CHAPTER 3

FUZZY MATRIX

3.1 Introduction

This Chapter mainly discussed to study the matrix game with fuzzy pay-offs via ranking (defuzzication) function approach. A two person zero sum game is taken and every two person zero sum game is equivalent to two linear programming problems which are dual to each other. Thus solving such a game amounts to solving any one of two mutually dual linear programming problems and obtaining solution of the other by using duality theory. There are two cases of fuzziness, one is players are having fuzzy goals and the other, in which the elements of the pay-off matrix are given by fuzzy numbers. There is a vast literature in the theory and applications of crisp matrix and some of which have been documented here from Aubin [1], Butnariu [5], campos [7], Chen [11], Liu [35], and Parthasarthy [46].

Definition 3.1.1

A matrix $A= (a_{ij})$ is called a fuzzy matrix if $a_{ij} \in [0, 1]$.

Example: 3.1.2

$$A = \begin{pmatrix} 0.1 & 0.3 & 0.4 \\ 0.5 & 0.7 & 0.01 \\ 0.2 & 0.7 & 0.3 \end{pmatrix}$$

every element in $A$ is in the unit interval $[0, 1]$.

Remark: 3.1.3

1. The interval $[0, 1]$ is called fuzzy interval.

2. All fuzzy matrices are matrices but converse is not true.
3.2 Duality in linear programming

In this section we will discuss certain important results from duality theory of crisp linear programming problem

Let dual of standard linear programming problem (primal problem)

Max $c^T x$ (linear primal)

Subject to $Ax \leq b$, $x \geq 0$,

Min $b^T y$ (linear dual)

Subject to $A^T y \geq c$, $y \geq 0$,

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A$ is an $(m \times n)$ real matrix. The above primal-dual pair is symmetric in the sense that the dual of linear dual –linear primal. Therefore out of these two problems anyone could be solved called a primal and other as its dual.

3.3 Two person zero-sum matrix games

Let $\mathbb{R}^n$ be a $n$-dimensional Euclidean space and $\mathbb{R}^n_+$ be its non-negative orthant.

Let $A \in \mathbb{R}^{m \times n}$ be a $(m \times n)$ real matrix and $e^T = (1, 1, \ldots, 1)$ be a vector of “ones” whose dimension is specified as per the specific context.

By a two person zero –sum matrix game $G$ is a triplet of $S^m$, $S^n$, $A$.

(ie) $G = (S^m, S^n, A)$ where $S^m = \{x \in \mathbb{R}^m_+, / e^T x = 1\}$ and $S^n = \{x \in \mathbb{R}^n_+, / e^T y = 1\}$ be a strategy space for player I and player II respectively and $A$ is called pay-off matrix. Therefore elements of $S^m$ and $S^n$ are of the form $x = \{0, 0, \ldots, 1, \ldots, 0\}^T = e_i$ where $1$ is at the $i^{th}$ place. $y = \{0, 0, \ldots, 1, \ldots, 0\}^T = e_j$ where $1$ is at the $j^{th}$ place are pure strategies of player I and II respectively. If the player I chooses $i^{th}$ pure strategy and player II chooses $j^{th}$ pure strategy,
then $a_{ij}$ is the amount paid by player II to player I. If game is zero sum, then $-(a_{ij})$ is the amount paid by player I to player II. \textit{i.e.} gain of one player is loss of another player.

The meaning of the solution of the game $G = (S^m, S^n, A)$ is best understood in terms of maximin and minimax principles for player I and player II respectively. According to this principle each player adopts that strategy which results in the best of the worst outcomes. In other words, player-I (maximizing player) decides to play that strategy which corresponds to the maximum of the minimum gain for his different courses of action. This is known as maxi-min principle.

Similarly player-II (minimizing player) also likes to play safe and in that case he selects that strategy which corresponds to the minimum of the maximum losses for his different course of action. This is known as mini-max principle.

Employing the maxi-min principle for player I we obtain

\[ v = \max \min \left( x^T A y \right) \text{ called lower value of the game.} \]

Similarly the min-max principle for player-II gives

\[ \bar{\nu} = \min \max \left( x^T A y \right) \text{ called upper value of the game.} \]

It is well known that $\bar{\nu} = \nu$. The main result of two-person zero sum game theory asserts that, in fact, these are equal, \textit{i.e.} $\bar{\nu} = \nu = v^*$ which is called the value of the game.

**Definition 3.3.1**

Let $E : S^m \times S^n \to \mathbb{R}$ given by $E(x, y) = x^T A y$ is called expected pay function

where $S^m$ refers to the mixed strategy of player I and probability distribution over $1, 2, \ldots, m.$

where $S^n$ refers to the mixed strategy of player I and probability distribution over $1, 2, \ldots, n$ and $A$ refers to the pay-off matrix.
**Theorem 3.3.2**

If there exist \((x^*, y^*, v^*)\) in \(S^m \times S^n \in \mathbb{R}\) such that

1) \(E(x^*, y) \geq v^* \quad \forall \ y \in S^n\)

2) \(E(x, y^*) \leq v^* \quad \forall \ x \in S^m\).

Then \(\ddot{v} = \ddot{v} = v^*\) and conversely.

