CHAPTER 2

FUZZY NUMBER AND FUZZY ARITHMETIC

2.1 Introduction

Fuzzy arithmetic or arithmetic of fuzzy numbers is generalisation of interval arithmetic, where rather than considering intervals at one constant level only, several levels are considered in [0, 1]. This is because, the definition of a fuzzy set which allows degree of membership for an element of the universal set. It plays an important role in many applications, say, fuzzy control, decision making, approximate reasoning, optimization and statistics with imprecise probabilities. Some appropriate reference for this chapter are Dubois and Prade [14] and [15], Kaufmann and Gupta [21] and [22] and Moore [39].

2.2 Fuzzy arithmetic

2.2.1 Interval arithmetic

The fundamentals of fuzzy arithmetic is nothing but interval arithmetic. In a given closed interval R, how to add, subtract, multiply and divide. These interval in R is also called an interval of confidence as its limits the uncertainty of data to an interval.

Let A = \([a_1, a_2]\) and B = \([b_1, b_2]\) be two closed interval in R then we have following definitions.

**Definition: 2.2.2 (Addition and Subtraction)**

If \(x \in [a_1, a_2]\) and \(y \in [b_1, b_2]\),

\[
\text{then } x + y \in [a_1 + b_1, a_2 + b_2]\text{ and } x - y \in [a_1 - b_1, a_2 - b_2]
\]
therefore

\[ A + B = [a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]. \]

\[ A - B = [a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]. \]

**Example: 2.2.3**

\[ [2, 5] + [1, 3] = [3, 8]. \]

\[ [0, 1] - [-6, 5] = [-5, 7]. \]

**Definition 2.2.4 (Image of an interval)**

If \( x \in [a_1, a_2] \) then its image \( -x \in [-a_2, -a_1] \). Therefore the image of \( A \) is denoted by \( \overline{A} \) is defined as

\[ \overline{A} = [a_1, a_2] = [-a_2, -a_1]. \]

**Definition: 2.2.5 (Multiplication)**

The multiplication of two closed intervals \( A = [a_1, a_2] \) and \( B = [b_1, b_2] \) of \( \mathbb{R} \) denoted by \( A \cdot B \) is defined as

\[ A \cdot B = [\min (a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2), \max (a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)]. \]

**Example: 2.2.6**

\[ [-1, 1] \cdot [-2, 0.5] = [-\min(2, 0.5, -2, -0.5), \max(2, 0.5, -2, -0.5)] \]

\[ = [-2, 2]. \]
Definition: 2.2.7 (Scalar Multiplication and Inverse)

Let $A = [a_1, a_2]$ be a closed interval in $\mathbb{R}$ and $k \in \mathbb{R}$ identifying the scalar $k$ as the closed interval $[k, k]$, the scalar multiplication $k \cdot A$ is defined as

$$k \cdot A = [k, k] \cdot [a_1, a_2] = [ka_1, ka_2].$$

for $A = [a_1, a_2] \in \mathbb{R}$ if $x \in [a_1, a_2]$ and if $0 \not\in [a_1, a_2]$ then $\frac{1}{x} \in \left[ \frac{1}{a_2}, \frac{1}{a_1} \right].$

Therefore the inverse of $A$ is denoted by $A^{-1}$ and it is defined as

$$A^{-1} = [a_1, a_2]^{-1} = \left[ \frac{1}{a_2}, \frac{1}{a_1} \right] \text{ provided } 0 \not\in [a_1, a_2].$$

Definition: 2.2.8 (Division)

The division of two closed intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ of $\mathbb{R}$ denoted by $A/B$ is defined as the multiplication $[a_1, a_2]$ and $\left[ \frac{1}{b_2}, \frac{1}{b_1} \right]$ provided $0 \not\in [b_1, b_2].$

