CHAPTER 1

INTRODUCTION

1.1 Basic Concept of Fuzzy Sets

The concept of fuzzy set was introduced by L.A. Zadeh in his paper “Fuzzy sets” in 1965. In 1970 Bellman and Zadeh found a class of objects in many situations are not clearly defined. For an instance, instead of describing the weather today in terms of the exact percentage of cloud cover, we could say that it is sunny, which is more uncertain and less precise. The term sunny may have vagueness, it doesn’t exhibit its characteristic. The basic concept of the fuzzy set facilitates both simple and intuitive forms and this lead to the generalization of the classical or crisp set.

The crisp set is defined to dichotomize the individuals in some given universe of discourse into two groups, members and non-members. A sharp, unambiguous distinction exists between the members and non-members of the class is represented by the crisp set, for an instance, a class of tall people, expensive cars, highly contagious diseases numbers much greater than 1 or sunny days, do not exhibit this characteristic. Instead, their boundary seems vague and the transition from member to non-member appears gradual rather than abrupt. Thus the fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse, a value representing its grade of membership in the fuzzy set. This grade corresponds to the degree at which that individuals is similar or compatible with the concept represented by the fuzzy set. Thus, individuals may belong in the fuzzy set to a greater or lesser as indicated by a larger or smaller membership grade. These membership grades are very often represented by real-number values ranging in the closed interval between 0 and 1. Thus a fuzzy set representing our concept of sunny might
assign a degree of membership of 1 to cloud cover of 0 percent, 0.8 to a cloud cover of 20 percent, 0.4 to a cloud cover of 30 percent and 0 to a cloud cover of 75 percent. These grades signify the degree to which each percentage of cloud cover approximates our subjective concept of sunny, and the set itself models the semantic flexibility inherent in such a common linguistic term. Because full membership and non-membership in the fuzzy set can still be indicated by the values of 1 and 0, respectively, we can consider the crisp set to be a restricted case of the more general fuzzy set for which only these two grades of membership are allowed.

The theory of fuzzy sets and the various mathematical representations and measurement of uncertainty and information have virtually unrestricted applicability. Indeed, possibilities for application include any field that examines how we process or act on information, make decisions, recognize patterns, or diagnose problems or any field in which the complexity of the necessary knowledge requires some form of simplification. Successful applications have, in fact, been made in fields as numerous and diverse as engineering, psychology, artificial intelligence, medicine decision theory, pattern recognition, sociology and meteorology.

In Chapter 1, discusses basic definitions such as fuzzy sets, operations on fuzzy sets and its illustrations.

Chapter 2 deals with five main sections, namely fuzzy numbers and their representations, arithmetic of fuzzy numbers, Zadeh’s extension principle, special types of fuzzy numbers, fuzzy arithmetic and ranking of fuzzy numbers.

In Chapter 3, presents a two person zero sum game with fuzzy goals. It is observed that complete equivalence between a matrix game with fuzzy pay-offs via ranking (defuzzication) function and fuzzy goals. But two person zero sum fuzzy matrix games do
not have nature of fuzzy linear programming duality. Here similar to fuzzy linear programming problems, fuzziness in matrix games, can also appear in so many ways, but two cases of fuzziness seem to be very vital which is discussed here. Chapter 3 is divided into five sections, namely duality in linear programming problem, two person zero sum matrix games using fuzzy goals and fuzzy pay-offs, fundamental theorem of matrix game, linear programming and matrix game equivalence.

Chapter 4 deals with basic definitions and operations of bi-matrix. Fuzzy bi-matrix, bi-matrix games with fuzzy pay-offs is introduced. Here, a brief review of the concept of equilibrium solution of bi-matrix game and a fuzzy expected pay-off in a bi-matrix game is defined. The relation between the equilibrium solution and optimal solution to certain mathematical programming problem is established, where all the membership functions of fuzzy goals and reference functions of fuzzy numbers are expressed by linear.

It is proved with two-person non-zero sum bi-matrix game with fuzzy payoffs and fuzzy goals with a new solution concept. A degree of attainment of the fuzzy goal is defined and the maxi-min strategy is employed to examine the fuzzy goal. If all the membership of the fuzzy payoffs and fuzzy goals are linear, then the maxi-min solution is formulated as a non-linear programming to obtain the solution through a relaxation procedure introduced by Charnes and Copper [10].

