Fractional Black - Scholes Equation

6.1 Introduction

The pricing of options is a central problem in quantitative finance. It is both a theoretical and practical problem since the use of options - bounds in the financial industry nowadays. In option pricing theory, the Black - Scholes equation is one of the most effective models for pricing options. The original derivation of the Black - Scholes equation with time-varying parameters can be found in [74]. A method of reducing this PDE to the heat equation is described in [112]. Black - Scholes model has Gaussian shocks which underestimate the probability of an extreme movement in the stock price than that these model suggests. So more realistic models have been proposed to model the movements in the stock price. One such model is called FMLS (Finite Moment Log Stable) model which falls in the class of Lévy models whose process can be written as fractional partial diffusion type equations. Mainardi et al [65] find the fundamental solution of space - time fractional diffusion equation and expressed it in terms of H-functions. Kumar et al. [53] derived the analytical solution to fractional Black - Scholes equation using Laplace transformation method. The analytical solution of the Black - Scholes equation using Adomian decomposition method is obtained by Bohner et al. [15]. Gülkac [38] used homtopy perturbation method to solve the Black - Scholes equation analytically.

In this chapter, we find the analytical solution of fractional Black - Scholes
equation by using Adomian decomposition method (ADM). Although ADM provides a series solutions, its convergence to exact solution when $\alpha \in \mathbb{N}$ shows the efficiency of the method. Due to the availability of mathematical packages for Mittag-Leffler function, the implementation of this series solution in softwares is easy so that finance problems can be solved exactly. It is observed that the results can be easily extended to both time and space fractional Black-Scholes equation.

This chapter is organised as follows: Section 6.2 has the transformation technique of converting Black-Scholes equation into heat equation. Solution to time fractional Black-Scholes equation is given at Section 6.3 and that for space fractional Black-Scholes equation is given in Section 6.4.

6.2 Black-Scholes Equation

The Black-Scholes equation and boundary conditions for an European call option with value $C(S, \tau)$ is

$$\frac{\partial C}{\partial \tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0,$$

with

$$C(0, \tau) = 0, \quad C(S, \tau) \sim S, \text{ as } S \to \infty,$$

and

$$C(S, T) = \max(S - E, 0),$$

where $\sigma$ is the volatility of the underlying asset, $E$ is the exercise price, $T$ is the expiry time and $r$ is the risk free interest rate.

Equation (6.2.1) is a diffusion equation but each time $C$ is differentiated with respect to $S$ and it is multiplied by $S$ giving non-constant coefficients. So we set

$$S = E e^{x}, \quad \tau = T - \frac{t}{(1/2)\sigma^2}, \quad C = Ev(x, \tau).$$

This results in the equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (k - 1) \frac{\partial v}{\partial x} - kv,$$
where \( k = 2r/\sigma^2 \). The initial condition becomes

\[
v(x, 0) = \max(e^x - 1, 0).
\]

The exact solution of the problem (6.2.2) is

\[
\max(e^x - 1, 0)e^{-kt} + \max(e^x, 0)(1 - e^{-kt}).
\]  

(6.2.3)

In this chapter we solve the time fractional and space fractional Black - Scholes equation where (6.2.2) will become the special case of those equations.

### 6.3 Time Fractional Black - Scholes Equation

In this section, we consider the fractional Black - Scholes option pricing equation of the form

\[
\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^2 v}{\partial x^2} + (k - 1) \frac{\partial v}{\partial x} - kv,
\]

(6.3.1)

with the initial condition \( v(x, 0) = e^x - 1, \ x > 0 \).

For applying the Adomian Decomposition method we rewrite the equation (6.3.1) as

\[
\frac{\partial v}{\partial t} = \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left[ \frac{\partial^2 v}{\partial x^2} + (k - 1) \frac{\partial v}{\partial x} - kv \right].
\]

Now integrating on both sides with respect to \( t \), we get

\[
v(x, t) = v(x, 0) + \int_0^t \left( \frac{\partial^{1-\alpha}}{\partial s^{1-\alpha}} \left[ \frac{\partial^2 v_0(x, s)}{\partial x^2} + (k - 1) \frac{\partial v_0(x, s)}{\partial x} - kv_0(x, s) \right] \right) \, ds.
\]

Here we take \( v(x, 0) \) as \( v_0 \) and

\[
v_1(x, t) = \int_0^t \left( \frac{\partial^{1-\alpha}}{\partial s^{1-\alpha}} \left[ \frac{\partial^2 v_0(x, s)}{\partial x^2} + (k - 1) \frac{\partial v_0(x, s)}{\partial x} - kv_0(x, s) \right] \right) \, ds.
\]

