CHAPTER 1
INTRODUCTION

1.1 Introduction

Geometric Function Theory is a branch of complex analysis, which began to take shape in the second decade of 20th century. Geometric Function Theory, concerned with the interplay between the geometric properties of the image domain and the analytic properties of the mapping function. Its origin (apart from the Riemann mapping Theorem) can be traced to the 1907 paper of Koebe to Gronwall's proof of the Area Theorem in 1914 and this led Bieberbach [22] in 1916, to propose a conjecture. For many years, this conjecture, known as Bieberbach Conjecture (or coefficient problem), stood as a challenge to geometric function theorists and inspired the development of many new techniques in this field. Finally, de Branges [25] proved this conjecture to be true, in 1985, by making use of special functions. Since then, Geometric Function Theory was a subject in its own right. Geometric function theory is a classical subject. Yet it continues to find new applications in an ever-growing variety of areas such as modern mathematical physics, more traditional fields of physics such as fluid dynamics, nonlinear integrable systems theory and the theory of partial differential equations. Detailed treatment of univalent functions are available in the standard books of Nehari [94], Schober [121], Duren [39] and Goodman [49].

A function $f$ analytic in a domain $\Omega$ of the complex plane $\mathbb{C}$ is said to be univalent or one-to-one in $\Omega$ if it never takes the same value more than once in $\Omega$. That is, for any two distinct points $z_1$ and $z_2$ in $\Omega$, $f(z_1) = f(z_2)$. The choice of the unit disc $U = \{z : |z| < 1\}$ as a domain for the study of analytic univalent functions is a matter of convenience to make the computations simpler and leads to elegant formulae. There is no loss of generality in this choice, since Riemann Mapping Theorem [2] asserts that any simply connected proper subdomain of $\mathbb{C}$ can be mapped onto the unit disc by univalent transformation [39, 94].
The class of all analytic functions in the open unit disc $U$ with normalization 
$f(0) = 0$ and $f'(0) = 1$ will be denoted by $A$, consisting of functions of the form 
$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \quad (1.1)$$

Geometrically, the normalization $f(0) = 0$ amounts to only a translation 
of the image domain and $f'(0) = 1$ corresponds to rotation and stretching or 
shrinking of the image domain. We denote the class of all analytic univalent 
functions with the above normalization by $S$.

The function $k(z)$, called the Koebe function, is defined by 
$$k(z) = \frac{z}{(1 - z)^2} = z + 2z^2 + 3z^3 + \ldots$$

which maps $U$ onto the complex plane except for a slit along the negative real 
axis from $-\infty$ to $-1/4$, is a leading example of a function in $S$. It plays a very 
important role in the study of the class $S$. In fact, the Koebe function and its 
rotations $e^{i\alpha}k(e^{i\alpha}z), \ \alpha \in \mathbb{R}$ are the only extremal functions for various extremal 
problems in $S$. The study of univalent functions was initiated by Koebe (1907). 
He discovered that the ranges of all functions in $S$ contain a common disc $|\omega| \leq \frac{1}{4}$, 
later named as the Koebe domain for the class $S$ in honour of him.

For functions $f$ in the class $S$, it is well known that the following growth 
and distortion estimates hold respectively as 
$$\frac{r}{(1 + r)^2} \leq |f(z)| \leq \frac{r}{(1 - r)^2},$$

and 
$$\frac{1 - r}{(1 + r)^2} \leq |f'(z)| \leq \frac{1 + r}{(1 - r)^2},$$

where $z = re^{i\varphi}, \ 0 \leq r < 1$.

Now, certain significant classes of study in the field which can be charac-
terized by simple geometric properties are put forth.

1.2 Certain Subclasses of Analytic Functions

Various interesting subclasses of $S$ defined by natural geometric conditions 
have been extensively investigated. Two such subclasses are starlike and convex
functions.