Chapter-5 deals with bi-matrix game using fuzzy pay-off by upper and lower measures. Extreme measures are used to as a tool to characterize an equilibrium solution of bi-matrix game using fuzzy pay-offs are. It is introduced that aggregation of fuzzy goals using bi-matrix game on L-fuzzy set and the membership function value of the fuzzy goal can be interpreted as a degree of attainment of the fuzzy goal. The degree of attainment of the fuzzy goal by applying the concept of the fuzzy decision by Bellman and Zadeh [3, 4].
is defined. In this Chapter a monotone convergence theorems for a sequence of fuzzy integrals using fuzzy bi-matrix game is proved.

1.2 Basic definitions

Most of the results in this Chapter are without proofs. Some appropriate reference for this Chapter are Dumitrescu, Lazzerini and Jain [13], Klir and Yuan [23], Lin and Lee [34] and Zimmermann [64].

Definition: 1.2.1 (Fuzzy set)

Let X be the universe whose generic element be denoted by x. A fuzzy set A ∈ X is a function A : X → [0, 1].

We use μA for the function A and fuzzy set is characterised by its membership function μA : X → [0, 1].

It associates with each x ∈ X and a real number μA(x) ∈ [0, 1].

A fuzzy set A can be defined by ordered pairs

\{(x, μA(x)) : x ∈ X and μA(x) ∈ [0, 1]\}

Or

A fuzzy set can also be defined by \{(μA(x)/x ∈ X and μA(x) ∈ [0, 1]\}, where symbol ‘/’ is not a division sign but it indicates the top number μA(x) is the membership value of the element x in the bottom. A crisp set or ordinary subset A of X can also be viewed as a fuzzy set in X with membership function as its characteristic function.
Definition: 1.2.2 (Characteristics function or membership function)

A fuzzy set \( A \in X \) is defined by the membership function \( \mu_A(x) \) takes only two values and it is denoted by

\[
\mu_A(x) = \begin{cases} 
1 & x \in A \\
0 & x \notin A
\end{cases}
\]

Definition: 1.2.3 (Support of fuzzy Set)

Let \( A \) be a fuzzy set in \( X \) then the support of \( A \) is denoted by crisp set \( S_A(x) \) and it is defined by

\[
S_A(x) = \{ x \in X / \mu_A(x) > 0 \}.
\]

Definition: 1.2.4 (Normal fuzzy set)

Let \( A \) be a fuzzy set in \( X \), then the height of \( A \) is defined by

\[
h(A) = \sup_x \mu_A(x).
\]

If \( h(A) = 1 \), then the fuzzy set \( A \) is called a normal fuzzy set otherwise it is called subnormal. If \( 0 < h(A) < 1 \), then the subnormal fuzzy set \( A \) can be normalised. It can be made normal by redefining the membership function as \( \mu_A(x) / h(A) \), \( x \in X \).

1.3 Applications of fuzzy sets

Decision making by intersection of fuzzy goals and constraints. Decision making is characterized by selection or choice from alternatives which are available. Consider a simple decision-making model consisting of a goal described by a fuzzy set \( G \) with membership function \( \mu_G(x) \) and a constraint described by a fuzzy set \( C \) with membership function
\( \mu_C(x) \), where \( x \) is an element of the crisp set of alternatives \( A_{alt} \). The decision is a fuzzy set \( D \) with membership function \( \mu_D(x) \), expressed as intersection of \( G \) and \( C \).

\[
D = G \cap C = \{(x, \mu_D(x))\} \text{ where } \mu_D(x) = \min(\mu_G(x), \mu_C(x)) \text{ in } A_{alt}.
\]

Therefore maximizing decision = \( \max \min(\mu_G(x), \mu_C(x)) \)

**Example: 1.3.1**

A professional person, say John, is offered jobs by several companies \( c_1, c_2, \ldots, c_n \). The set of alternatives is defined by \( A_{alt} = \{c_1, c_2, \ldots, c_n\} \). John having the goal of a high salary, job within driving distance, company with future, opportunity for fast advancement etc. John expresses the goal of a high salary by a set \( G \) with membership function \( \mu_G(x) \). He constructs the set of constraints on the membership value according to his judgements. The salaries \( s_1, s_2, \ldots, s_n \) are offered by the company \( c_1, c_2, \ldots, c_n \).