The iterative scheme, for \( n > 1 \), to this problem is given by

\[
v_n(x, t) = \int_0^t \left( \frac{\partial^{1-\alpha}}{\partial s^{1-\alpha}} \left[ \frac{\partial^2 v_{n-1}(x, s)}{\partial x^2} + (k - 1) \frac{\partial v_{n-1}(x, s)}{\partial x} - kv_{n-1}(x, s) \right] \right) \, ds.
\]  

(6.3.2)
Hence, by Adomian decomposition method, the solution to (6.3.1) is given by

\[ v(x, t) = \sum_{n=0}^{\infty} v_n(x, t). \]

Here we consider the initial condition \( v(x, 0) = e^x - 1, \quad x > 0; \) hence \( v_0 = e^x - 1. \)

By using (6.3.2), we calculate the remaining terms of the solution.

\[
\begin{align*}
v_1(x, t) &= \int_{0}^{t} \left( \frac{\partial^{1-\alpha}}{\partial s^{1-\alpha}} \left[ \frac{\partial^2 v_0(x, 0)}{\partial x^2} + (k - 1) \frac{\partial v_0(x, 0)}{\partial x} - kv_0(x, 0) \right] \right) ds \\
&= -e^x \frac{-kt^\alpha}{\Gamma(\alpha + 1)} + (e^x - 1) \frac{-kt^\alpha}{\Gamma(\alpha + 1)} \\
v_2(x, t) &= \int_{0}^{t} \left( \frac{\partial^{1-\alpha}}{\partial s^{1-\alpha}} \left[ \frac{\partial^2 v_1(x, 0)}{\partial x^2} + (k - 1) \frac{\partial v_1(x, 0)}{\partial x} - kv_1(x, 0) \right] \right) ds \\
&= -e^x \frac{( -kt^\alpha)^2}{\Gamma(2\alpha + 1)} + (e^x - 1) \frac{( -kt^\alpha)^2}{\Gamma(2\alpha + 1)}
\end{align*}
\]

and so on. So, by Adomian decomposition method, the solution of the equation (6.3.1) is given by

\[
v(x, t) = \sum_{n=0}^{\infty} v_n(x, t)
= e^x \left( \frac{kt^\alpha}{\Gamma(\alpha + 1)} - \frac{(kt^\alpha)^2}{\Gamma(2\alpha + 1)} + \cdots \right) \\
+ (e^x - 1) \left( 1 - \frac{kt^\alpha}{\Gamma(\alpha + 1)} + \frac{(kt^\alpha)^2}{\Gamma(2\alpha + 1)} + \cdots \right) \\
= e^x \left( 1 - E_\alpha(-kt^\alpha) \right) + (e^x - 1) E_\alpha(-kt^\alpha),
\]

where \( E_\alpha(x) \) is Mittag - Leffler function in one parameter.

(6.3.3) is the exact solution of the equation (6.3.1). Moreover, by the property of Mittag - Leffler function when \( \alpha \to 1 \), the solution (6.3.3) coincides with the the solution (6.2.3) which is the exact solution of the Black - Scholes equation (6.2.2).

The Figure 6.1, 6.2 shows the effect of \( \alpha \) in the Black - Scholes equation and the solution graph \( v(x, t) \) for \( \alpha = 0.5 \) respectively.
Figure 6.1: $v(x, t)$ for varies values of $\alpha$ by fixing $x$ as 25.

Figure 6.2: $v(x, t)$ for varies values of $x$ and $t$ by fixing $\alpha = 0.5$. 
6.4 Space Fractional Black - Scholes Equation

In this section, we consider the space fractional Black - Scholes equation [68] with fractional derivative in the space variable in the sense of Riemann - Liouville. For an Log Stable (LS) process, the Lévy density is given by

\[ w_{LS}(x) = \begin{cases} 
Dq|x|^{-1-\alpha} & x < 0, \\
Dp x^{-1-\alpha} & x > 0,
\end{cases} \]

where \( D > 0, \ p, q \in [-1, 1], \ p + q = 1 \) and \( 0 < \alpha \leq 2 \). This process has infinite variance making it mathematically difficult to work with. But when we substitute \( p = 0 \) and \( q = 1 \), the process is transformed into the well known FMLS (Finite Moment Log Stable) process. This process has incur downward jumps. The resulting fractional partial differential equation for the price of a European claim is then given by

\[
\frac{\partial V(x,t)}{\partial t} + \left(r + \frac{1}{2} \sigma^\alpha \sec \left(\frac{\alpha \pi}{2}\right)\right) \frac{\partial V(x,t)}{\partial x} - \frac{1}{2} \sigma^\alpha \sec \left(\frac{\alpha \pi}{2}\right) \frac{\partial^\alpha V(x,t)}{\partial x^\alpha} = rV(x,t)
\]

(6.4.1)

where \( \frac{\partial^\alpha f(x,t)}{\partial x^\alpha} \) is the left Riemann - Liouville fractional derivative, \( r \) denotes the risk free rate and \( \sigma \) is the volatility. When \( \alpha \to 2 \), (6.4.1) coincides with the regular Black - Scholes equation [62,2].