Therefore $A/B = [a_1, a_2] / [b_1, b_2].$

$$= [a_1, a_2] \left[ \frac{1}{b_2}, \frac{1}{b_1} \right].$$

$$= \left[ \min \left( \frac{a_1}{b_2}, \frac{a_2}{b_1} \right), \max \left( \frac{a_1}{b_2}, \frac{a_2}{b_1} \right) \right].$$

Example: 2.2.9

$$[4, 10] / [1, 2] = [2, 10].$$
Arithmetic operations on closed intervals satisfy some useful properties, to overview, let \( A = [a_1, a_2] \), \( B = [b_1, b_2] \), \( C = [c_1, c_2] \), \( 0 = [0, 0] \) and \( 1 = [1, 1] \), using these symbols, the properties are formulated as follows:

1. \( A+B = B+A \) and \( A \cdot B = B \cdot A \) (Commutative).
2. \( (A+B)+C = A+ (B+C) \) and \( A \cdot (B.C) = (A \cdot B) \cdot C \) (Associative).
3. \( A=0+A = A+0 \) and \( A=1 \cdot A = A \cdot 1 \) (Identity).
4. \( A \cdot (B+C) \subseteq A \cdot B+A \cdot C \) (Subdistributivity).
5. \( A \cdot B \leq C \) for every \( b \in B, \ c \in C \), then \( A \cdot (B+C) = A \cdot B + A \cdot C \) (Distributive).

Furthermore, if \( A= [a, a] \) then \( a \cdot (B+C) = a \cdot B +a \cdot C \).
6. \( 0 \) in \( A-A \) and \( 1 \) in \( A/A \).
7. \( A \subseteq E \) and \( B \subseteq F \) then \( A + B \subseteq E + F \), \( A - B \subseteq E - F \), \( A \cdot B \subseteq E \cdot F \), \( A / B \subseteq E / F \) (Inclusion monotonicity).

As an example we prove only the less obvious properties of Subdistributivity. First we have

\[
A \cdot (B+C) = \{a \cdot (b+c) / a \in A, \ b \in B, \ c \in C \}.
\]

\[
= \{a \cdot b+ a \cdot c / a \in A, \ b \in B, \ c \in C \}.
\]

\[
\subseteq \{ a \cdot b+a \cdot c/ a \cdot a \in A, \ b \in B, \ c \in C \}= A \cdot B + A \cdot C.
\]

Hence \( A \cdot (B+C) \subseteq A \cdot B+A \cdot C \).

Assume now without loss of generality, that \( b_1 \geq 0, c_1 \geq 0 \). Then we have to consider the following three cases:

1. If \( a_1 \geq 0 \) then \( A \cdot (B+C) = [a_1 \cdot (b_1+c_1), \ a_2 \cdot (b_2+c_2)] \).

\[
= [a_1 \cdot b_1+ a_2 \cdot b_2] + [a_1 \cdot c_1, \ a_2 \cdot c_2].
\]
\[ = A \cdot B + A \cdot C. \]

2. If \( a_1 < 0 \) and \( a_2 \leq 0 \) then \(-a_2 \geq 0\), \((-A)=[-a_2, -a_1]\) and
\[ (-A) \cdot (B+C) = (-A) \cdot B + (-A) \cdot C. \]
Hence \( A \cdot (B+C) = A \cdot B + A \cdot C. \)

3. If \( a_1 < 0 \) and \( a_2 > 0 \) then \( A \cdot (B+C) = [a_1 \cdot (b_2+c_2), a_2 \cdot (b_2+c_2)]. \)
\[ = [a_1 \cdot b_2, a_2 \cdot b_2] + [a_1 \cdot c_2, a_2 \cdot c_2]. \]
\[ = A \cdot B + A \cdot C. \]

To show that distributivity does not hold in general,

Let \( A = [0, 1], B = [1, 2], C = [-2, -1], \)
then \( A \cdot B = [0, 2], A \cdot C = [-2, 0], B+C = [-1, 1] \) and
\[ A \cdot (B+C) = [-1, 1] \subset [2, 2] = A \cdot B + A \cdot C. \]

