sum_{i=1}^m x'_i = 1 \& x'_i \geq 0 \ i=1,2,...m \). \( x' \) is mixed strategy of player-I and \( y' \) is also a mixed strategy of player-II. Because \( d_i(x, y) \), \( i=1,2 \) are continuous form the assumption a function \( T(x, y) = (x', y') \) is continuous.

We know that \( (x', y') = (x, y) \) if and only if is an equilibrium solution. If \( (x, y) \) is an equilibrium solution it is clear that for any \( x^\hat{i} \), \( d_i(x^\hat{i}, y) = d_i(x, y) \), \( i=1,2,...m \) and so \( c(x^\hat{i}) = 0 \) for all \( i \) hence \( x' = x \) and \( y' = y \).

Suppose on the other hand \( (x, y) \) is not an equilibrium solution. This means either that we can take some \( i \) such that \( d_i(x^\hat{i}, y) > d_i(x, y) \), or we can take some \( j \) such that \( d_2(x, y^\hat{j}) > d_2(x, y) \) we assume that first case holds. Hence for the \( i \), \( c(x^\hat{i}) > 0 \) therefore \( \sum_{i=1}^m c_k(x^\hat{k}) > 0 \) for any \( i \) except \( i \) we can set \( x^\hat{i} \) such that \( d_i(x^\hat{i}, y) \leq d_i(x, y) \) then for such \( i \) we have \( c_i(x^\hat{i}) = 0 \) and so
\[
\dot{x}_i = \frac{x_i + \theta}{1 + \sum_{k=1}^{m} c_k(x_k)} < x_i \text{ thus } \dot{x}_i \neq x_i \text{ similarly } y_j \neq y_j.
\]

Therefore \((x, y)\) is an equilibrium solution, when \((x, y)\) is mapped to \((x', y')\) by the function \(T\). Because \(X\) and \(Y\) is a compact and convex set, the function \(T (x, y) = (x', y')\) has the fixed point from Brouwer fixed point theorem and fixed point becomes an equilibrium solution.

The relation between the equilibrium solution and optimal solution to certain mathematical programming problem is examined. If all membership functions of fuzzy goals and reference functions of fuzzy numbers expressing fuzzy payoffs are linear, then, it should be noted that the functions \(d_i (x, y) i = 1, 2\) are continuous.

A linear function membership function of player I’s of fuzzy goal is represented by

\[
\mu_G(p) = \begin{cases} 
0 & \text{if } p \leq \underline{a} \\
\frac{(p-a)}{(\overline{a} - a)} & \text{if } \underline{a} \leq p \leq \overline{a} \\
1 & \text{if } a \leq p
\end{cases}
\]

Where \(\underline{a}\) and \(\overline{a}\) are the payoffs giving the worst and best degree of satisfaction of player-I and II respectively. Player-I is not satisfied by a payoff less than \(\underline{a}\) but fully satisfied by a payoff greater than \(\overline{a}\).

Let a reference function for fuzzy numbers be \(L(p) = R(p) = \max(0, 1-|p|)\) when player I and II choose pure strategies \(i\) in \(I\) and \(j\) in \(J\) respectively, payoff of player I is represented as the fuzzy number \(\tilde{a}_{ij}\) characterized by the membership function.
We assume that fuzzy goal is assessed such that for all \( x \) in \( X \) and \( y \) in \( Y \) the conditions \( 0 < d_1(x, y) < 1 \) and \( 0 < d_2(x, y) < 1 \) hold. Let the membership functions of the fuzzy goals and payoffs of player II be similar linear functions. For a pair of strategies \( x \) and \( y \) of players I and II, Player I’s degree of attainment of the fuzzy goal can be represented by

\[
\mu_{a_{ij}}(p) = \begin{cases} 
0 & \text{if } p < a_{ij} - a_{ij} \\
\frac{p - a_{ij} + a_{ij}}{a_{ij}} & \text{if } a_{ij} - a_{ij} \leq p \leq a_{ij} \\
\frac{a_{ij} + a_{ij} - p}{a_{ij}} & \text{if } a_{ij} \leq p \leq a_{ij} + a_{ij} \\
0 & \text{if } a_{ij} + a_{ij} < p
\end{cases}
\]

And player II can be represented similarly. Therefore an equilibrium solution with respect to the degree of attainment of the fuzzy goal is a pair of strategies \( x^* \) and \( y^* \) if, for any other strategies \( x \) and \( y \)

\[
\frac{x^T(A_1 + A_1)y - a}{a - a + x^T A_1 y^*} \geq \frac{x^T(A_1 + A_1)y^* - a}{a - a + x^T A_1 y^*}
\]

\[
\frac{x^T(A_2 + A_2)y - b}{b - b + x^T A_2 y^*} \geq \frac{x^T(A_2 + A_2)y^* - b}{b - b + x^T A_2 y^*}
\]

Optimal solution \( x^* \) and \( y^* \) to the following two mathematical programming problems are strategies \( x^* \) and \( y^* \) satisfying the above conditions
\[
\frac{x^T (A_1 + A_1) y^* - a}{a - a + x^T A_1 y^*} = \text{Maximize} \frac{x^T (A_1 + A_1) y^* - a}{a - a + x^T A_1 y^*}
\]
subject to \(e^m x - 1 = 0\) \(x \geq 0^m\)

\[
\frac{x^T (A_2 + A_2) y^* - b}{b - b + x^T A_2 y^*} = \text{Maximize} \frac{x^T (A_2 + A_2) y^* - b}{b - b + x^T A_2 y^*}
\]
subject to \(e^n y - 1 = 0\) \(y \geq 0^n\)

Where \(e^m\) and \(e^n\) are \(m\)-dimensional and \(n\)-dimensional column vectors every entry of which is a unit and \(0^m\) and \(0^n\) are \(m\)-dimensional and \(n\)-dimensional column vectors entry of which is a zero.

By applying the necessary condition of Kuhn-tucker to the above problems of maximization the following necessary conditions for \(x\) and \(y\) to be an equilibrium solution with respect to the degree of attainment of the fuzzy goal can be obtained

\[
\tilde{a} x^T (A_1 + A_1) y - a x^T A_1 y - (a - a + x^T A_1 y)^2 \psi = 0
\]
\[
b x^T (A_2 + A_2) y - b x^T A_2 y - (b - b + x^T A_2 y)^2 \xi = 0
\]
\[
(a - a + x^T A_1 y) A_1 y + (a - a + x^T A_1 y) A_1 y - (a - a + x^T A_1 y)^2 \psi e^m \leq 0^m
\]
\[
(b - b + x^T A_2 y) A_2 x + (b - b + x^T A_2 y) A_2 x - (b - b + x^T A_2 y)^2 \xi e^n \leq 0^n
\]
\[
e^m x - 1 = 0
\]
\[
e^n y - 1 = 0
\]
\[
x \geq 0^m
\]
\[
y \geq 0^n \text{ where } \psi \text{ and } \xi \text{ are scalars.}
\]

**Theorem: 5.2.2**

If \(x\) and \(y\) satisfy the Kuhn-tucker conditions of the above equations if and only if there exist an optimal solution to the following mathematical programming and \(x\) and \(y\) are components of the optimal solution
Maximize $\tilde{a} x^T (A_1 + A_1) y + \tilde{b} x^T (A_2 + A_2) y - a x^T A_1 y - b x^T A_2 y$

subject to

subject to

$x, y, \psi, \xi$ in $S$

Therefore Maximize

Let $(x^*, y^*, \psi^*, \xi^*)$ Satisfy the Kuhn-tucker conditions

\[
\tilde{a} x^* (A_1 + A_1) y^* + \tilde{b} x^* (A_2 + A_2) y^* - a x^* A_1 y^* - b x^* A_2 y^*
\]

\[
- (a - a + x^T A_1 y)^2 \psi^* - (b - b + x^T A_2 y)^2 \xi^* = 0
\]
Since Let \( (x^*, y^*, \psi^*, \xi^*) \) in S we have

\[
\tilde{a} x^T (A_1 + A_1) y^* + \tilde{b} x^T (A_2 + A_2) y^* - a x^T A_1 y^* - b x^T A_2 y^* - (a-a^*)A_1 y^* - (b-b^*)A_2 y^* \xi^* = 0
\]

Conversely suppose that Let \( (x^*, y^*, \psi^*, \xi^*) \) be an optimal solution to the problem then

\[
\tilde{a} x^T (A_1 + A_1) y^* + \tilde{b} x^T (A_2 + A_2) y^* - a x^T A_1 y^* - b x^T A_2 y^* - (a-a^*)A_1 y^* - (b-b^*)A_2 y^* \xi^* \leq 0
\]

Form the existence of equilibrium solutions and the conditions of Kuhn-tucker, there exist at least one \( (x, y, \psi, \xi) \) which satisfies

\[
\tilde{a} x^T (A_1 + A_1) y + \tilde{b} x^T (A_2 + A_2) y - a x^T A_1 y - b x^T A_2 y - (a-a^*)A_1 y - (b-b^*)A_2 y_2 \xi = 0
\]

So for \( (x^*, y^*, \psi^*, \xi^*) \) to be an global maximum in S must be

\[
\tilde{a} x^T (A_1 + A_1) y^* + \tilde{b} x^T (A_2 + A_2) y^* - a x^T A_1 y^* - b x^T A_2 y^* - (a-a^*)A_1 y^* - (b-b^*)A_2 y_2 \xi^* = 0
\]