**Definition 1.2.1.** [39] A domain $\Omega$ of the complex plane is said to be starlike with respect to a point $z_0 \in \Omega$, if for any $z \in \Omega$ and $0 \leq t \leq 1$, $tz_0 + (1 - t)z \in \Omega$. That is, for any fixed point $z_0 \in \Omega$, the line segment joining any point $z \in \Omega$ lies in $\Omega$ itself. A function in $S$ is said to be starlike with respect to the origin if it maps the unit disc conformally onto a starlike domain with respect to the origin. We denote this class of functions by $S^*$.

**Theorem 1.2.2.** [49] A necessary and sufficient condition for a function $f$ to be in the class $S^*$ is

$$R \frac{zf'(z)}{f(z)} > 0, \ z \in \mathbb{U}.$$ 

**Theorem 1.2.3.** [94] If $f \in S$, and is of the form (1.1), then

$$|a_n| \leq n, \ \forall n \geq 2,$$

with equality being attained for the Koebe function $\frac{z}{(1-z)}$ or one of its rotations.

**Definition 1.2.4.** [111] A function $f \in A$ is said to be starlike of order $\alpha$, $0 \leq \alpha < 1$, if and only if

$$R \frac{zf'(z)}{f(z)} > \alpha, \ z \in \mathbb{U}. \ \ (1.2)$$

This class is denoted by $S^*(\alpha)$. Note that $S^*(\alpha) \subset S$ and $S^*(0) = S^*$.

**Definition 1.2.5.** [39] A domain $\Omega$ of the complex plane is said to be convex if for any $z_1, z_2 \in \Omega$ implies that $tz_1 + (1 - t)z_2 \in \Omega$. That is, the line segment joining any two points of the domain $\Omega$ lies in the domain itself. A function $f \in S$ is said to be convex if it maps the unit disc conformally onto a convex domain. This class of functions are denoted by $K$.

**Theorem 1.2.6.** [111] A necessary and sufficient condition for a function $f$ to belong to $K$ is

$$R \frac{-zf''(z)}{f'(z)} > 0, \ z \in \mathbb{U}.$$
Theorem 1.2.7. [111](Alexander’s Theorem) Let \( f \in S \). Then \( f \in K \) if and only if \( zf'' \in S^* \).

Theorem 1.2.8. [94] If \( f(z) = z + \sum_{m=2}^{\infty} a_m z^m \in K \), \( z \in U \), then \( |a_n| \leq 1 \), for all \( n \) with equality being attained by the function \( \frac{z}{(1-z)} \).

Definition 1.2.9. [111] A function \( f \) of the form (1.1) is said to be convex in \( U \) of order \( \alpha \), if

\[
R \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad 0 \leq \alpha < 1, \quad z \in U
\]

and the class of all convex functions of order \( \alpha \) is denoted by \( K(\alpha) \). Clearly \( K(0) = K \), the class of convex univalent functions.

It follows from (1.2) and (1.3) that

\[
f \in K(\alpha) \iff zf'' \in S^*(\alpha).
\]

Definition 1.2.10. [111] A function \( f \in S \) is said to be close to convex of order \( \beta \) and type \( \alpha \), if there exists a function \( g \in S'(\alpha) \) such that

\[
R \left( 1 + \frac{zf''(z)}{g(z)} \right) > \beta, \quad 0 \leq \alpha, \beta < 1, \quad z \in U.
\]

Goodman (1991, 1991a) introduced an interesting subclass uniformly convex (uniformly starlike) of the class \( K \) of convex functions (\( S^* \) starlike functions) denoted by \( UCV(UST) \). A function \( f(z) \) is uniformly convex (uniformly starlike) in \( U \) if \( f(z) \) in \( K(S^*) \) has the property that for every circular arc \( \gamma \) contained in \( U \), with center \( \xi \) is also in \( U \), the arc \( f(\gamma) \) is a convex arc (starlike arc) with respect to \( f(\xi) \). In particular, Goodman gave the following analytic criterions