**Definition 2.2.10 (Max ∨ and min ∧ operations)**

Let \( A = [a_1, a_2] \) and \( B = [b_1, b_2] \) be two closed intervals in \( R \) then the Max ∨ and
min ∧ operations on \( A \) and \( B \) are defined as
\[ A \vee B = [a_1, a_2] \vee [b_1, b_2] = [a_1 \vee b_1, a_2 \vee b_2]. \]
\[ A \wedge B = [a_1, a_2] \wedge [b_1, b_2] = [a_1 \wedge b_1, a_2 \wedge b_2]. \]

**Note: 2.2.11**

It can be verified that addition and multiplication operations on closed intervals as
defined above are commutative and associative but subtraction and division are neither
commutative nor associative.

\[ A+ A' = [a_1, a_2] + [-a_2, -a_1] \neq [0, 0] = 0. \]
In case \( A = [a_1, a_2] \in \mathbb{R} \) and \( 0 \not\in [a_1, a_2] \), then \( A^{-1} = [a_1, a_2] \neq [1, 1] = 1 \).

**Note: 2.2.12**

In the next section, fuzzy numbers and fuzzy arithmetic are introduced. It is more significant in arithmetic of \( \alpha \)-cuts \( A_\alpha \) which are closed and bounded intervals of type \( A_\alpha = [a^L_\alpha, a^R_\alpha] \alpha \in (0,1] \), when fuzzy set in \( \mathbb{R} \) is a fuzzy number. Therefore all ideas presented in this section can be borrowed for intervals of type \( [a^L_\alpha, a^R_\alpha] \alpha \in (0,1] \) and meaningful arithmetic of fuzzy numbers can be developed.

2.3 **Fuzzy number and their representation**

There are many in real life situation, in which, the areas like decision making and optimization, where rather than dealing with crisp real numbers and crisp intervals one has to deal with approximation of numbers which are close to a given real number.

The purpose of this section is to understand, how fuzzy statement can be conceptualized by certain appropriate fuzzy sets in \( \mathbb{R} \) to be termed as fuzzy numbers.

Let us consider fuzzy statement “the numbers that are closed to a given number \( r \)” since \( r \) in \( \mathbb{R} \) is close to itself, any fuzzy set \( A \) in \( \mathbb{R} \) which represents the fuzzy statement should have property that \( \mu_A(r) = 1 \). Which imply \( A \) must be a normal fuzzy set. Also, just prescribing an interval around \( r \) is not enough. The interval should be considered at varying levels \( \alpha \in (0,1] \) to have proper gradation of \( \alpha \)-cuts in \( A \) and it must be closed interval of type \( [a^L_\alpha, a^R_\alpha] \) for \( \alpha \in (0,1] \). Further \( \alpha \in (0,1] \) must be of finite length and for that one needs that support of \( A \) is bounded.
Definition 2.3.1 (Fuzzy number)

A fuzzy set \( A \) in \( \mathbb{R} \) is called fuzzy number if it satisfies the following conditions:

1. \( A \) is normal fuzzy set.
2. \( A_\alpha \) is closed interval for every \( \alpha \in (0, 1] \).
3. Support of \( A \) is bounded.

Theorem 2.3.2

Let \( A \) be a fuzzy set in \( \mathbb{R} \) then \( a \) is a fuzzy number if and only if there exists a closed interval (which may be singleton) \([a, b]\) \( \neq \emptyset \) such that

\[
\mu_A(x) = \begin{cases} 
1 & \text{if } x \in [a, b] \\
\ell(x) & \text{if } x \in (-\infty, a) \\
r(x) & \text{if } x \in (b, \infty)
\end{cases}
\]

Where

i) \( \ell : (-\infty, a) \to [0, 1] \) is monotonic increasing, continuous from the right such that

\( \ell(x) = 0 \) for \( x \) in \( (-\infty, k) \), \( k_1 < a \).

ii) \( r : (b, -\infty) \to [0, 1] \) is monotonic decreasing, continuous from the left such that \( r(x) = 0 \) for \( x \) in \( (k_2, \infty) \), \( k_2 > b \).