\[
G_{alt} = \{(c_1, \mu_G(s_1)), (c_2, \mu_G(s_2)), \ldots, (c_n, \mu_G(s_n))\}.
\]

Assume that the John must choose one of three jobs offered to him by three different companies \( c_1, c_2 \) and \( c_3 \). The set of alternatives is \( A_{alt} = \{c_1, c_2, c_3\} \). The salaries in dollars per year are given in the table:

<table>
<thead>
<tr>
<th>Company</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( c_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Salary</td>
<td>40,000</td>
<td>35,000</td>
<td>30,000</td>
</tr>
</tbody>
</table>

John has the goal to earn high salary subject to the constraints

1. Interesting job
2. Job within a driving distance
3. Company with future
The constraints are defined by the discrete fuzzy sets. $C_1 = \{(c_1, 0.5), (c_2, 0.7) (c_3, 0.8)\}$, $C_2 = \{(c_1, 0.3), (c_2, 0.8) (c_3, 1)\}$ and $C_3 = \{(c_1, 0.3) (c_2, 0.7) (c_3, 0.5)\}$. The goal $G$ of a high salary is defined by the continuous membership function

$$
\mu_G(x) = \begin{cases} 
0 & \text{if } 0 < x < 25,000 \\
\frac{x - 25,000}{20,000} & \text{if } 25,000 < x < 45,000 \\
1 & \text{if } x \geq 45,000
\end{cases}
$$

For $x$, the salaries corresponding to the alternatives are

$$
\mu_G(40,000) = 0.75 \quad \mu_G(35,000) = 0.5 \quad \mu_G(30,000) = 0.25.
$$

The fuzzy set goal $G_{alt}$ on the set of alternatives is

$$
G_{alt} = \{(c_1, 0.75), (c_2, 0.5), (c_3, 0.25)\}.
$$

The decision is $D = G_{alt} \cap c_1 \cap c_2 \cap c_3$.

$$
= \{(c_1, 0.75), (c_2, 0.5), (c_3, 0.25) \cap (c_1, 0.5), (c_2, 0.7), (c_3, 0.8) \cap (c_1, 0.3), (c_2, 0.8), (c_3, 1) \cap (c_1, 0.3), (c_2, 0.7), (c_3, 0.5)\}.
$$

$$
= \{(c_1, \min (0.75, 0.5, 0.3, 0.3), (c_2, \min (0.5, 0.7, 0.8, 0.7), (c_3, \min (0.25, 0.8, 1, 0.5))\}.
$$

$$
= \{(c_1, 0.3), (c_2, 0.5), (c_3, 0.25)\}.
$$

The maximum membership value in the decision $D$ is 0.5. Hence John has to take the job with company $c_2$ if he wants to satisfy best his objectives.

**Example: 1.3.2**

Let $X = \{30, 50, 70, 90\}$ be possible speed (km/hr) at which cars can cruise over long distances. The fuzzy set $A$ of comfortable speed for long distances may be defined by a certain individual as $\mu(x=30) = 0.5$, $\mu(x=50) = 0.8$, $\mu(x=70) = 1$, $\mu(x=90) = 0.4$. 

This fuzzy set can also be represented as \( A = \{(30, 0.5), (50, 0.8), (70, 1), (90, 0.4)\} \).

**Definition: 1.3.3 (Empty fuzzy Set)**

A fuzzy set \( A \) is empty if its membership function is identically zero.

\[
\mu_A(x) = 0 \quad \text{for every } x \text{ in } X.
\]

**Definition: 1.3.4 (Fuzzy subset)**

A fuzzy set \( A \) is a fuzzy subset of a fuzzy set \( B \)

\[
\text{if } \mu_A(x) \leq \mu_B(x) \quad \text{for every } x \text{ in } X.
\]

**Definition: 1.3.5 (Equality of fuzzy set)**

Two fuzzy set \( A \) and \( B \) are said to be equal

\[
\text{if } \mu_A(x) = \mu_B(x) \quad \text{for every } x \text{ in } X.
\]