**Definition 3.3.3 (Saddle Point)**

Let \(E: S^m \times S^n \rightarrow \mathbb{R}\) given by \(E(x, y) = x^T Ay\). The function \(E\) is said to have saddle point \((x^*, y^*)\) if

\[
E(x^*, y) \geq E(x^*, y^*) \geq E(x, y^*) \quad \forall \ x \in S^m, y \in S^n.
\]

In the view of the above definition we have the following corollary for theorem 3.3.2

**Corollary 3.3.4**

A necessary and sufficient condition that \(\ddot{v} = \ddot{v}\) i.e. \(\min_{y \in S^n} \max_{x \in S^m} x^T Ay = \max_{x \in S^m} \min_{y \in S^n} x^T Ay\) is that the function \(E(x, y)\) has saddle point \((x^*, y^*)\). Here \(v^* = E(x^*, y^*) = v = \ddot{v}\).

Theorem 3.3.2 leads to the following definition of the solution of the game \(G\) if

1) \(E(x^*, y) \geq v^* \quad \forall \ y \in S^n\)

2) \(E(x, y^*) \leq v^* \quad \forall \ x \in S^m\)

Here \(x^*\) is called an optimal strategy for player-I, \(y^*\) is called an optimal strategy for player-II and \(v^*\) called the value of the game \(G\).

In the view of theorem 3.3.2 and corollary: 3.3.4 \((x^*, y^*, v^*)\) is a solution of the game \(G\) if and only if \((x^*, y^*)\) is a saddle point of \(E\) and in that case \(v^* = E(x^*, y^*)\). Such a saddle point is guaranteed to exist if \(\ddot{v} = \ddot{v}\) and conversely. Here it may be noted that only
the existence of \((\tilde{x}, \tilde{y})\) in \(S^m \times S^n\) such that \(\min_{y \in S^n} \max_{x \in S^m} x^T Ay = \max_{x \in S^m} \min_{y \in S^n} x^T Ay = \tilde{x}^T \tilde{A} \tilde{y}\), is not a sufficient condition in order that \((\tilde{x}, \tilde{y})\) be solution of the matrix game \(G\), this may not imply that \((\tilde{x}, \tilde{y})\) constitutes an optimal pair of strategies.

**Example 3.3.5**

If \(G = (S^2, S^2, A)\) with \(A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\) then \(v = v = 1\).

Also \(x^* = \left(\frac{1}{2}, \frac{1}{2}\right)^T = y^*\) constitutes a saddle point of \(E\) and therefore a pair of optimal strategies. However \(\tilde{x} = \left(\frac{1}{2}, \frac{1}{2}\right)^T\), \(\tilde{y} = (1, 0)^T\) also gives \(E(\tilde{x}, \tilde{y}) = 1\) but \(\tilde{y}\) is obviously not optimal to player-II. The main reason being that \((x^*, y^*)\) is a saddle point of \(E(x, y)\) but \((\tilde{x}, \tilde{y})\) is not a saddle point.

Next we assume existence of a solution for the game \(G\). The following theorem is very fundamental in this context as it asserts that every two-person zero sum matrix game \(G\) always has a solution.

**Theorem: 3.3.6 (Fundamental theorem of matrix game)**

Let \(G = (S^m, S^n, A)\), then \(\min_{y \in S^n} \max_{x \in S^m} x^T Ay\) and \(\max_{x \in S^m} \min_{y \in S^n} x^T Ay\) both exists and are equal. Here the problem \(\max_{x \in S^m} \min_{y \in S^n} x^T Ay\) (respectively \(\min_{y \in S^n} \max_{x \in S^m} x^T Ay\) ) is called player I’s (respectively player II’s) problem. If there exists \((i_0, j_0) \in I \times J\) so that \(a_{i_0,j_0} \geq a_{i,j_0} \geq a_{i_0,j}\) for all \(i\) and \(j\) then \((i_0, j_0)\) is called a pure saddle point and in that case we say that the game \(G\) has a solution in the pure form.

In this situation \(\left(\min_{y \in S^n} \max_{x \in S^m} a_{ij}\right) \left(\max_{x \in S^m} \min_{y \in S^n} a_{ij}\right) = a_{i_0,j_0}\).
Therefore \((i_o, j_o)\) gives an optimal strategy of player I and II respectively and \(a_{i_o,j_o}\) becomes the value of the game \(G\). In this case it may be noted that \(a_{i_o,j_o}\) is the smallest element in the \(i_o\)th row and the largest element in the \(j_o\)th column.

The above theorem 3.3.2 guarantees that every two person zero sum matrix game \(G\) has a solution. If there is no solution in the pure form then there is certainly a solution in the mixed form.

### 3.4 Linear Programming and matrix game equivalence

We shall now establish a relation between two person zero sum matrix game \(G = (S^m, S^n, A)\) and a pair of primal-dual linear programming problems.