Note: 2.3.3

In case the membership function of the fuzzy set \( A \) in \( \mathbb{R} \) takes the form

\[
\mu_A(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{if } x \neq a
\end{cases}
\]
It becomes the characteristic function of the singleton set \{a\} and therefore it represents the real number \(a\). The following figure exhibits both continuous function \(\ell(x)\) and \(r(x)\) are having continuity in the membership.

![Graph showing \(\ell(x)\) and \(r(x)\) with continuity in membership](image)

**Note: 2.3.4**

Fuzzy set \(A\) in \(X\) is a convex set if and only if all its \(\alpha\)-cuts \(A_\alpha\) are convex crisp sets for \(\alpha \in [0, 1]\). Further when the membership function of the convex fuzzy set \(A\) in \(R\) is upper semi-continuous, then all these \(\alpha\)-cuts \(A_\alpha\) for \(\alpha \in (0, 1]\) are closed intervals. Since the basic requirement to define a fuzzy number is that all it \(\alpha \in (0, 1]\) are closed and bounded intervals. The following is an alternate definition of fuzzy number.

**Definition: 2.3.5**

A fuzzy subset \(A\) of the real line \(R\) with membership function \(\mu_A : R \to [0, 1]\) is called a fuzzy number if

1. \(A\) is normal there exists an element \(x_0 \in A\) such that \(\mu_A(x_0) = 1\).

2. \(A\) is convex, \(\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \mu_A(x_1) \land \mu_A(x_2) \land (x_1, x_2) \in R\) and \(\lambda \in [0, 1]\).
3. is upper semi-continuous.

4. Support of A is bounded, where Supp A = \{x \in R: \mu_A(x) > 0\}.

2.4 Arithmetic of fuzzy numbers

Arithmetic of fuzzy numbers are defined by the following methods:

1. Interval arithmetic on \(\alpha\)-cuts of given fuzzy numbers.

2. Mathematical approach which decomposes a fuzzy set A in terms of a special fuzzy set \(\alpha A_\alpha\), \(\alpha \in (0, 1)\).


2.4.1 Approach on \(\alpha\)-cuts

Let A and B be any two fuzzy numbers and \(A_\alpha = [a_\alpha^l, a_\alpha^r]\) and \(B_\alpha = [b_\alpha^l, b_\alpha^r]\) be a \(\alpha\)–cuts where \(\alpha \in (0, 1)\) of A and B respectively. Let * denote any arithmetic operations \(\wedge, \vee\) on fuzzy numbers, then we have following definitions:

**Definition 2.4.2 (Operations on two fuzzy numbers)**

Let A and B be any two fuzzy sets. Let \(A_\alpha\) and \(B_\alpha\) be fuzzy numbers, then * operation on fuzzy numbers gives a fuzzy numbers in \(\mathbb{R}\)

\[ A \ast B = \bigcup_\alpha \alpha (A \ast B)_\alpha \text{ and } (A \ast B)_\alpha = A_\alpha \ast B_\alpha, \alpha \in (0, 1]. \]

Here \(A \ast B\) be a fuzzy number but not general fuzzy set. The sets \((A \ast B)_\alpha\), \(A_\alpha\), \(B_\alpha\), are all closed intervals for \(\alpha \in (0, 1]\). Also for given in \((0, 1]\), the closed interval \((A \ast B)_\alpha\) can be computed by applying the interval arithmetic on the closed intervals \(A_\alpha\) and \(B_\alpha\) with respect to the operations *.
In particular, \( A_\alpha + B_\alpha = [a_\alpha^L + b_\alpha^L, a_\alpha^R + b_\alpha^R] \).

\[
A_\alpha - B_\alpha = [a_\alpha^L - b_\alpha^L, a_\alpha^R - b_\alpha^R].
\]

Further, fuzzy number \( A \) in \( \mathbb{R} \), \( k > 0 \) can again be defined as earlier in the context of interval arithmetic \( (kA_\alpha)_\alpha = kA_\alpha = [ka_\alpha^L, ka_\alpha^R] \).