**Definition: 1.3.6 (Standard complement)**

The standard complement of a fuzzy set \( A \) is another fuzzy set denoted by \( A' \) whose membership function is given by

\[
\mu_A(x) = 1 - \mu_A(x) \quad \text{for every } x \text{ in } X.
\]

**Definition: 1.3.7 (Standard union)**

The standard union of two fuzzy sets \( A \) and \( B \) is a fuzzy set \( C \) whose membership function is given by

\[
\mu_C = \max\left(\mu_A(x), \mu_B(x)\right) \quad \text{for every } x \text{ in } X.
\]
Definition: 1.3.8 (Standard intersection)

The standard intersection of two fuzzy sets $A$ and $B$ is a fuzzy set $D$ whose membership function is given by

$$
\mu_D = \min(\mu_A(x), \mu_B(x))
$$

for every $x$ in $X$.

Definition of union and intersection can be extended to any finite number of fuzzy sets in an obvious manner, since associativity of max and min operations.

1.4 Properties of Crisp Set or Fuzzy set

1. $A \cup B = B \cup A$ (Commutativity).

2. $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ (Associativity).

3. $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$ (Demorgan’s law).

4. $(A \cup (B \cap C)) = (A \cup B) \cap (A \cap C)$ and $(A \cap (B \cup C)) = (A \cap B) \cup (A \cup C)$

(Distributive).

1.5 Properties of $\alpha$-Cuts

In the following, certain crisp sets, called $\alpha$ – cuts, are introduced for a given fuzzy set $A$ in $X$. These crisp set plays an important role in the study of fuzzy set theory because every fuzzy set $A$ in $X$ can uniquely be represented by a family of such sets associated with $A$. It can be extended to fuzzy arithmetic.

Definition: 1.5.1 ($\alpha$-cut)

Let $A$ be fuzzy set in $X$ and $\alpha \in (0, 1]$. The $\alpha$ – cut of the fuzzy set $A$ is the crisp set
\( A_\alpha = \{ x \in X / \mu_A(x) \geq \alpha \} \).

In Example 1.3.2 \( A_{0.5} = \{ 30, 50, 70 \} \) and \( A_{0.8} = \{ 50, 70 \} \).

For any given \( \alpha_1, \alpha_2 \in (0, 1) \) and \( \alpha_1 \leq \alpha_2 \), one can find \( A_{\alpha_1} \subseteq A_{\alpha_2} \). Hence \( \alpha \)-cuts of any fuzzy sets form families of crisp sets which can be used to represent given fuzzy set \( A \) in \( X \).

1.6 Convex Fuzzy Set

**Theorem: 1.6.1**

Let \( A \) be a fuzzy set in \( X \) with the membership function \( \mu_A(x) \). Let \( A_\alpha \) be the \( \alpha \) cuts of \( A \) and \( \chi_{A_\alpha}(x) \) be the characteristic function of crisp set \( \alpha \in (0, 1] \), then

\[
\mu_A(x) = \sup_{\alpha \in (0, 1]} \left( \alpha \wedge \chi_{A_\alpha}(x) \right) \text{ for each } x \in X.
\]

**Note: 1.6.2**

1. For a fuzzy set \( A \in X \), we can define \( \alpha A_\alpha \), where \( \alpha \in (0, 1] \). The membership function is defined as \( \mu_{\alpha A_\alpha}(x) = (\alpha \wedge \chi_{A_\alpha}(x)) \) for some \( x \) in \( X \).

The above theorem can be expressed in the form of \( A = \bigcup_{\alpha \in \Lambda} (\alpha A_\alpha) \), where \( \bigcup \) denotes typical fuzzy union. This results is called resolution principle of fuzzy sets. The essence of resolution principle is that a fuzzy set \( A \) is decomposed into \( \alpha A_\alpha \) fuzzy sets, \( \alpha \in (0, 1] \).

1.6.3 Convex Fuzzy Sets

Convexity of crisp sets in \( R^n \) is a key factor in crisp mathematical programming and game theory. It is having an application in fuzzy number and fuzzy arithmetic.
**Definition: 1.6.4 (Convex Fuzzy set)**

A fuzzy set $A$ in $\mathbb{R}^n$ is said to be a convex fuzzy set if its $\alpha$-cuts $A_\alpha$ are crisp convex set for every $\alpha$ in $(0, 1]$.