Let us consider the player I’s (respectively Player II’s) problem

\[
\max x^T Ay \quad \min s^T y
\]

(respectively \(\min x^T Ay \quad \max s^T x\)). Since \(S^m\) and \(S^n\) are compact convex sets and for a given \(x\) (respectively given \(y\)), the function \(E(x, y)\) is a linear function of \(y\) (respectively \(x\)), the \(\min x^T Ay\) (respectively \(\max x^T Ay\)) will be attained at an extreme point of \(S^n\) (respectively \(S^m\))

Therefore \(\forall \ x \in S^m, \ \min x^T Ay = \min y \in S^n x^T Ae_j\)

where \(e_j = (0, 0, \ldots, 1, \ldots, 0)^T\) with 1 at the \(j\)th place, is the \(j\)th pure strategy of player-II

Thus \(\max \min x^T Ay = \max \min x \in S^m y \in S^n (\sum_{i=1}^{m} a_{ij} x_i)\)

if we take \(v = \min_{i \leq j \leq n} (\sum_{i=1}^{m} a_{ij} x_i)\), then the maximin value for player I is obtained by solving the following linear programming problem
\[
\begin{align*}
\text{max } & \quad v \\
\text{subject to } & \quad \sum_{i=1}^{m} a_{ij} x_i \geq v \quad (j = 1, 2, \ldots, n) \\
& \quad e^T x = 1, \quad x \geq 0.
\end{align*}
\]

Similarly the mini-max value for player II is obtained as a solution of the following linear programming problem

\[
\begin{align*}
\text{min } & \quad w \\
\text{subject to } & \quad \sum_{j=1}^{n} a_{ij} y_j \leq w \quad (i = 1, 2, \ldots, m) \\
& \quad e^T y = 1, \quad y \geq 0.
\end{align*}
\]

where \( w = \max_{1 \leq i \leq m} \left( \sum_{j=1}^{n} a_{ij} y_j \right) \).

Now it can be verified that both (1) and (2) constitute a primal-dual pair of linear programming problems. Since both maximin and minimax are attained, these two LPP’s have optimal solutions \((\bar{x}, \bar{y})\) and therefore by linear programming duality, the optimal values of (1) and (2) will be equal. Let this common value be \(\bar{v}\). 1 and 2 have been constructed in that way it is obvious that \(\sum_{i=1}^{m} a_{ij} \bar{x}_i \geq \bar{v} \quad (j = 1, 2, \ldots, n)\) and \(\sum_{j=1}^{n} a_{ij} \bar{y}_j \leq \bar{v}\)

\( (i = 1, 2, \ldots, m)\)

\[e^T A \bar{y} \leq \bar{v}\]

implying that \( x^T A \bar{y} \geq \bar{v}\).

For all \( y \in S^n \) and \( x^T A \bar{y} \leq \bar{v} \) for all \( \forall \; x \in S^m \).

The above discussion then leads to the following equivalence theorem.
Theorem 3.4.1

The triplet \((\hat{x}, \hat{y}, \hat{v}) \in S^m \times S^n \times R^n\) is a solution of the game \(G\) if and only if \(\hat{x}\) is optimal to (1), \(\hat{y}\) is optimal to (2) and \(\hat{v}\) is the common value of linear programming problem (1) and (2).

Thus we concluded that matrix game \(G = (S^m, S^n, A)\) is equivalent to the primal-dual linear programming problems (1) and (2).

3.5 Duality in linear programming under fuzzy environment

Two person zero sum game have discussed using duality in the last section and the same can extent in the fuzzy environment so that fuzzy matrix game can be solved. It is known that classical linear programming duality can be understood in terms of the maximin and the minimax problems of the associated Lagrangian function \(L(x, u)\).

\[
\text{Max } c^T x
\]

Subject to \(Ax \leq b, x \geq 0\).

We associate the \(L: R^n \times R^m \rightarrow R\) is given by \(L(x, u) = c^T x + u^T (b-Ax)\) and linear primal and its dual as maximin \(L(x, u)\) and minimax \((x, u)\) respectively. This can be simplified to

\[
\text{Min } b^T u
\]

Subject to \(A^T u \geq c, u \geq 0\).

The classical maximin and minimax problems of Zimmermann [68], which are associated with \(L(x, u)\). We take this into fuzzy environment. In the crisp case for every decision say \(x\) of primal (say, industry and denote it by I) there exist a decision \(u\) of the dual (say, market and denote it by M) and vice versa. By fuzzifying the case, there is a fuzzy set on the solution space \(X\), that is, to each \(x\) in \(X\), there corresponds a grade of membership which represents the satisfaction of the decision marker I with decision \(x\). Also corresponding to each \(x\) in \(X\)
there exist a grade of membership for decision \( u \), which represents the satisfaction of the decision maker \( I \) with the decision \( u \).

Thus we have a fuzzy set \( \{(x, \mu^I(x)) / x \in X\} \) on \( X \) and a family of fuzzy sets \( \{\{(x, \mu^I_x(u)) / u \in U\} \) on \( U \) with the parameter \( x \) in \( X \). Here we assume \( \mu^I(x) \) is the membership function on \( I \) on \( x \) and \( \mu^I_x(u) \) is the membership function on \( I \) on \( u \) for any given \( x \), where \( x, u \geq 0 \).