Form the above equation first and second constraint of the problem

\[
\tilde{a} x^T (A_1 + A_1) y^* - a x^T A_1 y^* - (a-a^*)A_1 y_2 \psi = 0
\]
\[
\tilde{b} x^T (A_2 + A_2) y^* - b x^T A_2 y^* - (b-b^*)A_2 y_2 \xi = 0
\]

Hence \( (x^*, y^*, \psi^*, \xi^*) \) satisfies the condition of Kuhn and Tucker.
Theorem: 5.2.3

For a bi-matrix game \((A_1, A_2)\) with fuzzy pay-offs, let membership functions of fuzzy pay-offs and fuzzy goals of player I and II be linear. If the condition \(0 < d_1(x, y) < 1\) and \(0 < d_2(x, y) < 1\) is satisfied then the necessary conditions for \(x\) and \(y\) to be an equilibrium solution with respect to the degree of attainment of the fuzzy goal is that \(x\) and \(y\) are components of an optimal solution to the mathematical programming problem

\[
\begin{align*}
\text{Maximize} & \quad \tilde{x}^T (A_1 + A_2) y + \tilde{b} x^T (A_2 + A_2) y - \tilde{a} x^T A_1 y - \tilde{b} x^T A_2 y \\
\text{subject to} & \quad (\tilde{a} - a + x^T A_1 y) A_1 y + (\tilde{a} - a + x^T A_1 y) A_1 y - (\tilde{a} - a + x^T A_1 y)^2 \psi e^m \leq 0^m \\
& \quad (\tilde{b} - b + x^T A_2 y) A_2 x + (\tilde{b} - b + x^T A_2 y) A_2 x - (\tilde{b} - b + x^T A_2 y)^2 \xi e^n \leq 0^n \\
& \quad e^m x - 1 = 0 \\
& \quad e^n y - 1 = 0 \\
& \quad x \geq 0^m \\
& \quad y \geq 0^n \quad \text{where } \psi \text{ and } \xi \text{ are scalars.}
\end{align*}
\]

By finding the multiple solution of the above problem and checking whether each of the optimal solutions to the above problem satisfies the equilibrium conditions

\[
\begin{align*}
\frac{x^* T (A_1 + A_1) y^* - a}{a - a + x^* T A_1 y^*} & \geq \frac{x^* T (A_1 + A_1) y^* - a}{a - a + x^* T A_1 y^*} \\
\frac{x^* T (A_2 + A_2) y^* - b}{b - b + x^* T A_2 y^*} & \geq \frac{x^* T (A_2 + A_2) y^* - b}{b - b + x^* T A_2 y^*}
\end{align*}
\]

We can obtain an equilibrium solution with respect to the degree of attainment of the fuzzy goal. Let \((x^*, y^*)\) be an optimal solution to the above problem, if an optimal value of the problem are equal to
\[
\frac{x^T (A_1 + A_1) y^* - a}{a - a + x^T A_1 y^*} = \text{Maximize } \frac{x^T (A_1 + A_1) y^* - a}{a - a + x^T A_1 y^*}
\]
subject to \(e^{mT} x - 1 = 0 \quad x \geq 0^m\)

\[
\frac{x^T (A_2 + A_2) y^* - b}{b - b + x^T A_2 y^*} = \text{Maximize } \frac{x^T (A_2 + A_2) y^* - b}{b - b + x^T A_2 y^*}
\]
subject to \(e^{mT} y - 1 = 0 \quad y \geq 0^n\)

\[
\frac{x^T (A_1 + A_1) y^* - a}{a - a + x^T A_1 y^*} \quad \text{and} \quad \frac{x^T (A_2 + A_2) y^* - b}{b - b + x^T A_2 y^*}
\]
respectively,
then solution \((x^*, y^*)\) is an equilibrium solution. If the objective function of the problem

\[
\frac{x^T (A_1 + A_1) y^* - a}{a - a + x^T A_1 y^*} = \text{Maximize } \frac{x^T (A_1 + A_1) y^* - a}{a - a + x^T A_1 y^*}
\]
subject to \(e^{mT} x - 1 = 0 \quad x \geq 0^m\).

\[
\frac{x^T (A_2 + A_2) y^* - b}{b - b + x^T A_2 y^*} = \text{Maximize } \frac{x^T (A_2 + A_2) y^* - b}{b - b + x^T A_2 y^*}
\]
subject to \(e^{mT} y - 1 = 0 \quad y \geq 0^n\).

are concave with respect to \(x\) and concave with respect to \(y\), necessary conditions of Kuhn and tucker to the problems are also sufficient therefore above theorem gives the necessary and sufficient conditions.

Consider the concavity of the objective function of the problem
\[
\begin{align*}
\frac{x^*(A_1 + A_1) y^* - a}{a - a + x^T A_1 y^*} &= \text{Maximize} \frac{x^T (A_1 + A_1) y^* - a}{a - a + x^T A_1 y^*}, \\
\text{subject to } e^{mT} x - 1 = 0 \quad x \geq 0^n
\end{align*}
\]
\[
\begin{align*}
\frac{x^*(A_2 + A_2) y^* - b}{b - b + x^T A_2 y^*} &= \text{Maximize} \frac{x^T (A_2 + A_2) y - b}{b - b + x^T A_2 y}, \\
\text{subject to } e^{mT} y - 1 = 0 \quad y \geq 0^n.
\end{align*}
\]

For two strategies \(x^1, x^2\) in \(X\) of player I to a strategy \(\hat{y}\) in \(Y\) of player II let

\[
x^\lambda = \lambda x^1 + (1 - \lambda) x^2 \quad 0 \leq \lambda \leq 1
\]

\[
d_1(x, y) = \frac{x^T (A_1 + A_1) y - a}{a - a + x^T A_1 y}.
\]

Then if any for \(x_1, x_2\) in \(X\) and \(\hat{y}\) in \(Y\)

\[
d_1(x^\lambda, \hat{y}) - \lambda d_1(x^1, \hat{y}) - (1 - \lambda) d_1(x^2, \hat{y}) \geq 0
\]

\(d_1(x, y)\) is concave with respect to \(x\). Let \(x^T (A_1 + A_1) \hat{y}_1 = x^T A \hat{y} = 1, 2\) then the condition the inequality

\[
d_1(x^\lambda, \hat{y}) - \lambda d_1(x^1, \hat{y}) - (1 - \lambda) d_1(x^2, \hat{y}) \geq 0
\]

holds

\[
\lambda (1 - \lambda) \left\{ \alpha_2 \beta_1 (\beta_2 - \beta_1) + \alpha_1 \beta_2 (\beta_2 - \beta_1) + (a - a) \alpha_1 (\beta_2 - \beta_1) \right\} + (a - a)^2 \left( \lambda \alpha_1 + (1 - \lambda) \alpha_2 - a \right) \geq 0
\]

The function \(d_1(x, y)\) is concave with respect to \(x\), if \(a_{ij} = a\) for all \(i, j\) in \(J\). The function \(d_2(x, y)\) has a similar property.

5.3 An equilibrium solution in multi-objective bi-matrix games incorporating fuzzy goals:

For \(k^{th}\) pay-off Let player-I fuzzy goal \(\tilde{G}_{ik} \quad k = 1, \ldots, r\)
be a fuzzy set on the real line characterized by the membership function \( \mu_{G_i} : R \rightarrow [0,1] \).

Similarly, the \( l \)th fuzzy goal \( G_2^l \) is also a fuzzy set on the real line characterized by the membership function \( \mu_{G_2^l} \). The degree of attainment of fuzzy goal for player I can be represented as follows:

\[
\max_{\mu_{x_1}} \min_{\mu_{y_1}} \left( \mu_{x_1}^1(p^1), \mu_{y_1}^1(p^1) \right), \quad \max_{\mu_{x_2}} \min_{\mu_{y_2}} \left( \mu_{x_2}^2(p^2), \mu_{y_2}^2(p^2) \right), \ldots, \quad \max_{\mu_{x_s}} \min_{\mu_{y_s}} \left( \mu_{x_s}^s(p^s), \mu_{y_s}^s(p^s) \right)
\]

To aggregate multiple fuzzy goals, we incorporate weighting coefficients method.

Let weighting coefficients for fuzzy goals of player I and II are \( v \) in \( \{ v \in R^+ / \sum_{k=1}^{r} v_k = 1 \} \) and \( w \) in \( \{ w \in R^+ / \sum_{j=1}^{l} w_j = 1 \} \) respectively. Player aggregated fuzzy goals is represented by \( \sum_{k=1}^{r} v_k \max_{\mu_{x_k}} \min_{\mu_{y_k}} \left( \mu_{x_k}^k(p^k), \mu_{y_k}^k(p^k) \right) \). The degree of attainment of the \( l \)th fuzzy goal for player II can be defined in a similar way.