**Theorem: 1.6.5**

A fuzzy set $A$ in $\mathbb{R}^n$ is said to be a convex fuzzy set if and only if for all $x_1, x_2$ in $\mathbb{R}^n$ and $0 \leq \lambda \leq 1$, $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$.

**Remark: 1.6.6**

The convexity of a fuzzy set does not mean that its membership function $\mu_A$ is a convex function in the crisp sense.
If $A$ and $B$ are two convex fuzzy set in $\mathbb{R}^n$, then the intersection is also convex fuzzy set. But union of $A$ and $B$ need not be a convex fuzzy set.

1.6.7 Procedure for Fuzzify a crisp function:

The Zadeh’s extension principle plays a vital role in the fuzzy set theory. It provides a procedure to fuzzify a crisp function or a crisp relation. It facilitates to find a relationship between fuzzy entities and the real life fuzzy system. Its application will be discussed in the next Chapter.
**Definition: 1.6.8**

Let \( f: X \to Y \) be a crisp function and \( F(x) \) (respectively \( F(Y) \)) be the set of all fuzzy set (fuzzy power set) of \( X \), Similarly \( Y \) and \( f(y) \). The function \( f : X \to Y \) induces two functions \( f: F(X) \to F(Y) \) and \( f^{-1}: F(Y) \to F(X) \) and the extension principle of Zadeh gives formulae to compute the membership function of fuzzy sets \( f(A) \) in \( Y \) (respectively \( f^{-1}(B) \) in \( X \)) in terms of membership function of fuzzy set \( A \) in \( X \) (respectively \( B \) in \( Y \)).

1. \[ \mu_{f(A)}(y) = \sup_{x \in X \atop f(x) = y} \mu_A(x) \] for all \( A \in F(X) \).

2. \[ \mu_{f^{-1}(B)}(y) = \mu_B(f(x)) \] for all \( B \in F(Y) \).

This can be extended to set of all \( n \)-tuples in \( X \) to \( Y \).

i.e \( X = X_1 \times X_2 \times \ldots \times X_n \) and \( f : X \to Y \) defined by \( y = f(x_1, x_2, \ldots, x_n) \).

Let \( A_1, A_2, \ldots, A_n \) be a \( n \)-fuzzy sets in \( X_1, X_2, \ldots, X_n \) respectively. The extension principle of Zadeh allows to extend the crisp function \( y = f(x_1, x_2, \ldots, x_n) \) to act on \( n \) fuzzy subsets of \( X \) namely \( A_1, A_2, \ldots, A_n \) so that \( B = f(A_1, A_2, \ldots, A_n) \).

Here the fuzzy set \( B \) is defined by

\[ B = \{(y, \mu_B(y)) \mid y = f(x_1, x_2, \ldots, x_n), (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n \} \]

and \[ \mu_B(y) = \sup_{x \in X \atop f(x) = y} \mu_{A_i}(x_i), \ldots, \mu_{A_n}(x_n) \].

**Example: 1.6.9 (Lin and Lee [34])**

Let \( X = \{-2, -1, 0, 1, 2\} \) and \( A \) be a fuzzy set in \( X \) given by

\[ A = \{(-1, 0.5), (0, 0.8), (1, 1), (2, 0.4)\} \].
Let \( f : X \rightarrow R \) be a function given by \( y = f(x) = x^2 \) from this data one can make the following calculations:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \mu_A(x) )</th>
<th>( y = f(x) = x^2 )</th>
<th>( \mu_B(x) = \mu_{f(A)}(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.5</td>
<td>1</td>
<td>Max (0.5, 1) = 1.</td>
</tr>
<tr>
<td>0</td>
<td>0.8</td>
<td>0</td>
<td>0.8.</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>Max (0.5, 1) = 1.</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>4</td>
<td>0.4.</td>
</tr>
</tbody>
</table>

Here \( x = -1 \) and \( x = 1 \) both are mapped to the point \( y = 1 \) under \( y = x^2 \) and therefore the membership of \( y = 1 \) is taken as Max (0.5, 1) = 1.

Therefore \( B = f(A) = \{(1, 1), (0, 0.8), (4, 0.4)\} \).