Now we define another fuzzy set on \( U \) is called mixture of \( \mu^I \) and \( \mu^I_x \) having the membership function as

\[
v^I(u) = \max \min (\mu^I(x), \mu^I_x(u)).
\]

This mixture having membership function \( v^I(u) \), demands the family of decisions \( x(u) \) such that for each decision \( u \geq 0 \) of the competitor (M), the optimum of \( \mu^I \) and \( \mu^I_x \) is reached. Similarly if \( \mu^M(u) \) is the membership function of M on \( u \) and \( \mu^M_x(x) \) is the membership function of M on \( x \) for any given \( u \), then for M, we have to consider the mixture of fuzzy set \( \mu^M \) and \( \mu^M_x \), we get max min (\( \mu^M(x), \mu^M_x(x) \)).

\[
\mu^I(x) = \begin{cases} 
1, & c^T x^0 \leq c^T x \\
1 - (c^T x^0 - c^T x), & \text{otherwise}
\end{cases}
\]

\[
\mu^I_x(x) = \begin{cases} 
0, & u^T (b - Ax) \leq 0 \\
u^T (b - Ax), & \text{otherwise}
\end{cases}
\]

\[
\mu^M(x) = \begin{cases} 
1, & b^T u \leq b^T u^0 \\
1 - (b^T u - b^T u^0), & \text{otherwise}
\end{cases}
\]

and

\[
\mu^M_x(x) = \begin{cases} 
0, & x^T (c - A^T u) \leq 0 \\
x^T (c - A^T u), & \text{otherwise}
\end{cases}
\]
Here $c^Tx^0$ is the aspiration level of I and $b^Tu^0$ is the aspiration level of M. Substituting the above membership functions $\mu^I, \mu^I_x, \mu^M, \mu^M_x$ as above, we obtain the following linear programming problems for I and M respectively.

Max $\lambda_1$  
(fuzzy primal)

Subject to $\lambda_1 \leq (1 - (c^Tx^0 - c^Tx))$,

$\lambda_1 \leq (u^T(b - Ax))$ for any $x, u \geq 0$.

Min $-\lambda_2$  
(fuzzy dual)

Subject to $\lambda_2 \geq ((b^Tu - b^Tu^0) - 1),

\lambda_2 \leq (x^T(c - A^Tu))$ for any $x, u \geq 0$.

Here $\lambda_1 = \min (\mu^I(x), \mu^I_x(u))$ and $-\lambda_2 = \min (\mu^M(u), \mu^M_x(x))$. The above pair is called fuzzy primal-dual pair of linear programming problems.

3.6 Matrix game with fuzzy goals

Two person zero sum game and duality in linear programming under fuzzy environment has discussed in the section 3.4 and 3.5. Therefore let $S^m, S^n$ and $A$ be the matrix game, let $v_0, w_0$ be a scalars representing the aspiration level of player I and II respectively, then two person zero sum matrix game with fuzzy goals denoted by

$\text{FG} = (S^m, S^n, A, v_0, \geq, w_0, \leq)$ where $\leq$ and $\geq$ are defined in the crisp sense, max and min are strict imperative. (Constraints might be vague the sign $\leq$ might not be meant in the strict mathematical sense but subjectively determined violations may be acceptable. For example the decision maker might say try to contact 1300 customers but it will be too bad if less than 1200 customers are contacted). Where $\geq, \leq$ are fuzzified version therefore fuzzy game are fixed only when specific choice of membership function are made to define $\geq, \leq$ in Zimmerman [68].
Let \( t \) be a variable and \( a \) in \( \mathbb{R} \). Let \( p > 0 \) then the fuzzy set \( F \) defining the fuzzy statement \( t \sim_p a \) to be read as “\( t \) essentially greater than or equal to \( a \) with tolerance error \( p \)” is to be understood in terms of the following membership function

\[
\mu_p(t) = \begin{cases} 
1 & \text{if } t \geq a \\
1 - \frac{a - t}{p} & \text{if } a - p \leq t \leq a \\
0 & \text{if } t < (a - p)
\end{cases}
\]

**Theorem: 3.6.1**

Let \( t_1 \sim_p a, \ t_2 \sim_p a, \ \alpha, \beta \geq 0, \ \alpha + \beta = 1, \Rightarrow \alpha t_1 + \beta t_2 \sim_p a. \)

**Proof:** Relations \( t_1 \sim_p a \) and \( t_2 \sim_p a \) can be written as

\[
t_1 \geq a - (\lambda - 1)p \\
t_2 \geq a - (\lambda - 1)p \quad \text{then} \quad \alpha t_1 + \beta t_2 \geq (\alpha + \beta) a - (\lambda - 1) (\alpha + \beta) p
\]

Which gives \((\alpha t_1 + \beta t_2) \sim_p a. \)

In the view of the above discussion we include the tolerance \( p_0 \) and \( q_0 \) for player-I and player-II respectively in our definition of the fuzzy game \( FG \) and therefore take

\[
FG = (\mathbb{S}^m, \mathbb{S}^n, \mathbb{A}, \mathbb{V}_0, \geq, \ p_0, w_0, \leq, q_0).
\]

We now define the meaning of the solution of the fuzzy matrix game \( FG \).

**Definition: 3.6.2 (Solution of the fuzzy matrix game FG)**

A point \((\tilde{x}, \tilde{y})\) is called a solution to the fuzzy matrix game \( FG \) if
\[
\begin{pmatrix} x \end{pmatrix}^T Ay \geq p_0 \quad \forall \quad y \in S^n \quad \text{and} \quad x^T Ay \leq q_0 \quad \forall \quad x \in S^m.
\]

Since \( S^m \) and \( S^n \) are convex polytopes, for the choice of membership functions of type \( \mu_f(t) \) by theorem 3.6.1 guarantees that in the definition 3.6.2.

It is sufficient to consider only the extreme points (pure strategies) of \( S^m \) and \( S^n \).