Theorem 1.2.11. [51] A function \( f \in S \), of the form (1.1) is in \( UST \) if and only if

\[
R \left( 1 + \frac{f(z) - f(\xi)}{(z - \xi)f'(z)} \right) \geq 0, \quad (z, \xi) \in U \times U, \quad z = \xi.
\]

Theorem 1.2.12. [50] A function \( f \in S \), of the form (1.1) is in \( UCV \) if and only if

\[
R \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq 0, \quad (z, \xi) \in U \times U.
\]
Goodman established a number of results relating to these functions, besides raising a number of questions and showed that Alexander’s theorem does not hold between the classes \(UCV\) and \(UST\). In exploring the possibility of an analogous result of Alexander’s theorem between the classes \(UST\) and \(UCV\), Rønning [113] introduced a new class \(S_p\), namely the class of parabolic starlike functions. Geometrically, \(S_p\) is the class of functions for which \(\frac{z}{f(y)}\) has values in the interior of the parabola in the right half plane symmetric about the real axis with vertex \((\frac{1}{2}, 0)\). A function \(f \in S_p\) if and only if
\[
R \frac{zf''(z)}{f'(z)} > \frac{zf'(z)}{f(z)} - 1^+, \quad z \in U.
\]
The function
\[
\phi(z) = 1 + \frac{2}{\pi^2} \log \frac{1 + \frac{\sqrt{z}}{2}}{1 - \frac{\sqrt{z}}{2}}
\]
maps the open unit disc \(U\) onto the parabolic region \(\Omega = \{w : R(w) > |w - 1|\}\) which lies in the sector \(-\pi/4 < \arg(w) < \pi/4\) and hence is in \(S_p\).

Further, Ma and Minda [85] and Rønning [113] independently obtained a one-variable characterization of functions in the class \(UCV\), as follows:

**Theorem 1.2.13.** [113] \(f \in UCV\) if and only if
\[
R \frac{1 + zf''(z)}{f'(z)} > \frac{zf'(z)}{f(z)}, \quad z \in U.
\]

Moreover, generalizing the class \(S_p\), Rønning [112] defined and investigated the class \(S_p(\alpha)\) \((-1 < \alpha < 1\) of starlike functions and an associated class \(UCV(\alpha)\) \((-1 < \alpha < 1\) of convex functions. The classes \(UCV\) and \(S_p\) were studied extensively in the literature ([66, 67, 68, 69, 89] and [134, 135]) as the classes \(k - UCV\) and \(k - ST\).

**Definition 1.2.14.** [67] A function \(f\) of the form (1.1) is said to be in the class \(k - UCV\), the class of functions that are \(k\)-uniformly convex in \(U\) if
\[
R \frac{1 + zf''(z)}{f'(z)} \geq k \frac{zf'(z)}{f(z)}, \quad 0 \leq k < \infty.
\]
**Definition 1.2.15.** [68] A function $f$ of the form (1.1) is said to be in the class $k-ST$, the class of functions that are $k$-starlike in $U$ if

$$R \frac{zf'(z)}{f(z)} > k \frac{zf'(z)}{f(z)} - 1, \quad 0 \leq k < \infty.$$  

The class $k-UCV$ was considered by Kanas and Wiśniowska [67] and Subramanian et al., [134] where its geometrical definition and connections with the conic domains were considered. The class $k-ST$ was investigated by Kanas and Wiśniowska [68] and further properties were studied by Kanas and Srivastava [66]. In particular, when $k = 1$, we obtain $UCV$ and $S_p$ the familiar classes of uniformly convex functions and parabolic starlike functions in $U$, respectively.

