Chapter 4

Fractional Burgers Equation

4.1 Introduction

One of the major challenges in the field of complex systems is a thorough understanding of the phenomenon of turbulence. Burgers equation was proposed as a model of turbulent fluid motion by Burgers in a series of several articles, the results of which are collected in [15].

\[ u_t + uu_x = \mu u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \]

with the initial condition \( u(x, 0) = f(x) \). Although it is a special case of the system he originally described, it is this equation that has come to be known as Burgers equation. It is important in a variety of applications, perhaps most notably as a simplification of the Navier-Stokes equation which models fluid dynamics.

When \( k = 0 \) Burgers equation is one of the simplest nonlinear conservation laws and when \( k > 0 \), it is one of the simplest nonlinear dissipative PDEs, due to the resulting decay of energy. With the addition of stochastic forcing, it has played an important role in the theoretical development of stochastic PDEs. These simple settings make Burgers equation a strong mathematical model.

As for as fractional Burgers equation is concerned it describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative in the equation results from the memory effect of the wall friction through the boundary layer. The same form can be found
in other systems such as shallow - water waves and waves in bubbly liquids. Hence
the fractional Burgers equation is used to model the shallow water problems.

Plenty of articles are available in the literature for explaining the behaviour of
Burgers equation. Here we restrict ourselves to fractional Burgers equation.

Miskinis [77] considered the space fractional Burgers equation with Riemann -
Liouville fractional derivative and derived the explicit form of a solution. Existence
of travelling wave solution and conservation laws are considered in this paper.

Yildirim [113] developed a scheme to study numerical solution of the space -
time fractional Burgers equation by the homotopy analysis method. Momani [78]
used the Adomian decomposition method for solving the space - time fractional
Burgers equation. Inc [44] applied the variational iteration method to derive the
numerical and exact solutions of the space - time fractional Burgers equation.

In this chapter, we use Cole - Hopf transformation technique to solve the initial
and boundary value problem for fractional order Burgers equation.

4.2 Initial Value Problem

The initial value problem for the Burgers equation

\[
\frac{\partial^\alpha u}{\partial t^\alpha} + u \frac{\partial u}{\partial x} = k \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, \quad t > 0, \quad (4.2.1)
\]

\[
u(x, 0) = F(x), \quad x \in \mathbb{R}. \quad (4.2.2)
\]

is considered in this section. Using Cole - Hopf transformation, we reduce the
fractional Burgers equation into a fractional linear diffusion equation. First we
write (4.2.1) in a form similar to a conservation law

\[
\frac{\partial^\alpha u}{\partial t^\alpha} + \left( \frac{1}{2} u^2 - k \frac{\partial u}{\partial x} \right)_x = 0. \quad (4.2.3)
\]

We assume that there exists a function \( \psi \) such that

\[
\begin{align*}
u = \frac{\partial \psi}{\partial x}, & \quad (4.2.4) \\
x \frac{\partial u}{\partial x} - \frac{1}{2} u^2 = \frac{\partial^\alpha \psi}{\partial t^\alpha}. & \quad (4.2.5)
\end{align*}
\]
We substitute the value of $u$ from (4.2.4) and (4.2.5) to obtain

$$k \frac{\partial^2 \psi}{\partial x^2} - \frac{1}{2} \psi_x^2 = \frac{\partial^\alpha \psi}{\partial t^\alpha}. \quad (4.2.6)$$

Next we introduce $\psi = -2k \log \phi$ so that

$$u = \frac{\partial \psi}{\partial x} = -\frac{2k}{\phi} \left( \frac{\partial \phi}{\partial x} \right). \quad (4.2.7)$$

This is called the Cole-Hopf transformation, which, by differentiation gives

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{2k}{\phi^2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{2k}{\phi} \frac{\partial^2 \phi}{\partial x^2}$$

and

$$\frac{\partial^\alpha \psi}{\partial t^\alpha} \approx -\frac{2k}{\phi} \left( \frac{\partial^\alpha \phi}{\partial t^\alpha} \right). \quad (4.2.8)$$

The chain rule for Caputo fractional derivative is an infinite series. Hence for computation purpose, it is approximated as above and the remaining terms can be taken as an error called paraproduct error which is of lower order than the main term in some sense. One popular way to make this formula precise is the Bony linearisation formula. This is a part of more general theory called paradifferential calculus. For more details of paradifferential calculus, one can refer the monograph of Taylor [106].