The following model is based on the maximin and minimax principles of the crisp matrix game theory, for any pair \((x, y)\) in \( S^m \times S^n \), \( x^T Ay \) is the expected pay-off and the solution of the game is defined via the maxi-min and mini-max principles.

Now we define the meaning of fuzzy goal and we try to explain how the players will play the game in a fuzzy environment.

**Definition 3.6.3 (Fuzzy goal)**

Let \( D = \{ x^T Ay / x \in S^m, y \in S^n \} \subseteq \mathbb{R} \), Then a fuzzy goal for player I is a fuzzy set on \( D \) characterised by the membership function \( \mu_1 : D \rightarrow [0, 1] \). Similarly, a fuzzy goal for player II is also a fuzzy set on \( D \), characterized by a membership function

\[
\mu_2 : D \rightarrow [0, 1].
\]

A membership function value for a fuzzy goal can be interpreted as the degree of attainment of the fuzzy goal for the pay-off. Therefore when a player has two different pay-offs, he prefers possessing the higher membership function value in comparison to the other. It means that player I aims to maximize the degree of attainment for his fuzzy goal.

We assume that player I supposes that player II will choose a strategy \( y \) so as to minimize player I degree of attainment of the fuzzy goal. Assuming that player I chooses \( x \) in \( S^m \), the least degree of attainment of his goal will be \( v(x) = \min_{y \in S^n} \mu_1(x^T Ay) \). Hence player I will choose a strategy so as to maximize his degree of attainment of the fuzzy goal \( v(x) \).
In short we assume that player I behaves according to the maximin principle in terms of degree of attainment of his fuzzy goal. Similar argument holds for player II as well.

**Definition: 3.6.4 (Maxi-min value)**

The maxi-min value with respect to a degree of attainment of the fuzzy goal for player I is defined as \( \max_{x \in S} \min_{y \in \mathcal{Y}} \mu_1(x^T Ay) \) similarly maxi-min value with respect to a degree of attainment of the fuzzy goal for player-I is defined as \( \max_{x \in S} \min_{y \in \mathcal{Y}} \mu_1(x^T Ay) \) and for player II is defined as \( \max_{x \in S} \min_{y \in \mathcal{Y}} \mu_2(x^T Ay) \).

Here it may be noted that in Hannan [19] for player II, the mini-max value is suggested but later it has been corrected to maxi-min value for player-II as it should be. Thus player I (respectively player II) wishes to determine \( x^* \) in \( S^m \) (respectively \( y^* \) in \( S^n \)) such that the maxi-min value with respect to the degree of attainment of the fuzzy goal for player I (respectively). This is another interpretation of the definition 3.6.4 for fuzzy game. We analyse the optimization problem for player I and player II so as to obtain a solution of the given fuzzy game. For this game we assume that membership functions of fuzzy goals for player I and II are linear.

**3.6.5 Optimization problem for player I**

Consider the membership function of the fuzzy goal for player-I is defined as

\[
\mu_1(x^T Ay) = \begin{cases} 
0 & \text{if } x^T Ay \leq a \\
1 - \frac{\tilde{a} - x^T Ay}{\tilde{a} - a} & \text{if } a \leq x^T Ay \leq \tilde{a} \\
1 & \text{if } \tilde{a} < x^T Ay
\end{cases}
\]
Where \( a \) & \( \bar{a} \) are pay-offs giving best and worst degree of satisfaction to player-I. Assume 

the parameter \( a = \min_x \min_y x^T Ay = \min_x \min_j a_{ij} \) and \( \bar{a} = \max_x \max_y x^T Ay = \max_x \max_j a_{ij} \)

**Theorem: 3.6.6**

For two person zero sum fuzzy game FG let the membership function \( \mu_1 \) for player-I be linear as described above. Then player-I’s maxi-min solution with respect to the degree of attainment of fuzzy goal is obtained by solving the following linear programming problem

Max \( \lambda \)

subject to

\[
\sum_{i=1}^{m} \frac{a_{ij}}{\bar{a} - a} \left( x_i - \frac{a}{\bar{a} - a} \right) \geq \lambda \quad (j = 1, 2, \ldots, n)
\]

\[
e^T x = 1, \quad x \geq 0
\]

**Proof:** The maximin problem for player I is \( \max_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^n} (x^T Ay) \)

Which can be transformed into

\[
\max_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^n} \left( 1 - \frac{\bar{a} - x^T Ay}{\bar{a} - a} \right) = \max_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^n} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j + c \right)
\]

\[
= \max_{x \in \mathbb{R}^n} \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} x_i + c \right) y_j
\]

\[
= \max_{x \in \mathbb{R}^n} \left( \sum_{i=1}^{m} a_{ij} x_i + c \right)
\]
Where \( \tilde{a}_{ij} = \frac{a_{ij}}{a-a} \) and \( c = \frac{a}{a-a} \) therefore taking \( \min_j \left( \sum_{i=1}^m \tilde{a}_{ij}x_i + c \right) = \lambda \) the maximin problem for player I reduces to the desired linear programming problem.