Next, we consider equilibrium solutions with respect to the degree of attainment of the aggregated fuzzy goal. A pair of strategies \( x^*, y^* \) is said to be an equilibrium solution with respect to the degree of attainment of fuzzy goal if for any mixed strategies \( x \) and \( y \)

\[
\sum_{k=1}^{r} v_k \max_{\mu_{x_k}} \min_{\mu_{y_k}} \left( \mu_{x_k}^k(p), \mu_{y_k}^k(p) \right) \geq \sum_{k=1}^{r} v_k \max_{\mu_{x_k}} \min_{\mu_{y_k}} \left( \mu_{x_k}^k(p), \mu_{y_k}^k(p) \right)
\]

And \( \sum_{j=1}^{l} w_j \max_{\mu_{x_j}} \min_{\mu_{y_j}} \left( \mu_{x_j}^j(p), \mu_{y_j}^j(p) \right) \geq \sum_{j=1}^{l} w_j \max_{\mu_{x_j}} \min_{\mu_{y_j}} \left( \mu_{x_j}^j(p), \mu_{y_j}^j(p) \right) \).

We will examine a relation between equilibrium solutions and optimal solutions to a certain mathematical programming where the membership functions of fuzzy goals and reference functions are linear.
When player I and II choose pure strategies \( i=1, 2, \ldots, m \) and \( j=1, \ldots, n \) respectively, the \( k \)-th pay-off of player I is represented as the fuzzy number \( a_{ij} \) characterized by the membership function

\[
\mu_{g_i}(p) = \begin{cases} 
0 & \text{if } p \leq a_k^- \\
\frac{p - a_k^-}{a_k^-} & \text{if } a_k^- \leq p \leq a_k^+ \\
1 & \text{if } a_k^+ \leq p.
\end{cases}
\]

we assume that \( 0 < d_1(x, y) < 1 \) and \( 0 < d_2(x, y) < 1 \).

Let membership functions of the fuzzy goals and pay-offs of player II similar linear functions. Applying aggregation by weighting coefficients for a pair of \( x \) and \( y \) Player I degree of attainment of the aggregated fuzzy goal can be represented by

\[
W d_1(x, y) = \sum_{k=1}^{r} \sum_{i=1}^{m} \sum_{j=1}^{n} a_k^- (a_{ij} + a_{ij}) x_i y_j - a_k^- \]

And player II also be represented similarly.

A pair of optimal solutions \( x^* \) and \( y^* \) to the following two mathematical programming problems is an equilibrium solution.
\[ W d_1(x^*, y^*) = \max_x W d_1(x, y^*) \]

Subject to \( e_{mT}x - 1 = 0, \ x \geq 0^m \)

\[ W d_2(x^*, y^*) = \max_y W d_2(x^*, y) \]

Subject to \( e_{nT}y - 1 = 0, \ y \geq 0^n \).

By applying the necessary conditions of Kuhn and Tucker to the above problems, a necessary conditions that a pair of \( x \) and \( y \) be an equilibrium solution with respect to the degree of attainment of fuzzy goals becomes that there exist scalar values \( \alpha \) and \( \beta \) such that \( x, y, \alpha, \beta \).

\[
\sum_{k=1}^{r} v_k \left( \frac{-k}{(a-a^k+x^T A_1^k y)^2} \right) - \alpha = 0
\]

\[
\sum_{s=1}^{r} w_e \left( \frac{-e}{(b-b^e+x^T A_2^e y)^2} \right) - \beta = 0
\]

\[
\sum_{k=1}^{r} v_k \left( \frac{-k}{(a-a^k+x^T A_1^k y)^2} \right) - \alpha e^m \leq 0^m
\]

\[
\sum_{s=1}^{r} w_e \left( \frac{-e}{(b-b^e+x^T A_2^e y)^2} \right) - \beta e^n \leq 0^n
\]

\[ e^{mT}x - 1 = 0, \ e^{nT}y - 1 = 0, \ x \geq 0^m, \ y \geq 0^n \]

**Theorem 5.3.1**

For bi-matrix game \((\widetilde{A}_1, \widetilde{A}_2)\) with multiple fuzzy payoff matrices, let membership functions of the fuzzy pay-offs and fuzzy goals of players I and II be linear. If the condition
*) is satisfied then necessary conditions for \(x\) and \(y\) to be an equilibrium solution with respect to the degree of attainment of the fuzzy goal aggregated by weighting coefficients is that \(x\) and \(y\) are components of an optimal solution to the following mathematical programming problem:

\[
\text{Max} \sum_{k=1}^{r} \left \{ \frac{\alpha}{(a - a_k^+ x^T A_1 y)^2} \left( a x^T (A_1^k + A_1) y - a_k^+ x^T A_1^k y \right) \right \} - \sum_{\ell=1}^{s} \frac{\beta}{(b - b_{\ell}^+ x^T A_2 y)^2} \left( b x^T (A_2^\ell + A_2) y - b_{\ell}^+ x^T A_2^\ell y \right)\]

Subject to

\[
\sum_{k=1}^{r} \left \{ \frac{\alpha}{(a - a_k^+ x^T A_1 y)^2} \left( a x^T (A_1^k + A_1) y - a_k^+ x^T A_1^k y \right) \right \} - \alpha e^m \leq 0^m \]

\[
\sum_{\ell=1}^{s} \frac{\beta}{(b - b_{\ell}^+ x^T A_2 y)^2} \left( b x^T (A_2^\ell + A_2) y - b_{\ell}^+ x^T A_2^\ell y \right) - \beta e^n \leq 0^n. \quad (1)
\]

\[e^m x = 0, \quad e^n y = 0, \quad x \geq 0^m, \quad y \geq 0^n.\]

**Proof:**

First we suppose that constraints of the above problem (1) is same as components of the Kuhn and tucker conditions, then \((x^*, y^*, \alpha, \beta)\) Satisfy the Kuhn-tucker conditions, Let \(S\) denote the feasible region of the above problem we have

\[
\sum_{k=1}^{r} \left \{ \frac{\alpha}{(a - a_k^+ x^T A_1 y)^2} \left( a x^T (A_1^k + A_1) y - a_k^+ x^T A_1^k y \right) \right \} - \alpha \sum_{\ell=1}^{s} \frac{\beta}{(b - b_{\ell}^+ x^T A_2 y)^2} \left( b x^T (A_2^\ell + A_2) y - b_{\ell}^+ x^T A_2^\ell y \right)\]

This equals
\[
\sum_{k=1}^{r} V_k \left\{ x^T \left( a - a_k^k + x^T A_1 y \right) A_1^k y + (a - x^T A_1^k) A_1 y \right\} - \alpha e^m
\]

\[
+ \sum_{l=1}^{s} W_{l} \left\{ x^T \left( b - b_1^l + x^T A_2 y \right) A_2^l y \right\} - \beta e^s \leq 0.
\]

(2)

For any \((x, y, \alpha, \beta)\) in \(S\)

\[
\text{Max} \sum_{k=1}^{r} V_k \left\{ a x^T (A_1^k + A_1) y - a_k^k x^T A_1^k y \right\} - \alpha + \sum_{l=1}^{s} W_{l} \left\{ b x^T (A_2^l + A_2) y - b_1^l x^T A_2^l y \right\} - \beta \leq 0.
\]

(3)

Let \((x^*, y^*, \alpha^*, \beta^*)\) satisfy the Kuhn-tucker conditions

\[
\sum_{k=1}^{r} V_k \left\{ a x^*^T (A_1^k + A_1^*) y^* - a_k^k x^*^T A_1^k y^* \right\} - \alpha^*
\]

\[
+ \sum_{l=1}^{s} W_{l} \left\{ b x^*^T (A_2^l + A_2^*) y^* - b_1^l x^*^T A_2^l y^* \right\} - \beta^* = 0.
\]

(4)

from the above (3) and (4) and the fact that \((x^*, y^*, \alpha^*, \beta^*)\) in \(S\), we have

\[
\sum_{k=1}^{r} V_k \left\{ a x^*^T (A_1^k + A_1^*) y^* - a_k^k x^*^T A_1^k y^* \right\} - \alpha^*
\]

\[
+ \sum_{l=1}^{s} W_{l} \left\{ b x^*^T (A_2^l + A_2^*) y^* - b_1^l x^*^T A_2^l y^* \right\} - \beta^*
\]
\[
\begin{align*}
= \text{Max} \sum_{k=1}^{r} v_k \frac{a^{T}(A^k_1 + A_1)y - a^k x^T A^k_k y}{(a - a^k + x^T A_1 y)^2} - \alpha \sum_{\ell=1}^{s} w_\ell \frac{b^{T}(A^\ell_2 + A_2)y - b^\ell x^T A_1 y}{(b - b^\ell + x^T A_2 y)^2} - \beta \leq 0.
\end{align*}
\]

(5)

Conversely suppose that \((x^*, y^*, \alpha^*, \beta^*)\) be an optimal solution to the problem \((1)\) and it is satisfied by \((5)\), from \((3)\)

\[
\begin{align*}
\sum_{k=1}^{r} v_k \frac{-a^{T}(A^k_1 + A_1)y^* - a^k x^T A^k_k y^*}{(a - a^k + x^T A_1 y^*)^2} - \alpha^* \\
+ \sum_{\ell=1}^{s} w_\ell \frac{-b^{T}(A^\ell_2 + A_2)y^* - b^\ell x^T A_1 y^*}{(b - b^\ell + x^T A_2 y^*)^2} - \beta^*
\end{align*}
\]