### 1.3 The Class of Bi-univalent Functions

It is well known that every function $f \in S$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z, \quad z \in U$$

and

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f); \quad r_0(f) \geq \frac{1}{4},$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 2a_2a_3 + a_4)w^4 + \ldots.$$  

A function $f \in A$ is said to be bi-univalent in $U$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $U$. Let $\Sigma_B$ denote the class of bi-univalent functions in $U$ given by (1.1). Examples of functions in the class $\Sigma_B$ are

$$z \frac{z}{1-z}, -\log(1-z), \quad \frac{1}{2} \log \frac{1+z}{1-z}$$

and so on. However, the familiar Koebe function is not a member of $\Sigma_B$. Other common examples of functions in $S$ such as

$$z - \frac{z^2}{2}$$

and

$$\frac{z}{1 - z^2}$$
are also not members of $\Sigma_B$ (see [45, 128]).

In 1967, Lewin [76] investigated the bi-univalent function class $\Sigma_B$ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [27] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [96], on the other hand, showed that $\max_{f \in \Sigma_B} |a_2| = \frac{4}{3}$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N}$ \{1, 2\}, $\mathbb{N} := \{1, 2, 3, \ldots \}$ is presumably still an open problem.

Brannan and Taha [28] introduced certain subclasses of the bi-univalent function class $\Sigma_B$ similar to the familiar subclasses $S^*(\alpha)$ and $K\alpha$ of starlike and convex functions of order $\alpha$ ($0 \leq \alpha < 1$), respectively (see [26]). Thus, following Brannan and Taha [28], a function $f \in A$ is in the class $SS^\Delta_{B\alpha}(\alpha)$ of strongly bi-starlike of order $\alpha$ ($0 < \alpha \leq 1$), if each of the following condition is satisfied:

$$f \in \Sigma_B, \quad \arg \frac{zf'(z)}{f(z)} < \frac{\alpha \pi}{2}, \ z \in U; \ 0 < \alpha \leq 1$$

and

$$\arg \frac{w g'(w)}{g(w)} < \frac{\alpha \pi}{2}, \ w \in U; \ 0 < \alpha \leq 1,$$

where $g$ is the extension of $f^{-1}$ to $U$.

The classes $SS^\Delta_{B\alpha}(\beta)$ and $K\Sigma_B(\beta)$ of bi-starlike functions of order $\beta$ ($0 \leq \beta < 1$) and bi-convex functions of order $\beta$ ($0 \leq \beta < 1$) corresponding (respectively) to the function classes $S^*(\beta)$ and $K(\beta)$ defined by (1.2) and (1.3), were also introduced analogously. For each of the function classes $SS^\Delta_{B\alpha}(\beta)$ and $K\Sigma_B(\beta)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details see [28]).

Recently Srivastava et al. [128] introduced certain subclass $H^\Delta_{B\alpha}$ of the bi-univalent functions class $\Sigma_B$ as defined below:

For a function $f(z)$ given by (1.1) is said to be in the class $H^\Delta_{B\alpha}$, if the following conditions are satisfied:

$$f \in \Sigma_B, \ |\arg(f'(z))| < \frac{\alpha \pi}{2}, \ z \in U; \ 0 < \alpha \leq 1$$

and

$$|\arg(g'(w))| < \frac{\alpha \pi}{2}, \ w \in U; \ 0 < \alpha \leq 1,$$
where the function $g$ is given

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - a_2a_3 + a_4)w^4 + \ldots$$  \hspace{1cm} (1.4)

### 1.4 The Subclass $\mathcal{T}$ of $\mathcal{S}$ with Negative Coefficients

In 1975, Silverman [120] opened up a new line of research in the theory of univalent functions by introducing the subclass $\mathcal{T}$ of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \; z \in \mathbb{U},$$  \hspace{1cm} (1.5)

that are analytic and univalent with negative coefficients, which enabled easy handling of difficult problems, but still there are many open problems as listed by Silverman [122]. There have been several significant contributions such as coefficient estimates, distortion bounds, closure properties, etc., by way of extension or generalization.