The equation (4.2.6) reduces to the linear diffusion equation

$$k \left[ \frac{2k}{\phi^2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{2k}{\phi} \frac{\partial^2 \phi}{\partial x^2} \right] - \frac{1}{2} \left( \frac{-2k}{\phi^2} \left( \frac{\partial \phi}{\partial x} \right) \right)^2 = -\frac{2k}{\phi} \left( \frac{\partial^\alpha \phi}{\partial t^\alpha} \right).$$

On simplification

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} = k \frac{\partial^2 \phi}{\partial x^2}. \quad (4.2.9)$$

The discussion of above equation is given in detail in the last chapter.

Now we solve (4.2.9) subject to the initial conditions

$$\phi(x, 0) = \Phi(x), \quad x \in \mathbb{R}. \quad (4.2.10)$$

This can be written in terms of initial value $u(x, 0) = F(x)$ by using (4.2.7)

$$F(x) = u(x, 0) = \frac{-2k}{\phi(x, 0)} \left( \frac{\partial \phi(x, 0)}{\partial x} \right).$$
Integrating this result, we get

\[ \phi(x, 0) = \Phi(x) = \exp \left\{ -\frac{1}{2k} \int_0^x F(\tau)d\tau \right\}. \tag{4.2.11} \]

The joint Laplace and Fourier transformation technique can be used to solve the linear initial-value problem \((4.2.9), (4.2.10)\) which is done by Saxena et al. [96].

**Theorem 4.2.1** ([96]). *Consider the one dimensional fractional diffusion equation given by*

\[
\frac{\partial^\alpha N}{\partial t^\alpha} = k \frac{\partial^2 N}{\partial x^2}, \quad x \in \mathbb{R}, \ t > 0, \ 0 < \alpha \leq 1,
\]

\[ N(x, 0) = N_0(x), \quad x \in \mathbb{R}. \]

*The solution of the above equation subject to the initial condition is*

\[ N(x, t) = \int_{-\infty}^{\infty} G(x - \zeta)N_0(\zeta)d\zeta, \]

*where the Green function \(G(x, t)\) is given by*

\[ G(x, y) = \frac{1}{2|x|}H_{1,1}^{1,0} \left[ \frac{|x|}{(kt^\alpha)^{1/2}} \left[ \left(1, \frac{\alpha}{2}\right) \right] \right], \]

*where \(H_{1,1}^{1,0}(z)\) is the H-function.*

Hence, by using the above theorem, the solution of \((4.2.9)\) is given by

\[ \phi(x, t) = \int_{-\infty}^{\infty} \Phi(\zeta) \frac{1}{2|x - \zeta|}H_{1,1}^{1,0} \left[ \frac{|x - \zeta|}{(kt^\alpha)^{1/2}} \left[ \left(1, \frac{\alpha}{2}\right) \right] \right] d\zeta, \]

*where \(\Phi(\zeta)\) is given by \((4.2.11)\).*

Thus

\[ \frac{\partial \phi(x, t)}{\partial x} = \int_{-\infty}^{\infty} \Phi(\zeta) \frac{1}{2|x - \zeta|^2}H_{2,2}^{1,1} \left[ \frac{|x - \zeta|}{(kt^\alpha)^{1/2}} \left[ \left(1, 1, \frac{\alpha}{2}\right) \right] \right] d\zeta. \]
Therefore the solution of the fractional Burgers initial value problem is obtained from (4.2.7) in the form

\[ u(x, t) = -2k \int_{-\infty}^{\infty} \Phi(\zeta) \frac{1}{2|x-\zeta|} \left. H^{1,0}_{1,1} \left( \frac{|x-\zeta|}{(kt)^{1/2}} \left[ \begin{array}{c} (1, \alpha/2) \\ (1, 1) \end{array} \right] \right) \right) d\zeta \]

\[ + \int_{-\infty}^{\infty} \Phi(\zeta) \frac{1}{2|x-\zeta|^2} \left. H^{1,1}_{2,2} \left( \frac{|x-\zeta|}{(kt)^{1/2}} \left[ \begin{array}{c} (1,1)(1, \alpha/2) \\ (1,1,2) \end{array} \right] \right) \right) d\zeta \]

The physical interpretation of this exact solution can hardly be given unless a suitable simple form of \( F(x) \) is specified.

**Special Case:**

If we take \( \alpha = 1 \), then

\[ \phi(x, t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} \exp \left( -\frac{f^2}{2k} \right) d\zeta. \]

Similarly

\[ \frac{\partial \phi(x, t)}{\partial x} = -\frac{1}{4k\sqrt{\pi kt}} \int_{-\infty}^{\infty} \frac{(x - \zeta)}{t} \exp \left( -\frac{f^2}{2k} \right) d\zeta. \]

where

\[ f(\zeta, x, t) = \int_{0}^{\zeta} F(\alpha) d\alpha + \frac{(x - \zeta)^2}{2t}. \]