Now we are going to solve the equation (6.4.1) with the initial condition \( V(x,0) = e^x - 1 \) using Adomian decomposition method.

We rewrite (6.4.1) as

\[
\frac{\partial V(x,t)}{\partial t} = s \frac{\partial^\alpha V(x,t)}{\partial x^\alpha} - (r + s) \frac{\partial V(x,t)}{\partial x} + rV(x,t),
\]

where \( s = \frac{1}{2} \sigma^\alpha \sec \left(\frac{\alpha \pi}{2}\right) \). Now integrating both sides with respect to \( t \) we get

\[
V(x,t) = V(x,0) + \int_0^t \left(s \frac{\partial^\alpha V(x,s)}{\partial x^\alpha} - (r + s) \frac{\partial V(x,s)}{\partial x} + rV(x,s)\right) ds.
\]

Here we take \( V(x,0) \) as \( V_0(x,0) \) and

\[
V_1(x,t) = \int_0^t \left(s \frac{\partial^\alpha V_0(x,s)}{\partial x^\alpha} - (r + s) \frac{\partial V_0(x,s)}{\partial x} + rV_0(x,s)\right) ds,
\]
and so on. The general iterative scheme for this problem is given by

\[ V_n(x, t) = \int_0^t \left( s \frac{\partial^\alpha V_{n-1}(x, s)}{\partial x^\alpha} - (r + s) \frac{\partial V_{n-1}(x, s)}{\partial x} + r V_{n-1}(x, s) \right) ds. \] (6.4.2)

Then, by Adomian decomposition method the solution of the problem is given by

\[ V(x, t) = \sum_{n=0}^{\infty} V_n(x, t). \]

Let us take the initial condition \( V(x, 0) = e^x - 1 \) as \( V_0(x, t) = e^x - 1 \). Then by using (6.4.2) we calculate the remaining terms of the solution. For, \( V_1(x, t) \)

\[ V_1(x, t) = \int_0^t \left( s \frac{\partial^\alpha V_0(x, \tau)}{\partial x^\alpha} - (r + s) \frac{\partial V_0(x, \tau)}{\partial x} + r V_0(x, \tau) \right) d\tau \]

\[ = st \left[ x^{3-\alpha} E_{1,4-\alpha}(x) + \frac{x^{2-\alpha}}{\Gamma(3 - \alpha)} + \frac{x^{1-\alpha}}{\Gamma(1 - \alpha)} - E_{1,1}(x) \right] - rt \]

\[ = st \left[ x^{1-\alpha} E_{1,2-\alpha} - E_{1,1}(x) \right] - rt, \]

and so on. Then, by Adomian decomposition method, the solution of (6.4.1) is given by

\[ V(x, t) = \sum_{n=0}^{\infty} V_n(x, t) \]

\[ = E_{1,1}(x) - 1 + st \left[ x^{1-\alpha} E_{1,2-\alpha}(x) - E_{1,1}(x) \right] - rt \]

\[ + \frac{(st)^2}{2} \left[ x^{-2\alpha} (xE_{1,2-\alpha}(x) - E_{1,1-2\alpha}(x)) - x^{-1-\alpha} (xE_{1,1-\alpha}(x) - E_{1,\alpha}(x)) \right] \]

\[ - rst \left[ x^{-2\alpha} E_{1,1-2\alpha}(x) - x^{-1-\alpha} E_{1,\alpha}(x) - E_{1,2-\alpha}(x) + E_{1,1}(x) \right] + \cdots \]
where $E_{\alpha,\beta}(x)$ is the generalised Mittag-Leffler function. By using the properties of Mittag-Leffler function, we can easily club the terms when $\alpha \to 2$. Also when we take $s = 1$, $r = -k$ and $\alpha \to 2$, most of the terms get canceled and we get the simplified solution as

$$V(x, t) = e^x - e^{-kt}$$

which is the exact solution of the Black-Scholes equation (6.2.2) with the initial condition $v(x, 0) = e^x - 1.$

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