### 2.5 Approach using Zadeh’s extension principle

Let \( A \) and \( B \) two fuzzy numbers and \( * \) be any of the arithmetic operations described above. Using extension principle, fuzzy number \( A*B \) is defined as

\[
\mu_{A*B}(z) = \sup_{z=x+y} \min(\mu_A(x), \mu_B(y)) \quad \forall z \in \mathbb{R}.
\]

In particular we have

\[
\mu_{A+B}(Z) = \sup_{z=x+y} \min(\mu_A(x), \mu_B(y));
\]

\[
\mu_{A-B}(z) = \sup_{z=x-y} \min(\mu_A(x), \mu_B(y));
\]

\[
\mu_{A*B}(z) = \sup_{z=x \cdot y} \min(\mu_A(x), \mu_B(y));
\]

\[
\mu_{A \div B}(z) = \sup_{z=x \div y} \min(\mu_A(x), \mu_B(y)).
\]

### 2.5.1 Special types of fuzzy numbers and their arithmetic

Some special types of fuzzy numbers and fuzzy arithmetic are discussed here, which will be used extensively in later Chapters on fuzzy mathematical programming and fuzzy games.
**Definition 2.5.2 (Triangular fuzzy number (TFN))**

A fuzzy number $A$ is called a triangular fuzzy number (TFN) if its membership function $\mu_A$ is defined by

$$
\mu_A(x) = \begin{cases} 
0 & x < a_l, \ x > a_u \\
\frac{x-a_l}{a-a_l} & a_l \leq x \leq a \\
\frac{a_u-x}{a_u-a} & a < x \leq a_u 
\end{cases}
$$

The TFN $A$ is denoted by triplet ($3$-tuples) $A = (a_l, a, a_u)$ and has the shape of a triangle shown below.

Further the $\alpha-$cuts of the TFN $A = (a_l, a, a_u)$ is the closed interval.

$$
A_\alpha = [a_{l\alpha}, a_{u\alpha}] = ((a-a_l)\alpha + a_l, -(a_u-a)\alpha + a_u) \quad \alpha \in (0, 1].
$$

Next $A = [a_l, a, a_u]$ and $B = [b_l, b, b_u]$ two TFN then using one can compute $A*B$ where operation $*$ may be $+$, $-$, $\cdot$, $\div$, $\&$, $\wedge$, $\vee$.

In this context

$$
A + B = (a_l + b_l, a + b, a_u + b_u),
$$

$$
-A = (-a_u, -a - a_l).
$$
\[ kA = (ka_l, ka, ka_u), \quad k > 0, \]

\[ A - B = (a_l - b_u, a - b, a_u - b_l). \]

are TFN but \( A \cdot B, \quad A^{-1}, \quad A / B, \quad A \cup B, \quad A \cap B \) need not be a TFN.