Therefore the exact solution of the Burgers initial value problem is obtained as

\[ u(x, t) = \int_{-\infty}^{\infty} \frac{(x-\zeta)}{t} \exp \left( \frac{-f^2}{2k} \right) d\zeta \]

\[ + \int_{-\infty}^{\infty} \frac{\Phi(\zeta)}{2\sqrt{\pi kt}} d\zeta, \]

which is the same as the result obtained by Debnath [27].
4.3 Boundary Value Problem

In this section, we solve Burgers equation

\[
\frac{\partial^\alpha u}{\partial t^\alpha} + u \frac{\partial u}{\partial x} = k \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, t > 0, 0 < \alpha \leq 1,
\]

with the following initial and boundary conditions

\[
\begin{align*}
  u(x, 0) &= u_0 \sin \left( \frac{\pi x}{l} \right), \quad 0 < x \leq l, \\
  u(0, t) &= u(l, t) = 0, \quad t > 0.
\end{align*}
\]

(4.3.1) (4.3.2)

It follows from Cole - Hopf transformation (4.2.7) that the problem of solving fractional burgers equation is transformed into the problem of solving fractional diffusion equation

\[
\frac{\partial^\alpha \phi}{\partial t^\alpha} = k \frac{\partial^2 \phi}{\partial x^2}.
\]

The solution to fractional Burgers equation is given by

\[
u(x, t) = \frac{-2k}{\phi} \left( \frac{\partial \phi}{\partial x} \right).
\]

By the above equation and (4.2.11), the initial condition (4.3.1) is transformed into

\[
\phi(x, 0) = \exp \left[ - \left( \frac{u_0 l}{2\pi k} \right) \left( 1 - \cos \frac{\pi x}{l} \right) \right].
\]

so that the boundary conditions (4.3.2) are satisfied.

The standard fractional linear diffusion equation is solved in the last chapter and we recall the solution as

\[
\phi(x, t) = a_0 + \sum_{n=1}^{\infty} a_n E_{\alpha} \left( -(\frac{n\pi}{l})^2 k t^\alpha \right) \cos \left( \frac{n\pi}{l} x \right).
\]

where

\[
\begin{align*}
  a_0 &= \frac{1}{l} \int_0^l \exp \left[ - \left( \frac{u_0 l}{2\pi k} \right) \left( 1 - \cos \frac{\pi x}{l} \right) \right] dx \\
  &= \exp \left( \frac{-u_0 l}{2\pi k} \right) I_0 \left( \frac{u_0 l}{2\pi k} \right)
\end{align*}
\]
and

\[ a_n = \frac{2}{l} \int_0^l \exp \left[ - \left( \frac{u_0 l}{2\pi k} \right) \left( 1 - \cos \frac{\pi x}{l} \right) \right] \cos \left( \frac{n\pi x}{l} \right) \]

\[ = 2 \exp \left( -\frac{u_0 l}{2\pi k} \right) I_n \left( \frac{u_0 l}{2\pi k} \right) \]

where \( I_0 \) and \( I_n \) are the modified Bessel functions of the first kind. Similarly

\[ \frac{\partial \phi(x, t)}{\partial x} = \left( -\frac{n\pi}{l} \right) \sum_{n=1}^{\infty} 2 \exp \left( -\frac{u_0 l}{2\pi k} \right) I_n \left( \frac{u_0 l}{2\pi k} \right) E_{\alpha} \left( -\left( \frac{n\pi}{l} \right)^2 k t^\alpha \right) \sin \left( \frac{n\pi}{l} x \right). \]

Hence the solution of the fractional Burgers equation is given by

\[ u(x, t) = \frac{\left( \frac{4\pi k}{l} \right) \sum_{n=1}^{\infty} n I_n \left( \frac{u_0 l}{2\pi k} \right) E_{\alpha} \left( -\left( \frac{n\pi}{l} \right)^2 k t^\alpha \right) \sin \left( \frac{n\pi}{l} x \right)}{I_0 \left( \frac{u_0 l}{2\pi k} \right) + 2 \sum_{n=1}^{\infty} I_n \left( \frac{u_0 l}{2\pi k} \right) E_{\alpha} \left( -\left( \frac{n\pi}{l} \right)^2 k t^\alpha \right) \cos \left( \frac{n\pi}{l} x \right)} \]

At \( t = 0 \), the value of the denominator of the above equation is \( \exp \left[ \left( \frac{u_0 l}{2\pi k} \right) \cos \left( \frac{\pi x}{l} \right) \right] \).

We use this result combined with initial condition and the value of \( \phi_x(0) \) to verify that the solution satisfies the initial/boundary conditions. We note here that the quantity \( \left( \frac{u_0 l}{2\pi k} \right) = R \) represents the Reynolds number.

Thus fractional Burgers equation with initial/boundary conditions is completely solved.

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