(6)

Therefore by the existence of equilibrium solutions and the conditions of Kuhn-tucker, there exist at least one \((x, y, \alpha, \beta)\) in \(S\) which satisfies

\[
\begin{align*}
\sum_{k=1}^{r} v_k \frac{-a^{T}(A^k_1 + A_1)y - a^k x^T A^k_k y}{(a - a^k + x^T A_1 y)^2} - \alpha + \sum_{\ell=1}^{s} w_\ell \frac{b^{T}(A^\ell_2 + A_2)y - b^\ell x^T A_1 y}{(b - b^\ell + x^T A_2 y)^2} - \beta = 0.
\end{align*}
\]

For \((x^*, y^*, \alpha^*, \beta^*)\) be maximum in \(S\) in the above \((6)\)

\[
\begin{align*}
\sum_{k=1}^{r} v_k \frac{-a^{T}(A^k_1 + A_1)y - a^k x^T A^k_k y^*}{(a - a^k + x^T A_1 y^*)^2} - \alpha^* \\
+ \sum_{\ell=1}^{s} w_\ell \frac{b^{T}(A^\ell_2 + A_2)y - b^\ell x^T A_1 y^*}{(b - b^\ell + x^T A_2 y^*)^2} - \beta^*
\end{align*}
\]

(7)
from the above equation and the first and second constraints of the problem (1)

\[
\sum_{k=1}^{r} V_k \left( \frac{-k a x^T (A^k + A_1) y^* - a^k x^T A^k y^*}{(-a^k + x^T A_1 y^*)^2} \right) - \alpha^* = 0.
\]

\[
\sum_{\ell=1}^{s} W_\ell \left( \frac{-\ell b x^T (A^\ell + A_2) y^* - b^\ell x^T A^\ell y^*}{(-b^\ell + x^T A_2 y^*)^2} \right) - \beta^* = 0.
\]

Therefore \((x^*, y^*, \alpha^*, \beta^*)\) satisfies the Kuhn-tucker conditions.

5.4 New Solution concept in two person non zero sum with fuzzy goals and fuzzy pay-offs:

A new method for computing the maximin solution of two person non-zero sum game. Let the membership functions of the fuzzy goals and shape functions for fuzzy numbers representing the fuzzy pay-offs are linear. A membership function player-I fuzzy goal are represented by

\[
\mu_a(p) = \begin{cases} 
  0 & \text{if } p \leq a \\
  \frac{p-a}{a-a} & \text{if } a \leq p \leq a \\
  1 & \text{if } a \leq p.
\end{cases}
\]

Similarly for player-II

\[
\mu_b(p) = \begin{cases} 
  0 & \text{if } p \leq b \\
  \frac{p-a}{a-a} & \text{if } b \leq p \leq b \\
  1 & \text{if } b \leq p.
\end{cases}
\]
Where $\tilde{a}$, $a$ are best and worst degree of satisfaction of Player-I and $\tilde{b}$, $b$ are best and worst degree of satisfaction of Player-II. Let $\tilde{a}_{ij} = (a_{ij}, a_{ij}, a_{ij})_{LR}$ be the shape function of LR fuzzy number and shape of fuzzy number be $L(p) = R(p) = \max(0, 1 - |p|)$.  

When players I and II choose pure strategies $i \in I$ and $j \in J$ respectively, a pay-off for player-I is represented as $a_{ij} = (a_{ij}, a_{ij}, a_{ij})_{LR}$ and it is characterized by the membership function.

$$
\mu_{a_{ij}}(p) = \begin{cases} 
0 & \text{if } p^* \leq a_{ij} - a_{ij} \\
\frac{p - a_{ij} + a_{ij}}{a_{ij}} & \text{if } a_{ij} - a_{ij} \leq p \leq a_{ij} \\
\frac{a_{ij} + a_{ij} - p}{a_{ij}} & \text{if } a_{ij} \leq p \leq a_{ij} + a_{ij} \\
0 & \text{if } a_{ij} + a_{ij} \leq p. 
\end{cases} 
$$  

(4)

Let the membership functions of the fuzzy goals and payoffs of player-II are similar.

This method corresponds to the intersection of all of the fuzzy sets and a solution is defined as the maximum of the intersection of the fuzzy expected pay-off $\mu_{xA,\gamma}$ and fuzzy goal $\mu_{G}$. 

Player-I maximin value with respect to degree of attainment of the aggregated fuzzy goal is  

$$
\max_{x \in X} \min_{y \in Y} \{\mu_{G}(p^*)\} = \min_{x \in X, y \in Y} \max_{p} \min_{\mu_{G}(x, y)(p)} \{\mu_{G}(p^*)\} 
$$  

(5)

Where membership function are linear, maximin strategy with respect to degree of attainment of fuzzy goals can be obtained by solving the following mathematical linear programming problem.
Theorem: 5.4.1

For single objective two person non zero sum games, if the membership functions of the fuzzy goals and shape function of LR fuzzy numbers for fuzzy pay offs are linear such as (1) and (4) then for the first player, maximin solution with respect to a degree of attainment of the aggregated fuzzy goal is equal to an optimal solution to the nonlinear programming problem:

\[
M d_1(x, y) = \max_{x, \sigma} \sigma \quad \text{and} \quad M d_2(x, y) = \max_{x, \delta} \delta
\]

\[
\begin{align*}
M d_1(x, y) & = \max_{x, \sigma} \sigma, \\
M d_2(x, y) & = \max_{x, \delta} \delta
\end{align*}
\]

Subject to

\[
\begin{align*}
& x^T (A_1^k + A_1) y - a_i \geq \sigma, \quad e^{mT} x = 1, \quad x \geq 0^m, \quad k = 1,2,\ldots, r, \\
& x^T A_1 y + a - a_i \geq \delta, \quad e^{aT} y = 1, \quad y \geq 0^n, \\
& x^T (A_2^u + A_2) y - b \geq \delta, \\
& x^T A_2 y + b - b \geq \delta
\end{align*}
\]

Where \( \sigma, \delta \) are scalar variable.

If the optimal value \( \sigma^*, \delta^* \) satisfies \( 0 \leq \sigma \leq 1, \quad 0 \leq \delta \leq 1 \) problem (6) is a non-linear programming problem noted that the first constraint of (6) must hold for any \( y \in Y \), since the constraint are maximizing the decision variable \( x \) and minimizing the decision variable \( y \) in (6) are separated from each other, we can solve (6) by relaxation procedure by Shimizu and Aiyoshi [55] and then we obtain the maximin solution.

Consider the following relaxed problem for the above problem (6) by taking \( L \) & \( H \) points \( x^h, y^f \) where \( \ell = 1,2,\ldots,L \) and \( h = 1,2,\ldots, H \), satisfying

\[
\begin{align*}
x^h & \in X, \quad y^f \in Y, \quad e^{mT} x^h = 1, \quad x_i^h \geq 0^m, \quad j = 1,2,\ldots, n \quad \text{and} \quad e^{aT} y^f = 1, \quad y_j^f \geq 0^n, \quad j = 1,2,\ldots, n
\end{align*}
\]
\[
\text{Md}_1(x, y) = \max_{x, \sigma} \text{ and } M_d(x, y) = \max_{x, \delta}
\]

Subject to

\[
\begin{align*}
& x^T (A_1^k + A_1) y^\ell - a \geq \sigma, \quad e^{mT} x = 1, \quad x \geq 0^m, \quad \ell = 1, 2, \ldots, L. \\
& x^T A_1 y^\ell + a - a \geq \delta, \quad e^{nT} y = 1, \quad i = 1, 2, \ldots, m, \quad h = 1, 2, \ldots, H.
\end{align*}
\]  

(7)

Let an optimal solution to the relaxed problem be denoted by \((x^h, y^\ell, \sigma^\ell, \delta^h)\) and if \((x^h, y^\ell, \sigma^\ell, \delta^h)\) be a feasible solution to the original problem (6) it must be optimal to (6) by solving the minimization problem, we can accomplish the violated constraints

\[
\begin{align*}
& \text{minimize} \quad \frac{x^{L+1T} (A_1^k + A_1) y^{L+1} - a}{x^{L+1T} A_1 y^{L+1} + a - a} \quad \text{and} \quad \frac{x^{L+1T} (A_2^u + A_2) y^H - b}{x^{L+1T} A_2 y^H + b - b} \\
& \text{subject to} \quad e^{nT} y = 1, \quad e^{mT} x = 1, \quad x_j = 0, \quad y_j \geq 0, \quad j = 1, 2, \ldots, n
\end{align*}
\]  

(8)

Let an optimal solution to the minimization problem (8) denoted by \(y^{L+1} = y(x^L)\), \(x^{H+1} = x(y^H)\). If \((x^H, y(x^L), \sigma^L, \delta^H)\) and \((x(y^H), y^L, \sigma^L, \delta^H)\) satisfies the constraints of the original problem (6) then we reached the optimal solution. If it does not satisfy, add the constraint

\[
\begin{align*}
& \frac{x^T (A_1^k + A_1) y^{L+1} - a}{x^T A_1 y^{L+1} + a - a} \geq \sigma \quad \text{and} \quad \frac{x^{(H+1)T} (A_2^u + A_2) y^H - b}{x^{(H+1)T} A_2 y^H + b - b} \geq \delta.
\end{align*}
\]  

(9)
To the relaxed problem (7) and solve it again. The constraint (9) violates to greatest extent. The optimal solution to the problem (6) can be obtained by repeating this procedure for finite number of iterations, but it is supposed that solving the relaxed problem (7) it is still difficult because it has non-linear constraints.