**Example: 2.5.3 [21, 22]**

Let \( A = (-3, 2, 4), B = (-1, 0, 5) \) be two TFN’s then using the formulae for the addition and subtraction of TFN’s

\[
A + B = (-3, 2, 4) + (-1, 0, 5) = (-4, 2, 9).
\]

\[
A - B = (-3, 2, 4) - (-1, 0, 5) = (-8, 2, 5).
\]

Using the \( \alpha \) - cuts \( A_\alpha \) and \( B_\alpha \) for the given fuzzy number \( A \) and \( B \) we have

\[
A_\alpha = [a_{\alpha}^{L}, a_{\alpha}^{R}] = [(a - a_{\alpha}^{L}) \alpha + a_{\alpha}^{L}, -(a_{\alpha}^{R} - a) \alpha + a_{\alpha}^{R}] \quad \alpha \in (0, 1].
\]

\[
= [2 + 3\alpha - 3, -2\alpha + 4] \quad \alpha \in (0, 1].
\]

\[
= [5\alpha - 3, -2\alpha + 4] \quad \alpha \in (0, 1].
\]

and \( B_\alpha = [b_{\alpha}^{L}, b_{\alpha}^{R}] = [(b - b_{\alpha}^{L}) \alpha + b_{\alpha}^{L}, -(b_{\alpha}^{R} - b) \alpha + b_{\alpha}^{R}] \quad \alpha \in (0, 1].
\]

\[
= [\alpha - 1, -5\alpha + 5] \quad \alpha \in (0, 1].
\]

Therefore \( A_\alpha + B_\alpha = [5\alpha - 3, -2\alpha + 4] + [\alpha - 1, -5\alpha + 5] \quad \alpha \in (0, 1].
\]

\[
= [6\alpha - 4, -7\alpha + 9] = [c_{\alpha}^{L}, c_{\alpha}^{R}] \quad \text{(say)}, \quad \alpha \in (0, 1].
\]

To find the membership function of \( \mu_{A+B}(x) \), we have to find the range \( x \) in \( R \) where \( \alpha \) level sets are valid. Thus \( c_{\alpha}^{L}, \)
\[ x = \begin{cases} 
6\alpha - 4 & \text{if } \alpha = \frac{x + 4}{6}, \text{ from } c^l \alpha \\
-7\alpha + 9 & \text{if } \alpha = \frac{9 - x}{7}, \text{ from } c^r \alpha 
\end{cases} \]

Further \( \alpha \) becomes, for \( x = 2 \) the membership function becomes

\[ \mu_{A \cdot B}(x) = \begin{cases} 
0 & \text{if } x < -4 \text{ or } x > 9 \\
\frac{x + 4}{6} & \text{if } -4 \leq x \leq 2 \\
\frac{9 - x}{7} & \text{if } 2 < x \leq 9.
\end{cases} \]

Which is triangular fuzzy number (TFN) (-4, 2, 9) as shown in the below given diagram

The following example shows \( A \cdot B \) need not be TFN in general:

**Example: 2.5.4 [21, 22]**

Let \( A = (2, 3, 5) \) and \( B = (1, 4, 8) \) be two TFN in \( \mathbb{R} \). As noted earlier one has to compute \( A \cdot B \) by the first principle using \( \alpha \)-cuts \( A_\alpha \) and \( B_\alpha \)

where \( A_\alpha = [\alpha + 2, -2\alpha + 5] \) and \( B_\alpha = [3\alpha + 1, -4\alpha + 8] \).
\[ A_\alpha \cdot B_\alpha = [(\alpha + 2)(3\alpha + 1), (-2\alpha + 5)(-4\alpha + 8)] \]

\[ = [3\alpha^2 + 7\alpha + 2, 8\alpha^2 - 36\alpha + 8] = [c^L_\alpha, c^R_\alpha]. \]

For, \( \alpha = 0, A_0 \cdot B_0 = [2, 40] \) and for \( \alpha = 1, A_1 \cdot B_1 = [12, 12] \).

Therefore membership function \( A \cdot B \) takes 1 for \( x = 12 \) and 0 for \( x < 2 \) and also for \( x > 40 \). Also in between 2 and 12 and also between 12 and 40, the segments of the membership functions are not straight lines but it is a parabola. For this it is found that the range \( x \) in \( R \), in which \( \alpha \) level set are valid. This can be accomplished from \( c^L_\alpha \) by solving

\[ x = 3\alpha^2 + 7\alpha + 2. \]

\[ 3\alpha^2 + 7\alpha + 2 - x = 0. \]

\[ \alpha = \frac{-7 \pm \sqrt{25 + 12x}}{6} \]

Here only + sign will be taken because at \( x = 12, \alpha = 1 \) and \( \alpha \) cannot be negative.