### 3.6.7 Optimization problem for player II

Next, we consider player II maximin solution with respect to the degree of attainment of his fuzzy goal. The membership function of the fuzzy goal for player-II is defined

\[
\mu_2(x^T Ay) = \begin{cases} 
0 & \text{if } x^T Ay \leq a \\
1 - \frac{x^T Ay - a}{a - a} & \text{if } a \leq x^T Ay \leq \tilde{a} \\
1 & \text{if } \tilde{a} < x^T Ay.
\end{cases}
\]

**Theorem 3.6.8**

For two person zero sum fuzzy game FG let the membership function \( \mu_2 \) for player-II be linear as described above. Then player-II maximin solution with respect to the degree of attainment of fuzzy goal is obtained by solving the following linear programming problem

\[
\min \lambda
\]

Subject to

\[
\sum_{j=1}^n \frac{a_{ij}y_i}{a-a} - \frac{a}{a-a} \leq \lambda \quad (j = 1, 2, \ldots, n) \\
e^T y = 1, \; y \geq 0
\]
**Proof:** The maxi-min problem for player II is \( \max \min_{\mu \in \mathbb{R}} \mu \gamma (x^T A y) \).

Which can be transformed into

\[
\max \min_{\mu \in \mathbb{R}} \left\{ 1 - \frac{x^T A y - a}{a - a} \right\} = \max \min_{\mu \in \mathbb{R}} \left\{ -\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j y_j + 1 - c \right\}
\]

\[
= \max \min_{\mu \in \mathbb{R}} \left\{ -\sum_{j=1}^{n} a_{ij} y_j + 1 - c \right\}
\]

Where \( a_{ij} = a_{ij}^{\sim} \) and \( c = -\left( \frac{a}{a - a} \right) \). The strategy \( y^* \) satisfying the above is obtained by solving the following linear programming problem

\[
\text{Max } \eta \quad \text{subject to } \sum_{j=1}^{n} a_{ij}^{\sim} y_j - \frac{a}{a - a} \leq 1 - \eta (i = 1, 2, \ldots, m)
\]

\[
e^T y = 1, \ y \geq 0
\]

Taking \( 1 - \eta = \lambda \) is equivalent to the linear programming problem given in the statement of the theorem.

**3.6.9 Matrix game with fuzzy pay-offs**

In the last section matrix game with fuzzy goals discussed. An outline procedure to find the solution for the same also studied. In this section matrix game with fuzzy pay-offs will be taken, which has earlier studied by campos [7]. Fuzzy pay-offs can be extended to the bi-matrix game Hannan [19] in the next chapter for the multi-objective situation. Using
defuzzication function, we establish certain duality for linear programming with fuzzy parameters and employs the same to solve matrix games with fuzzy pay-offs.

### 3.6.10 Definitions and preliminaries

Let \( G = (S^m, S^n, A) \) be a crisp matrix game. We need to understand the concept of double fuzzy constraints which are expressed as fuzzy inequalities involving fuzzy numbers.

For this let \( N(R) \) be the set of all fuzzy numbers. Let \( \tilde{A}, \tilde{b}, \tilde{c} \) respectively be \((m \times n)\) matrix, \((m \times 1)\) and \((n \times 1)\) vector having entries from \( N(R) \) and the double fuzzy constraints under consideration be given by

\[
\begin{align*}
\tilde{A}x & \leq_{p} \tilde{b} \\
\tilde{A}^T y & \leq_{q} \tilde{c}
\end{align*}
\]

with adequacies \( \tilde{p}, \tilde{q} \) respectively.

Based on resolution method proposed by Yager [59], the constraints \( \tilde{A}x \leq_{p} \tilde{b} \) is expressed as

\[
\tilde{A}_i x \leq_{p} b_i + p_i (1 - \lambda), \quad \lambda \in [0, 1]
\]

where \( i = 1, 2, \ldots, m \). The \( i^{th} \) component of the fuzzy vector \( \tilde{p} \), namely \( \tilde{p}_i \), measures the adequacy between the fuzzy numbers \( \tilde{A}_i x \) and \( \tilde{b}_i \) which are the \( i^{th} \) component of the fuzzy vectors \( \tilde{A} x \) & \( \tilde{b} \).

Similarly \( \tilde{A}^T y \geq_{q} \tilde{c} \) is expressed as \( \tilde{A}_j y \geq_{q} c_j - q_j (1 - \eta), \quad \eta \in [0, 1] \) where for \( j = 1, 2, \ldots, n \) the \( j^{th} \) component of the fuzzy vector \( \tilde{q} \), namely \( \tilde{q}_j \), measures the adequacy between the fuzzy numbers \( \tilde{A}_j y \) and \( \tilde{c}_j \), which are the \( j^{th} \) component of the fuzzy vectors \( \tilde{A} y \) & \( \tilde{c} \) respectively.

Here \( \leq_{p}, \geq_{q} \) are relations between fuzzy numbers, which preserve the ranking, when fuzzy numbers are multiplied by positive scalars.
Example: 3.6.11

Let ranking function $F: N(R) \to R$ be defined by $F(\tilde{a}) \leq_p F(\tilde{b})$ if $\tilde{a} \leq_p \tilde{b}$, the function $F$ is used to defuzzify the given fuzzy linear programming problems. Here onwards it is called as defuzzification rather than a ranking function.