However we can reduce the relaxed problem (7) which is linear fractional programming problem, to a linear problem by using sakawa method. The variable in the relaxed problem satisfies \( \sigma, \delta \in [0,1] \) because the variable corresponds to the maximin value with respect to a degree of attainment of the fuzzy goal. Let \( \sigma \in [0,1] \) then the constraint of the relaxed problem (7) becomes as

\[
\begin{align*}
\max & \quad x^T (A^k_i + A_1) y' - a \\
\text{s.t.} & \quad x^T A^k_i y' + a - a \\
& \quad x^T A^k_2 y + b - b \\
& \quad e^T x = 1, \quad x \geq 0^n.
\end{align*}
\] (10)

We take first constraints of (7) is \( \frac{P_\ell}{Q_\ell} \geq \sigma^\ell, \frac{P_h}{Q_h} \geq \delta^H \), where \( \ell = 1,2,\ldots,L \), \( h=1,2,\ldots,H \).

Where \( P_\ell = x^T (A^k_i + A_1) y' - a \) and \( Q_\ell = x^T A^k_i y' + a - a \) and

\( P_h = x^T (A^u_2 + A_2) y - b \) and \( Q_h = x^T A^u_2 y + b - b \).

We can find the maximal constant value of the problem (10) by making use of Dinkelbach algorithm,
Max $r$
\[
\begin{aligned}
P_{\ell} - \sigma_{t} Q_{\ell} - r & \geq 0, \\
x^T (A_{1}^{k} + A_{1})y - a - P_{\ell} & = 0, \\
x^T A_{1} y + a - a - Q_{\ell} & = 0, \\
e^{mT} x = 1, x \geq 0^{m}, t = 1, ..., B.
\end{aligned}
\]

Max $s$
\[
\begin{aligned}
P_{h} - \delta_{g} Q_{h} - s & \geq 0, \\
x^{bT} (A_{2}^{u} + A_{1})y - b - P_{h} & = 0, \\
x^{bT} A_{2} y + b - b - Q_{h} & = 0, \\
e^{nT} y = 1, y \geq 0^{n}, g = 1, ..., G.
\end{aligned}
\]

Where $\hat{\sigma}_{t} = \min \left( \frac{P_{\ell}}{Q_{\ell}} \right)$ and $\hat{\delta}_{g} = \min \left( \frac{P_{h}}{Q_{h}} \right)$.

If $r = 0$ and $s = 0$ then, stop the iteration and the feasible solution $x^{*}$ and $y^{*}$ and maximum constant value $\hat{\sigma}, \hat{\delta}$ must be optimal solution $\left( x^{*}, \hat{\sigma} = \sigma^{*} \right)$, $\left( y^{*}, \hat{\delta} = \delta^{*} \right)$ of the relaxed problem (7) if maximum value of $r \neq 0$, $s \neq 0$. Set $t = t + 1$, $g = g + 1$ and solve it again. We can find maximal value by repeating this procedure for finite number of iterations.

The minimization problem (8) which generates most violated constraints, can be reduced to a linear programming problem by using the variable transformation by Charnes and Cooper [10].

Let
\[
\begin{aligned}
1 & = t \quad \text{and} \\
1 & = g
\end{aligned}
\]

\[
y_{j}(t) = z_{j}(t), \quad x_{i}(g) = f_{i}(g).
\]
The minimization problem (8) can be written as

\[
\min_{x,t} \left\{ (x^T (A_1 + A_2)) - a \right\} \quad t
\]
subject to \( e^a z = t, \quad x^T A_1 z + (a - a) \right\} t = 1, \quad z \geq 0, \)

\[
\min_{f,g} \left\{ f^T (A_2 - A_2) - a \right\} \quad y
\]
subject to \( e^m f = g, \quad x^T A_2 f + (b - b) \right\} g = 1, \quad f \geq 0.
\]

The above (13) is called linear programming problem which has decision variable \( z, f \) and \( t, g \) has two equality constraints and the non-negative conditions of decision variable.

### 5.5 Bi-matrix game using fuzzy pay-off by an extreme measures approach

In this section extreme measure approach is used to characterize an equilibrium solution of a bi-matrix game using fuzzy pay-offs. Here fuzzy pay-offs are given by symmetric triangular fuzzy numbers. Dubois and Prade studied the ranking of fuzzy numbers in the possibility theory. To construct this, two fuzzy numbers are required. By incorporating fuzzy pay-offs in the bi-matrix, the crisp inequality can be extended to obtain the truth value of the statement using Zadeh extension principle. This truth value is called the grade of possibility of dominance on fuzzy numbers.

Let \( a_{ij}^1, a_{ij}^2 \) are two fuzzy numbers. According to the extension principle of Zadeh, the crisp inequality \( x \leq y \) can be extended to obtain the truth value \( T \) (Here \( T \) is grade membership of possibility of dominance of \( a_{ij}^1 \) on \( a_{ij}^2 \). In the next section this will be interpreted in terms of fuzzy measures. The statement \( a_{ij}^1 \) is less than or equal to \( a_{ij}^2 \) is defined as

\[
g^*(a_{ij}^1 \leq a_{ij}^2) = \sup_{x \leq y} \min \left\{ \mu_{a_{ij}^1}(x), \mu_{a_{ij}^2}(y) \right\}
\]
This truth value \( T(a_{ij}^1 \leq a_{ij}^2) \) are also called the grade of dominance of \( a_{ij}^2 \) on \( a_{ij}^1 \) and it is denoted by \( g^*(a_{ij}^1 \leq a_{ij}^2) = \sup_{x \leq y} \min \{ \mu_{a_{ij}^1}(x), \mu_{a_{ij}^2}(y) \} \).

In this similar way the grade (or degree) of \( g^* \) that ascertain \( a_{ij}^1 \) is greater than or equal to \( a_{ij}^2 \) is given by \( g^*(a_{ij}^1 \geq a_{ij}^2) = \sup_{x \leq y} \min \{ \mu_{a_{ij}^1}(x), \mu_{a_{ij}^2}(y) \} \).

The above discussion motivates us to define \( a_{ij}^1 \leq a_{ij}^2 \Leftrightarrow g^*(a_{ij}^1 \leq a_{ij}^2) \geq g^*(a_{ij}^2 \leq a_{ij}^1) \)

Here it may be noted that for any case when \( a_{ij}^1 = (a_1^1, a_1^1, a_1^1), a_{ij}^2 = (a_2^1, a_2^2, a_2^2) \) are symmetric TFN then \( a_{ij}^1 \leq a_{ij}^2 \) gives \( g^*(a_{ij}^1 \leq a_{ij}^2) = 1 \) and \( g^*(a_{ij}^2 \leq a_{ij}^1) = \text{height}(a_{ij}^2 \cap a_{ij}^1) \leq 1 \).

Lower measure \( \hat{g}^* \) which measures the degree of dominance of \( A_2 \) on \( A_1 \) and it is given by \( g_*(a_{ij}^1 \leq a_{ij}^2) = 1 - g^*(a_{ij}^1 \geq a_{ij}^2) \).

The number \( g_*(a_{ij}^1 \leq a_{ij}^2) \) can also be used for ranking of fuzzy numbers, for this we define \( a_{ij}^1 \leq a_{ij}^2 \) if and only if \( g_*(a_{ij}^1 \leq a_{ij}^2) \geq g_*(a_{ij}^2 \leq a_{ij}^1) \)

In case \( A_1 = (a_1^1, a_1^1, a_1^1), A_2 = (a_2^2, a_2^2, a_2^2) \) are TFN then by actual computation of \( g_*(a_{ij}^1 \geq a_{ij}^2) \) it can be defined that \( a_{ij}^1 \leq a_{ij}^2 \) with respect to \( g_*(a_{ij}^1 \leq a_{ij}^2) \) approach if \( a_{ij}^1 + a_{ij}^1 \leq a_{ij}^2 + a_{ij}^2 \).

We present extreme measures approach to characterize an equilibrium solution of bi-matrix game with fuzzy pay-offs.

Let \( (X, A_1, A_2) \) be an extreme measurable space and \( (g_*, g^*) \) be a couple of extreme fuzzy measures defined over it \( (\sigma_*, \sigma^*) \) then fuzzy expected pay-offs of player I and II with respect to upper measurable space

\[ g^*(A_1 \geq A_2) = \max_{\lambda} \min \{ \mu_{A_1}(x), \mu_{A_2}(y) \} \]
\[ \mu_{F_1}(x, y) = \max_{p \in \mathcal{P}} \min_{y \in \mathcal{Y}} \left\{ \mu_{a^{1}_{ij}}(x), \mu_{a^{2}_{ij}}(y) \right\}. \]

Let \( a^{1}_{ij}, a^{2}_{ij} \) are TFN

If \( \alpha \in (0, 1) \) then \( g^*(a^{1}_{ij}, a^{2}_{ij}) \geq \alpha \Leftrightarrow a^{1L}_{ij} \geq a^{2L}_{ij} \)

where \([a^{1L}_{ij}, a^{2L}_{ij}]\) and \([a^{1R}_{ij}, a^{2R}_{ij}]\) are \( \alpha \) – cuts of \( a^{1}_{ij}, a^{2}_{ij} \) respectively

\[ g^*(a^{1}_{ij}, a^{2}_{ij}) \leq \alpha \Leftrightarrow a^{1R}_{ij} \leq a^{2L}_{ij}. \]

This is later result follows \( g_*(a^{1}_{ij} > a^{2}_{ij}) \geq \alpha \Leftrightarrow a^{1L}_{ij} \geq a^{2L}_{ij} \).