Similarly one has to solve \( 8\alpha^2 - 36\alpha + 8 - x = 0. \)

\[ \alpha = \frac{9 - \sqrt{1 - 2x}}{5} \]

Therefore membership function \( \mu_{A \cdot B} = \begin{cases} 0 & \text{if } x < 2 \text{ or } x > 40 \\ \frac{-7 \pm \sqrt{25 + 12x}}{6} & \text{if } 2 \leq x \leq 12 \\ \frac{9 - \sqrt{1 - 2x}}{5} & \text{if } 12 \leq x \leq 40. \end{cases} \)

Which is not the membership of a TFN and it can be seen from the diagram.
For more details in triangular approximations of fuzzy one can refer Kauffmann and Gupta [20, 21] and Dubois and Prade [14, 15].

**Definition 2.5.5 (Trapezoidal fuzzy number TrFN)**

A fuzzy number $A$ is called a trapezoidal fuzzy number (TrFN) if its membership function $\mu_A$ is given by

$$
\mu_A(x) =
\begin{cases}
0 & \text{if } x < a_l, \ x > a_u \\
\frac{x-a_l}{a-a_l} & \text{if } a_l \leq x \leq a \\
1 & \text{if } a \leq x \leq a \\
\frac{a_u-x}{a_u-a} & \text{if } a < x \leq a_u.
\end{cases}
$$

The TrFN $A$ is denoted by quadruplet (4-tuples) $A = \left[ a_l, a, a, a_u \right]$ and has the shape of a trapezoid.
The TrFN $A$ is denoted by quadruplet $A = (a_l, a_-, a_u, a_u)$ and has the shape of a trapezoid.

Further $\alpha$-cut of TrFN $A = (a_l, a_-, a_u, a_u)$ is the closed interval.

$$A_{\alpha} = [a^L_{\alpha}, a^R_{\alpha}] = \left( (a_l - a_l) \alpha + a_l, -(a_u - a_l) \alpha + a_u \right)$$ where $\alpha \in [0, 1]$.

Next $A = (a_l, a_, a_u)$ and $B = (b_l, b_, b_u)$ two TFN’s then using one can compute $A \ast B$ where operation $\ast$ may be $+, -, \ast, /, \land, \lor$

In this context it can be verified

$$A + B = \left( a_l + b_l, a_+ + b_-, a_l + b_u \right),$$

$$A - B = \left( a_l - b_l, a_+ - b_-, a_u - b_u \right)$$ and

$$kA = \left( ka_l, ka_-, ka_k, ka_u \right) \text{ for } k > 0.$$

are TrFN but $A^{-1}$, $A \cdot B$, $A / B$, $A \land B$ and $A \lor B$ need not be TrFN.
Definition 2.5.6 (Left – Right fuzzy number)

A fuzzy number $A$ is called an L-R fuzzy number if its membership function $\mu_A: X \rightarrow [0, 1]$ has the following form:

$$
\mu_A(x) = \begin{cases} 
L \left( \frac{x-a}{\alpha} \right) & \text{if } a - \alpha \leq x \leq a, \quad \alpha > 0 \\
1 & \text{if } a \leq x \leq b \\
R \left( \frac{x-b}{\beta} \right) & \text{if } b < x \leq b + \beta, \quad \beta > 0 \\
0 & \text{otherwise.}
\end{cases}
$$

The L-R fuzzy number $A$ as described above and it is denoted by $A = (a, b, \alpha, \beta)_{LR}$. Here $L$ and $R$ are called starting as the left and right reference functions, $a$ and $b$ are respectively called starting and end points of flat interval, $\alpha$ is called the left spread and $\beta$ is called the right spread. The general shape of an L-R fuzzy number will as follows:

Let $A(a_1, b_1, \alpha, \beta)_{LR}$ and $B(a_2, b_2, \gamma, \delta)_{LR}$ be two L-R fuzzy numbers then it can be verified that
\[ A + B = (a_1 + a_2, b_1 + b_2, \alpha + \gamma, \beta + \delta)_{LR}. \]

\[ A - B = (a_1 - b_2, b_1 - a_2, \alpha + \delta, \beta + \gamma)_{LR}. \]

Further, similar to TFN and TrFN, \( A^{-1}, A.B, A/B \) are not L-R fuzzy numbers in general and it needs certain L-R approximations if they are to be used as approximate L-R fuzzy numbers.

2.6 Ranking of fuzzy numbers

Ranking of fuzzy number plays important role in fuzzy mathematical programming and fuzzy games.

Let \( N(R) \) be the set of all fuzzy numbers in \( R \) and \( A, B \) in \( N(R) \).

Let \( F: N(R) \rightarrow R \), called a ranking function or ranking index is defined and \( F(A) \leq F(B) \) is treated as equivalent to \( A \leq B \). Since \( F(A) = F(B) \), in general, it does not mean \( A=B \). This ranking is to be understood in the sense of equivalence classes only. Yager [59] proposed the following indices

\[ h_1(A) = \frac{\int_{a_1}^{a_2} x\mu_A(x)dx}{\int_{a_1}^{a_2} \mu_A(x)dx} \text{ where } a_1 \text{ and } a_2 \text{ are the lower and upper limits of the support of } A. \]

of \( A \). The value \( h_1(A) \) represents the centroid of the fuzzy number \( A \) in \( N(R) \).

If in particular \( A = (a_1, a, a_2) \) is a TFN, then \( a_1 \) and \( a_2 \) are the lower and upper limits of the support of \( A \). Here \( A \) is the modal value.

In this case by actual substitution of the membership function of the TFN, it can be verified that
\[ h_1(A) = \frac{a_r + a + a_u}{3}. \]

Therefore for given two TFN’s \( A = (a_r, a, a_u) \), \( B = (b_r, b, b_u) \).

\( A \leq B \) with respect to \( h_1 \) if and only if \( (a_r, a, a_u) \leq (b_r, b, b_u) \).

\[ h_2(A) = \frac{\alpha_{\max}}{m[a^L, a^R]} \int_0^{\alpha_{\max}} m[a^L, a^R] d\alpha, \]

where \( \alpha_{\max} \) is the height of \( A \), \( A_{\alpha} = [a^L_{\alpha}, a^R_{\alpha}] \) is an \( \alpha \)-cut, where \( \alpha \) in \((0, 1]\) and \( m[a^L_{\alpha}, a^R_{\alpha}] \) is the mean value of elements of that an \( \alpha \)-cut. For TFN \( A = [a_r, a, a_u] \)

\( \alpha_{\max} = 1 \) and \( A_{\alpha} = [a^L_{\alpha}, a^R_{\alpha}] = ((a - a_r)\alpha + a_r, -(a_u - a)\alpha + a_u) \)

\[ m[a^L_{\alpha}, a^R_{\alpha}] = \frac{(2a - a_r - a_u)\alpha + (a_r + a_u)}{2}. \]

\[ h_2(A) = \frac{a_r + a_u + 2a}{4}. \]

In this view of the above, we conclude that given two TFN \( A = (a_r, a, a_u) \) and \( B = (b_r, b, b_u) \).

\( A \leq B \) with respect to \( h_2 \) if and only if \( (a_r + 2a, a_u) \leq (b_r + 2b, b_u) \).

The applications of the above fuzzy numbers shall be discussed in Chapter 3 and Chapter 5.