Therefore, the double fuzzy constraints of the type $\tilde{A} x \leq_p \tilde{b}$ and $\tilde{A}^T y \leq_p \tilde{c}$ are to be considered as $\tilde{A}_i x \leq_p \tilde{b}_i + p_i(1-\lambda)$, $\lambda \in [0,1]$ for $i=1, 2, ..., m$.

$$
\tilde{A}_j y \geq_q \tilde{c}_j - q_j(1-\eta)$, $\eta \in [0,1]$ for $j=1, 2, ..., n.$

$$
F(\tilde{A}_i x) \leq F(\tilde{b}_i) + F(p_i(1-\lambda)).
$$

$$
F(\tilde{A}^T_j y) \geq F(\tilde{c}_j) - F(q_j(1-\eta)).
$$

Let $\tilde{a}_j$, $\tilde{b}_i$, $p_i$, $\tilde{c}_j$ and $q_j$ are TFN and $F$ is the Yager [59] first index given by

$$
F(D) = \frac{\int_{d_u}^{d_d} x\mu_D(x)dx}{\int_{d_d}^{d_u} \mu_D(x)dx}.
$$

Where $d_d$ and $d_u$ are the lower and upper limits of the support of fuzzy number $D$. Then for the special case of TFN the constraints are $\tilde{A} x \leq_p \tilde{b}$ and $\tilde{A}^T y \leq_p \tilde{c}$ respectively mean

$$
\sum_{j=1}^{n} (a_{ij} + a_j) x_j \leq ((b_1)_i + b_1 + (b_2)_i) + (1-\lambda)((p_1)_i + p_1 + (p_2)_i)
$$

and

$$
\sum_{j=1}^{n} (a_{ij} + a_j) x_j \leq ((b_1)_i + b_1 + (b_2)_i) + (1-\lambda)((p_1)_i + p_1 + (p_2)_i)
$$

and
\[ \sum_{j=1}^{n} \left( (a_{ij} + a_{ii} + (a_{ij})^n) y_j \right) \geq ((c_i) + c_i + (c_i) + (1 - \eta)(q_i) + q_i + (q_i)) \cdot \]

For \( i=1,2,\ldots,m, \lambda \in [0,1] \) and \( j=1,2,\ldots,n, \eta \in [0,1] \).

\[ a_g = ((a_{ij}), a_{ii}, (a_{ij})^n), b_i = ((b_i), b_i, (b_i)_{(n)}), p_i = ((p_i), p_i, (p_i)_{(n)}), q_i = ((q_i), q_i, (q_i)_{(n)}) \]

\[ c_j = ((c_i), c_i, (c_i)_{(n)}) \] are TFN.

### 3.7 Two person zero sum matrix games with fuzzy pay-offs

Two person zero sum matrix game with fuzzy pay-offs is the triplet \( FG (S^m, S^n, \tilde{A}) \). We now define meaning of the solution of fuzzy matrix game \( FG \).

**Definition: 3.7.1 (Reasonable solution of the game \( FG \))**

Let \( \tilde{v}, \tilde{w} \) in \( N(R) \) then \( \tilde{v}, \tilde{w} \) is called a reasonable solution of the fuzzy matrix game \( FG \) if there exist \( x^* \) in \( S^m \) and \( y^* \) in \( S^n \) satisfying

\[ (x^*)^T \tilde{A} y \geq \tilde{v} \quad \forall \, y \in S^n \quad \text{and} \quad x^T Ay^* \leq \tilde{w} \quad \forall \, x \in S^m \]

If \( \tilde{v}, \tilde{w} \) is reasonable solution of \( FG \) then \( \tilde{v} \) (respectively \( \tilde{w} \)) is called a reasonable value for player-I (respectively player-II).

**Definition: 3.7.2 (Solution of the game \( FG \))**

Let \( T_1 \) and \( T_2 \) be the set of all reasonable values of \( \tilde{v} \) and \( \tilde{w} \) in \( N(R) \). Let there exist

\[ \tilde{v}^* \in T_1 \] and \[ \tilde{w}^* \in T_2 \] such that \( F(\tilde{v}^*) \geq F(\tilde{v}) \) for every \( \tilde{v} \) in \( T_1 \).

and \( F(\tilde{w}^*) \leq F(\tilde{w}) \) for every \( \tilde{w} \) in \( T_2 \).
Then \((x^*, y^*, \tilde{v}, \tilde{w})\) is called the solution of the game \(FG\) where \(\tilde{v}\) (respectively \(\tilde{w}\)) is the value of the game \(FG\) for player I (respectively player-II) and \(x^*\) (respectively \(y^*\)) is called an optimal strategy for player I (respectively Player II).

By using the above definitions for the game \(FG\) we now construct the following pair of fuzzy linear programming problems for player I and player II

\[
\begin{align*}
\text{Max} & \quad F(\tilde{v}) & \text{fuzzy primal} & (1) \\
\text{Subject to} & \quad x^T\hat{A}y \geq_{p} \tilde{v} \quad \forall y \in S^n & x \in S^m. \\
\text{min} & \quad F(\tilde{w}) & \text{fuzzy dual} & (2) \\
\text{Subject to} & \quad x^T\hat{A}y \leq_{q} \tilde{w} \quad \forall x \in S^m & y \in S^n.
\end{align*}
\]