Bi-matrix game with fuzzy pay-off \( BFP = \{S^m, S^n, A_1, A_2\} \) where \( a^{1}_{ij}, a^{2}_{ij} \) are fuzzy expected pay-off of matrices \( A_1, A_2 \) and they are symmetric TFN. Similarly we can define lower measurable space \( g_*(A_1 \leq A_2) = \inf \max_{x} \{1 - \mu_{A_1}(x), \mu_{A_2}(y)\} \).

**Definition: 5.5.1**

Let \((v^*, w^*) \) in \( R \) and an element \((x^*, y^*) \) \( S^m \times S^n \) called \( g^*(v^*, w^*) \) be Nash equilibrium strategy for upper fuzzy measure

\[
\begin{align*}
g^*(x^TA_1y^* \geq v^*) & \geq g^*(x^TA_1y^* \geq v^*) & \text{for all } x \in S^m \\
g^*(x^TA_2y^* \geq w^*) & \geq g^*(x^TA_2y^* \geq w^*) & \text{for all } y \in S^n.
\end{align*}
\]

Here \( v, w \) are fuzzy pay-off with respect to player-I and II.

**Definition: 5.5.2**

Let \((x^*, y^*) \) in \( S^m \times S^n \) is said to be an \((\alpha, \beta)\) Nash equilibrium strategy of BFP

\[
\begin{align*}
x^*T A_1 y^* & \geq x^T A_1 y^* & \text{for all } x \in S^m \\
x^*T A_2 y^* & \geq x^T A_2 y & \text{for all } y \in S^n.
\end{align*}
\]

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Here $A_{1a}^R$ is a matrix whose entries are $(a_{ij}^{1})_{\alpha}$ and $(a_{ij}^{1})_{\alpha^{T}}$ is the $\alpha$-cut of $(a_{ij}^{1})$ and $A_{2\beta}^R$ is a matrix whose entries are $(a_{ij}^{2})_{\beta}$ and $(a_{ij}^{2})_{\beta^{T}}$ is the $\beta$-cut of $(a_{ij}^{2})$.

**Theorem: 5.5.3**

Let $(v, w)$ in $R$ and an element $(x^*, y^*) S^m \times S^n$ called $g^*(v, w)$ be Nash equilibrium strategy for BFP. Let $\alpha = g^*(x^T A_1 y^* \geq v^*)$ and $\beta = g^*(x^T A_1 y^* \geq w^*) \alpha, \beta \in (0, 1)$. Then $(x^*, y^*)$ is also an $(\alpha, \beta)$ Nash equilibrium strategy.

In that case $v^* = x^* A_{1a}^R y^*$ and $w^* = x^* A_{2\beta}^R y^*$.

**Proof:**

Given that $(x^*, y^*)$ in $S^m \times S^n$ called $g^*(v, w)$ be Nash equilibrium strategy for BFP, then $(x^*, y^*)$ is also a $(v, w)$-possible Nash equilibrium strategy for BFP, we have

$$g^*(x^T A_1 y^* \geq v^*) \geq g^*(x^T A_1 y^* \geq v^*) \text{ for all } x \in S^m.$$

$$g^*(x^T A_2 y^* \geq w^*) \geq g^*(x^T A_2 y^* \geq w^*) \text{ for all } y \in S^n.$$

Let $\alpha = g^*(x^T A_1 y^* \geq v^*)$ and $\beta = g^*(x^T A_1 y^* \geq w^*)$.

The above inequalities imply that $g^*(x^T A_1 y^* \geq v^*) \leq \alpha$ for all $x \in S^m$.

By symmetric TFNs means $x^T A_{1a}^R y^* \geq v_a^L$ for all $x \in S^m$.

$x^T A_{2\beta}^R y^* \geq w_\beta^L$ for all $y \in S^n$.

But $\alpha = g^*(x^T A_1 y^* \geq v^*)$ means $x^T A_{1a}^R y^* \geq v_a^L = v$ as $v^* = (v, v, v)$

$x^T A_{1a}^R y^* \geq x^T A_{1a}^R y^*$ for all $x \in S^m$.

Similarly $x^T A_{2\beta}^R y^* \geq w_\beta^L = w$ and $x^T A_{2\beta}^R y^* \geq x^T A_{2\beta}^R y^*$ for all $y \in S^n$. 

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Implying that \((x^*, y^*)\) is a Nash equilibrium strategy of the fuzzy bi-matrix game.

Conversely suppose that let \((x^*, y^*)\) be an Nash equilibrium strategy we have to show that 

\((x^*, y^*), g^* (v, w)\) are Nash equilibrium strategy where \(v^* = x^* A_{1a}^R y^*\) and \(w^* = x^* A_{2b}^R y^*\).

Now \(\alpha = \beta = 1\) it is obvious that \((x^*, y^*)\) is a \(g^*(v, w)\) Nash equilibrium strategy without any loss of generality we assume that \(\alpha < 1\) if possible let there exist \(x \in S^m\), so that

\[
g^*(x^T A_1 y^* \geq v^*) = \alpha \Rightarrow \alpha^*
\]

\[
x^T A_{1a}^R y^* \geq x^T A_{1a}^R y^* \geq v^* = v = x^T A_{1a}^R y^*
\]

This is a contradiction therefore

\[
g^*(x^T A_1 y^* \geq v^*) \geq g^*(x^T A_1 y^* \geq v^*) \quad \text{for all} \quad x \in S^m.
\]

\[
g^*(x^T A_2 y^* \geq w^*) \geq g^*(x^T A_2 y^* \geq w^*) \quad \text{for all} \quad y \in S^n.
\]

\((x^*, y^*)\) is also a \(g^*(v, w)\) be Nash equilibrium strategy.

**Theorem: 5.5.4**

Every fuzzy bi-matrix game has a \((\alpha, \beta)\) Nash equilibrium strategy \((x^*, y^*)\) for every \((\alpha, \beta) \in [0, 1]\).

**Proof:**

Let \((x^*, y^*)\) in \(S^m \times S^n\) is said to be an \((\alpha, \beta)\) Nash equilibrium strategy of BFP

\[
x^* T A_{1a}^R y^* \geq x^* T A_{1a}^R y^* \quad \text{for all} \quad x \in S^m.
\]

\[
x^* T A_{2b}^R y^* \geq x^* T A_{2b}^R y^* \quad \text{for all} \quad y \in S^n.
\]

here \(A_{1a}^R\) is a matrix whose entries are \(\left(a_{ij}^1\right)^R_\alpha\) and \(\left(a_{ij}^1\right)^R_\alpha\) is the \(\alpha\)-cut of \(\left(a_{ij}^1\right)^R\) and \(A_{2b}^R\) is a matrix whose entries are \(\left(a_{ij}^2\right)^R_\beta\) and \(\left(a_{ij}^2\right)^R_\beta\) is the \(\beta\)-cut of \(\left(a_{ij}^2\right)^R\).

Therefore by definition 5.5.2 and theorem 4.4.3, \((x^*, y^*)\) in \(S^m \times S^n\) is said to be an \((\alpha, \beta)\) Nash equilibrium strategy of BFP, where \((\alpha, \beta) \in [0, 1]\).
5.6 Aggregation of fuzzy goals using bi-matrix game on L-fuzzy set

Definition: 5.6.1 (Fuzzy $\sigma$ - algebra)

A non-empty subclass of $\rho$ of $\rho_L(x)$ is called a fuzzy $\sigma$ – algebra. If it satisfies the following conditions

1. $\phi, X \in \rho$.
2. If $A \in \rho$ then $A^c \in \rho$.
3. If $\{A_n\} \subseteq \rho$, then $\bigcup_{n=1}^{\infty} A_n \in \rho$.

Evidently $\omega_L(X)$ and $\rho_L(X)$ fuzzy $\sigma$ - algebra. If $\rho$ is fuzzy $\sigma$ - algebra, then $B = \rho \cap \omega_L(X)$ is fuzzy $\sigma$ - algebra.

4. If $\{A_n\} \subseteq \rho$, then $\bigcap_{n=1}^{\infty} A_n \in \rho$.

Definition: 5.6.2 (Fuzzy measure)

Let $X$ be a set and $\rho \subseteq P(x)$ be an $\sigma$ – algebra on a universe $X$.

The function $g: \rho \rightarrow [0, 1]$ is said to be fuzzy measure $\Leftrightarrow$ it satisfies

1) $g(\phi) = 0, g(x) = 1$.

2) If $A, B \in \rho$ and $A \subseteq B$ then $g(A) \leq g(B)$.

3) If $A_n \in \rho$ and $A_1 \subseteq A_2 \subseteq \ldots$ then $\lim_{n \to \infty} g(A_n) = g(\lim_{n \to \infty} A_n)$.

4) If $A_n \in \rho$ and $A_1 \supseteq A_2 \supseteq \ldots$ then $\lim_{n \to \infty} g(A_n) = g(\lim_{n \to \infty} A_n)$.
3 and 4 say for any infinite sequence \( \{A_n\} \) of monotonic subset of \( X \) which converges to a set \( A \) where \( A = \bigcup_{n=1}^{\infty} A_n \) or \( A = \bigcap_{n=1}^{\infty} A_n \) for increasing and decreasing sequence respectively then the sequence \( g(A_1), g(A_2) \ldots \) must converge to \( g(A) \) which shows that \( g \) is continuous function. The fuzzy measure is monotone but its main characteristic is that additivity is not needed. Important classes of measures are the possibility measure, introduced by Zadeh [60].

**Definition: 5.6.3 (L-fuzzy set) [48]**

A mapping \( g : X \rightarrow (-\infty, \infty) \) is called measurable function on \( \rho \) if \( G_{1\alpha} \) in \( \rho \) for every in \( (-\infty, \infty) \), \( G_{1\alpha} \) is L-fuzzy set so that

\[
G_{1\alpha} = \begin{cases} 
1 & \text{if } g(p) \geq \alpha \\
0 & \text{if } g(p) < \alpha \quad \text{for any } p \in X.
\end{cases}
\]

\[
G_{1\alpha} = \begin{cases} 
1 & \text{if } g(p) > \alpha \\
0 & \text{if } g(p) \leq \alpha \quad \text{for any } p \in X.
\end{cases}
\]

Let \( g : X \rightarrow [0, \infty] \) is called measurable function on \( \rho \), if fuzzy goal of player-I is characterized by the membership function \( \mu_{G_{1\alpha}} : [0, \infty] \rightarrow [0,1] \) and \( \mu_{G_{1\alpha}} \in \rho \) is L fuzzy set for every \( \alpha \in [0,1] \).

where \( \mu_{G_{1\alpha}}(p) = \begin{cases} 
1 & \text{if } g(p) \geq a \\
0 & \text{if } g(p) < a
\end{cases} \).

Here expected pay-offs are represented by L-R fuzzy number \( \left( x^T A y, x^T A y, x^T A y \right)_{LR} \)
\[ X^T A_1 Y = \left( \sum_{i} a_1^i x_i y_i, \sum_{i} a_1^i x_i y_i, \sum_{i} a_1^i x_i y_i \right) \text{L-R.} \]

and for player-II
\[ X^T A_2 Y = \left( \sum_{i} a_2^i x_i y_i, \sum_{i} a_2^i x_i y_i, \sum_{i} a_2^i x_i y_i \right) \text{L-R.} \]

characterized by membership functions \( \mu_{xa_1 Y} : D \rightarrow [0,1] \) and \( \mu_{xa_2 Y} : D \rightarrow [0,1] \) where D is the domain of the payoffs. Let player-I specifies the finite value \( \underline{a} \) of the pay-off for which the degree of satisfaction is 0 and the finite value \( \overline{a} \) of the pay-off for which the degree of satisfaction is 1. For the undesired (smaller) than \( \underline{a} \) it is defined by \( \mu_{G_{ia}} (p) = 0 \) for the value \( p \) desired (Larger) than \( \underline{a} \) it is defined that \( \mu_{G_{ia}} (p) = 1 \) and \( \underline{a} \leq p \leq \overline{a} \). Here \( \mu_{G_{ia}} (p) \) is continuous and strictly increasing. The fuzzy goal of player-II is similar fuzzy set characterized by the membership function \( \mu_{G_{ia}} \).

**Definition 5.6.4 (Fuzzy Goal for player I and II)**

Let the domain of payoff for player I and II be by denoted D, the fuzzy goal \( G_{ia} \) and \( G_{ia} \) with respect to the payoff for player I and II are defined as the fuzzy set on the set D characterized by the membership function

\[ \mu_{G_{ia}} : D \rightarrow [0,1] \text{ where } D = \left\{ X^T A_1 Y / x \in S^n, y \in S^n \right\} \text{ and } \]

\[ \mu_{G_{ia}} : D \rightarrow [0,1] \text{ where } D = \left\{ X^T A_2 Y / x \in S^m, y \in S^n \right\}. \]

A membership function value of a fuzzy goal can be interpreted as a degree of attainment of the fuzzy goal. Then we assume that, for any pair of payoffs, a player prefers the payoff having the greater degree of attainment of the fuzzy goal to the other payoffs.
5.7 Degree of attainment of fuzzy goal

Degree of attainment of fuzzy goal is defined as the maximum of the intersection between the membership function of the above fuzzy expected pay-off and fuzzy goal.

The fuzzy integral of \( g \) on \( \mu_{E_I(x,y)}(\pi) = \mu_{a_{ij}^I(x,y)}(\pi) \in \rho \) with respected to \( \mu \) is defined by

\[
\operatorname{max}_p \left[ \mu_{E_I(x,y)}(\pi), \mu_{G_{1u}}(\pi) \right] \wedge \min p = \int g \, d\mu
\]

where

\[
\operatorname{max}_p \left[ \mu_{E_I(x,y)}(\pi), \mu_{G_{1u}}(\pi) \right] \wedge \min p = d_1(x, y).
\]

Similarly we can define for player-II

\[
\operatorname{max}_p \left[ \mu_{E_I(x,y)}(\pi), \mu_{G_{2u}}(\pi) \right] \wedge \min p = d_2(x, y).
\]

A membership function value of the fuzzy goal can be interpreted as a degree of attainment of the fuzzy goal. Then we assume that for any pair of pay-offs a player prefers the payoff having the larger degree of attainment of the fuzzy goal. We define the degree of attainment of the fuzzy goal by applying the concept of the fuzzy decision by Bellman and Zadeh [3, 4]. For any pair of mixed strategies \((x, y)\) let fuzzy expected pay-off for player-I be denoted by \( E_I(x, y) \) and let the fuzzy goal for player-I denoted by \( G_{1u} \) then the degree of attainment of the fuzzy goal is defined as the maximum of the intersection of the fuzzy expected pay-off and the fuzzy goal.

We now consider equilibrium solution problems with respect to the degree of attainment of fuzzy goal. A pair of strategies \( x^* \) and \( y^* \) is said to be an equilibrium solution with respect to the degree of attainment of the fuzzy goal if for any other mixed strategies \( x \) and \( y \)

\[
\operatorname{max}_p \left[ \mu_{E_I(x^*, y^*)}(p), \mu_{G_{1u}}(p) \right] \wedge \min p = \operatorname{max}_p \left[ \mu_{E_I(x, y^*)}(p), \mu_{G_{1u}}(p) \right] \wedge \min p \geq \operatorname{max}_p \left[ \mu_{E_I(x^*, y^*)}(p), \mu_{G_{1u}}(p) \right] \wedge \min p
\]
\[
\max_p \left[ \left( \mu_{E_2(x^*,y^*)}(p), \mu_{G_2n}(p) \right) \wedge \min g(p) \right] \geq \max_p \left[ \left( \mu_{E_2(x^*,y^*)}(p), \mu_{G_2n}(p) \right) \wedge \min g(p) \right]
\]

\[
d_1(x^*, y^*) \geq d_1(x, y^*) \quad \text{and} \quad d_2(x^*, y^*) \geq d_2(x^*, y) . \quad (1)
\]

We assume that the membership function is \( \mu_{a_{ij}}(p), \mu_{b_{ij}}(p) \) where \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \) are determined such that \( d_i(x, y) \) are continuous for \( i = 1, 2 \).

\( x^*, y^* \) be the optimal solution for the following mathematical programming problems which satisfies (1)

\[
d_1(x^*, y^*) = \max_x d_i(x, y^*)
\]

subject to \( e^{mT}x - 1 = 0, \quad x \geq 0^m \)

\[
d_2(x^*, y^*) = \max_y d_i(x^*, y)
\]

subject to \( e^{nT}y - 1 = 0, \quad y \geq 0^n \).

Where \( e^m \) and \( e^n \) are \( m \) and \( n \) dimensions column vectors respectively in which every entry is a unit.

### 5.8 Monotone convergence theorem for a sequence of fuzzy integrals using fuzzy bi-matrix game on L-fuzzy set

In this section, some convergences for sequence of measurable functions are discussed. The fuzzy goals for player-I and II also defined for the same and relations between these concepts are discussed. Monotone convergence theorem [49] for a sequence of fuzzy integrals using fuzzy bi-matrix game also discussed.
Definition: 5.8.1

Let \( \{g_n, g\} \subset M = \{g \mid g \text{ is measurable function on } \rho \} \) and \( k\)th fuzzy goal for player-I is defined by \( \mu_{G_{i_a}} \) in \( \rho \)

\[
\text{Define } D_{ia}(p) = 1 \quad \text{if } g_n(p) \to g(p) \\
0 \quad \text{if } g_n(p) \to g(p) \quad \left( \lim_{n \to \infty} g_n(p) \neq g(p) \right) \quad x \text{ in } X. \quad \alpha \in [0,1]
\]

If \( \mu_{G_{i_a}} \subset D_{ia} \) then we say that \( g_n(x) \) converges to \( g \) everywhere on \( \mu_{G_{i_a}} \) and it is denoted by \( g_n(p) \to g(p) \) on \( \mu_{G_{i_a}} \).