Recalling the explanation of the double fuzzy constraints as explained in the section 3.6.9 and noting the relation \(\leq_p\) and \(\geq_q\) preserve the ranking, when fuzzy numbers are multiplied by positive scalars, it makes sense to consider only the extreme points of sets \(S^m\) and \(S^n\) in the constraints of the above (1) and (2) fuzzy pay-offs. Therefore the above problems (1) and (2) will be converted into

\[
\begin{align*}
\text{Max} & \quad F(\tilde{v}) \\
\text{Subject to} & \quad \begin{cases} 
  x^T\hat{A}_j \geq_{p} \tilde{v} & (j = 1, 2, \ldots, n) \\
  e^T x = 1, x \geq 0
\end{cases}
\end{align*}
\]
\[
\min \quad F(\tilde{w}) \\
\text{Subject to } \begin{cases}
\tilde{A}_j y \geq \tilde{w} & (i = 1, 2, \ldots, m) \\
\mathbf{e}^T y = 1, \quad y \geq 0
\end{cases}
\]

Here \( \tilde{A}_i \) (respectively \( \tilde{A}_j \)) denotes the \( i \)th row (respectively \( j \)th column) of \( \tilde{A} \) \((i=1, 2, \ldots, m)\) and \( j=(1, 2, \ldots, n) \).

By using the resolution procedure for the double fuzzy constraints in the fuzzy pay-offs we obtain

\[
\max \quad F(\tilde{v}) \\
\text{Subject to } \begin{cases}
\sum_{i=1}^{m} \tilde{a}_{ij} x_i \geq \tilde{v} - (1-\lambda) \tilde{p} & (j = 1, 2, \ldots, n) \\
\mathbf{e}^T x = 1, \quad \lambda \leq 1, \quad x, \lambda \geq 0.
\end{cases}
\]

\[
\min \quad F(\tilde{w}) \\
\text{Subject to } \begin{cases}
\sum_{i=1}^{m} \tilde{a}_{ij} y_j \leq \tilde{w} + (1-\eta) \tilde{q} & (i = 1, 2, \ldots, m) \\
\mathbf{e}^T y = 1, \quad \eta \leq 1, \quad y, \eta \geq 0.
\end{cases}
\]

Now by using the defuzzication function \( F: \mathbb{N} \rightarrow \mathbb{R} \) for the above constraints (3) and (4) theses problems be written as

\[
\max \quad F(\tilde{v}) \\
\text{Subject to } \begin{cases}
\sum_{i=1}^{m} F(\tilde{a}_{ij}) x_i \geq F(\tilde{v}) - (1-\lambda)F(\tilde{p}) & (j = 1, 2, \ldots, n) \\
\mathbf{e}^T x = 1, \quad \lambda \leq 1, \quad x, \lambda \geq 0.
\end{cases}
\]
\[
\min \ F(\tilde{w})
\]
Subject to
\[
\sum_{j=1}^{m} F(a_{ij}) y_j \geq F(w) + (1 - \eta)F(q) \quad (1 = 1, 2, ..., m)
\]
\[
e^T y = 1, \quad \eta \leq 1, \quad y, \eta \geq 0.
\]

From the above discussion we observe that for solving the fuzzy matrix game FG we have to solve the above crisp linear programming problems (5) and (6) for player-I and player-II respectively. Also if \((x^*, \lambda^*, v^*\) is an optimal solution of (5) then for player-I, \(x^*\) is an optimal strategy, \(\tilde{v}^*\) is the fuzzy value and \((1 - \lambda^*) \bar{p}\) is the measure of adequacy level for the double fuzzy constraints in double fuzzy constraint (5). Similar interpretation can also be given to an optimal solution \((x^*, \eta^*, \tilde{w}^*\) of the problem (6).

**Theorem: 3.7.3**

The fuzzy matrix game FG described by FG \((S^m, S^n, \tilde{A})\) is equivalent to two crisp linear programming problems (5) and (6) which constitutes a primal pair in the sense of duality for linear programming with fuzzy parameters, if \((x, \lambda)\) be a feasible solution for (5) and \((y, \eta)\) be a feasible solution for (6), then

\[
F(\tilde{v}^T y) - F(\tilde{w}^T y) \geq (1 - \lambda) F(\tilde{p}^T y_j) - (1 - \eta) F(q^T x_i).
\]

**Proof:**

Since \((x, \lambda)\) be a feasible solution for (5) and \((y, \eta)\) be a feasible solution for (6), we have

\[
\sum_{i=1}^{m} F(a_{ij}) x_i \geq F(\tilde{v}) - (1 - \lambda) F(\tilde{p}) \quad \text{and}
\]
\[
\sum_{i=1}^{m} F(a_i) y_j \geq F(w) + (1 - \eta) F(q)
\]

Now because of properties of relations \( \leq_p \) and \( \geq_q \) the defuzzication function \( F \) preserves the ranking when fuzzy numbers are multiplied by non-negative scalars, the above relations imply

\[
F(x^T A^T y) \geq F(v^T y_j) - (1 - \lambda) F(p^T y_j) \quad \text{and}
\]

\[
F(y^T A^T x) \geq F(w^T x_i) - (1 - \eta) F(q^T x_i).
\]

Therefore

\[
F(v^T y_j) - (1 - \lambda) F(p^T y_j) \leq F(w^T x) - (1 - \eta) F(q^T x_i).
\]

Because \( F(x^T A^T y) = F(y^T A^T x) \) imply \( x^T A^T y = y^T A^T x \)

Combining the above we obtain

\[
F(v^T y) - F(w^T y) \geq (1 - \lambda) F(p^T y_j) - (1 - \eta) F(q^T x_i).
\]