To prove monotone convergence theorems for a sequence of fuzzy integrals using bi-matrix game on L-fuzzy sets.

Let \( n\)th fuzzy goal for player-I

\[
\mu_{O_{i_a}}(p) = 1 \quad \text{if } g_n(p) \geq \alpha \\
0 \quad \text{if } g_n(p) < \alpha
\]

\[
\mu_{G_{i_a}}(p) = 1 \quad \text{if } g_n(p) > \alpha \\
0 \quad \text{if } g_n(p) \leq \alpha
\]

\( \alpha \in [0,1] \)

In same manner we can define for player-II

Theorem: 5.8.2

Let \( \{g_n, g\} \subset M^+ = \{g \mid g \geq 0 \text{ on } \rho \} \) and \( \mu_{G_{i_a}} \) in \( \rho \)

1. If \( g_n(p) \to g(p) \) on \( \mu_{E_1} \) then \( \bigcap_{n=1}^{\infty} \left( \mu_{G_{i_a}} \cap \mu_{x_{A_1}y} \right) = \mu_{G_{i_a}} \cap \mu_{x_{A_1}y} \)

2. If \( g_n(p) \to g(p) \) on \( \mu_{E_1} \) then \( \bigcup_{n=1}^{\infty} \left( \mu_{G_{i_a}} \cap \mu_{x_{A_1}y} \right) = \mu_{G_{i_a}} \cap \mu_{x_{A_1}y} \)
Proof:

2. \( \mu_{x_{A_{1y}}}(p) \land \left( \bigvee_{n=1}^{\infty} G_{iA}^n \right) = \mu_{x_{A_{1y}}}(p) \land G_{iA} \)

\[ D_{iA}(p) = 1 \quad \text{if } g_n(p) \to g(p) \]
\[ 0 \quad \text{if } g_n(p) \to g(p) \]

Then \( D_{iA}(p) = 0 \) since \( \mu_{x_{A_{1y}}}(p) \subset D_{iA}(p) \) the equality is obviously true when \( D_{iA}(p) = 0 \)

If \( D_{iA}(p) = 1 \), \( g_n(p) \leq g(p) \)

if \( \bigvee_{n=1}^{\infty} G_{iA}^n(p) = 1 \) there exist \( n \) such that \( G_{iA}^n(p) = 1 \) namely \( g_n(p) > \alpha \)

Hence \( g(p) \geq g_n(p) > \alpha \) that is \( G_{iA}^n(p) = 1 \)

Thus we have \( \mu_{x_{A_{1y}}}(p) \land \left( \bigvee_{n=1}^{\infty} G_{iA}^n \right) \leq \mu_{x_{A_{1y}}}(p) \land G_{iA}(x) \)

On other hand if \( G_{iA}(x) = 1 \) namely \( g(p) > \alpha \) since \( g_n(p) \to g(p) \) there exist \( n_0 \)

such that \( g_{n_0}(p) > \alpha \). That is \( G_{iA}^{n_0}(p) = 1 \) and therefore we have \( \bigvee_{n=1}^{\infty} G_{iA}^n = 1 \)

Thus \( \mu_{x_{A_{1y}}}(p) \land \left( \bigvee_{n=1}^{\infty} G_{iA}^n \right) \geq \mu_{x_{A_{1y}}}(p) \land G_{iA}(x) \).

Theorem: 5.8.3 (Monotone convergence Theorem)

Let \( \{g_n, g\} \subset M^+ = \{g / g \geq 0 \text{ on } \rho \} \) and \( k^{th} \) fuzzy expected pay-off of player-I is

defined by \( \mu_{x_{A_{1y}}}(p) \) in \( \rho \) and if \( g_n(p) \to g(p) \) on \( \mu_{E^k_i}(p) \) then
\[ \int g_n \, d\mu \text{ Converges to } \int g \, d\mu. \]

**Proof:**

Let \( g_n \) converges to \( g \)

We have

\[ \int g_n \, d\mu \leq \int g_{n+1} \, d\mu \leq \int g \, d\mu, \quad n = 1, 2, \ldots \]

It is observed that \( \mu_{x_A \gamma} (p) \subseteq \mathcal{D}_{la}(p) \) then

Where \( D_{la}(p) = 1 \) if \( g_n(p) \rightarrow g(p) \)

\[ 0 \quad \text{if } g_n(p) \rightarrow g(p) \]

\[ \mu_{x_A \gamma} (p) \subseteq \mathcal{D}_{la}(p) \cap G_{la}^n \subseteq \mu_{x_A \gamma} (p) \cap \mathcal{D}_{la}(p) \cap G_{la}^{n+1} \subseteq \mu_{x_A \gamma} (p) \cap \mathcal{D}_{la}(p) \cap G_{la} \]

It follows immediately

\[ \int g_n \, d\mu = \int g_n \, d\mu = \int g_{n+1} \, d\mu = \int g_{n+1} \, d\mu, \quad n = 1, 2, \ldots \]

Assume \( c = \int g \, d\mu \). If \( c = 0 \) then \( 0 \leq \int g_n \, d\mu \leq \int g \, d\mu = 0, \quad n = 1, 2, \ldots \) so that for this case, the conclusion of the theorem is obliviously true. Now suppose that \( c > 0 \), it is clear that

\[ \lim_{n \to \infty} \int g_n \, d\mu \leq c. \]

if we assume that \( \lim_{n \to \infty} \int g_n \, d\mu < c \) then there exist \( c_0 < c \) such that

\[ \lim_{n \to \infty} \int g_n \, d\mu \leq c_0. \]

Hence \( \int g_n \, d\mu \leq c_0 \) for every \( n \),

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\[
\bigcup_{n=1}^{\infty} \left( \mu_{G_{0n} \cap \mu_{\mathcal{A}_{1}}} (p) \right) = \mu_{G_{0n} \cap \mu_{\mathcal{A}_{1}}} (p)
\]

by continuity below \( \lim_{n \to \infty} \left( \mu_{G_{0n} \cap \mu_{\mathcal{A}_{1}}} (p) \right) = \mu_{G_{0n} \cap \mu_{\mathcal{A}_{1}}} (p) \leq c_0 \)

We obtain \( \lim_{n \to \infty} \int g \, d\mu \leq c_0 < c \), this is a contradiction, similarly it can be proved for player-II.

**Theorem 5.8.4 (Monotone convergence Theorem)**

Let \( \{g_n, g\} \subset M^* = \{ g / g \geq 0 \ \text{on} \rho \} \) and \( k^{th} \) fuzzy expected pay-off of player-I is defined by \( \mu_{\mathcal{A}_{1}} (p) \) in \( \rho \) and if \( g_n (p) \to g(p) \) (continuity from above) on \( \mu_{\mathcal{A}_{1}} (p) \) then

\[
\int g_n \, d\mu_{\mathcal{A}_{1}} \text{ Converges to } \int g \, d\mu_{\mathcal{A}_{1}}
\]

**Proof:** Suppose that \( g_n (p) \to g(p) \) (continuity from above) on \( \mu_{\mathcal{A}_{1}} (p) \)

then by theorem 6.18 \( \int g \, d\mu_{\mathcal{A}_{1}} \leq \int g_n \, d\mu_{\mathcal{A}_{1}} \leq \int g_n \, d\mu_{\mathcal{A}_{1}} \), let \( c = \int g \, d\mu_{\mathcal{A}_{1}} \)

When \( c = \infty \) obviously \( \int g_n \, d\mu_{\mathcal{A}_{1}} \leq \int g_n \, d\mu_{\mathcal{A}_{1}} \leq \infty \) we see that conclusion of the theorem is evidently true.

In the following we assume that \( c < \infty \) clearly \( \lim_{n \to \infty} \int g_n \, d\mu_{\mathcal{A}_{1}} \geq c \)

Suppose

\[
\lim_{n \to \infty} \int g_n \, d\mu > c, \ c_0 > c \text{ such that } \lim_{n \to \infty} \int g_n \, d\mu > c_0 \text{ then } \int g_n \, d\mu > c_0
\]

By using theorem \( \mu_{G_{0n} \cap \mu_{\mathcal{A}_{1}}} (p) \geq c_0 \) for every \( n \).
by theorem $\bigcap_{n=1}^{\infty} \left( \mu_{G_{1n}} \cap \mu_{x_{A, y}}(p) \right) = \mu_{G_{1c}} \cap \mu_{x_{A, y}}(p)$.

By continuity from above of $g$ and condition

$$\left( \mu_{G_{1c}} \cap \mu_{x_{A, y}}(p) \right) \leq \left( \mu_{G_{1c}} \cap \mu_{x_{A, y}}(p) \right) < \infty .$$

$$\lim_{n \to \infty} \left( \mu_{G_{1n}} \cap \mu_{x_{A, y}}(p) \right) = \left( \mu_{G_{1c}} \cap \mu_{x_{A, y}}(p) \right) \geq c_0 .$$

Thus $\int g \, d\mu \geq c_0 > c$.

This is a contradiction. It can be proved similarly for player-II.