**Theorem 1.4.1.** [120] Let $f \in \mathcal{T}$. Then $f \in S^+(\alpha)$, if and only if

$$\sum_{n=2}^{\infty} (n-\alpha)a_n \leq 1 - \alpha.$$  

**Theorem 1.4.2.** [120] Let $f \in \mathcal{T}$. Then $f \in K(\alpha)$, if and only if

$$\sum_{n=2}^{\infty} n(n-\alpha)a_n \leq 1 - \alpha.$$  

**Theorem 1.4.3.** [120] If $f \in S^+(\alpha)$, then

$$a_n \leq \frac{1 - \alpha}{n - \alpha}.$$  

The result is sharp for the function

$$f_n(z) = z - \frac{\alpha}{n(n-\alpha)} z^n.$$  

**Theorem 1.4.4.** [120] If $f \in K(\alpha)$, then

$$a_n \leq \frac{1 - \alpha}{n(n-\alpha)}.$$
The result is sharp for the function
\[ f_n(z) = z - \frac{1 - \alpha}{n(n - \alpha)} z^n. \]

**Theorem 1.4.5.** [120] (Distortion Theorem) If \( f \in S^\alpha \), then for \(|z| = r < 1\),
\[ r - \frac{1 - \alpha}{2(2 - \alpha)} r^2 \leq |f(z)| \leq r + \frac{1 - \alpha}{2(2 - \alpha)} r^2 \]
and
\[ 1 - \frac{2(1 - \alpha)}{(2 - \alpha)} r \leq |f'(z)| \leq 1 + \frac{2(1 - \alpha)}{(2 - \alpha)} r. \]
The result is sharp for the function
\[ f(z) = z - \frac{1 - \alpha}{2} z^2. \]

**Theorem 1.4.6.** [120] (Distortion Theorem) If \( f \in K(\alpha) \), then for \(|z| = r < 1\),
\[ r - \frac{1 - \alpha}{2(2 - \alpha)} r^2 \leq |f(z)| \leq r + \frac{1 - \alpha}{2(2 - \alpha)} r^2 \]
and
\[ 1 - \frac{1 - \alpha}{(2 - \alpha)} r \leq |f'(z)| \leq 1 + \frac{1 - \alpha}{(2 - \alpha)} r. \]
The result is sharp for the function
\[ f(z) = z - \frac{1 - \alpha}{2(2 - \alpha)} z^2. \]

**Theorem 1.4.7.** [120] If \( f \in K(\alpha) \), then \( f \in S^\alpha \left( \frac{2}{3-\alpha} \right) \). The result is sharp, with
\[ f(z) = z - \frac{(1 - \alpha)}{2(2 - \alpha)} z^2 \]
being the extremal.

**Theorem 1.4.8.** [120] (Extreme Points) The extreme points of \( K(\alpha) \) are the functions
\[ f_1(z) = z \quad \text{and} \quad f_n(z) = z - \frac{1 - \alpha}{n(n - \alpha)} z^n, \quad n = 2, 3, \ldots. \]

Analogously, the class of uniformly convex and uniformly starlike functions with negative coefficients were studied recently by function theorists Bharati et al., [21], Kanas and Srivastava [66], Merkes and Salmassi [89] and Subramanian et al., [134, 135].
1.5 Harmonic Functions

Denote by $S_H$, the family of functions $f = h + g$ which are harmonic, univalent and orientation preserving in the open unit disc $U = \{z : |z| < 1\}$ so that $f$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + g \in S_H$, the functions $h$ and $g$ are analytic in $U$ that can be expressed in the following forms:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1.$$

We note that the family $S_H$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class $S$ of normalized univalent functions if the co-analytic part of $f$ is identically zero ($g \equiv 0$).

Also, we denote by $TS_H$, the subfamily of $S_H$ consisting of harmonic functions of the form $f = h + g$ such that $h$ and $g$ are of the form

$$h(z) = z - \sum_{n=2}^{\infty} \frac{a_n}{{|a_n|}^2} z^n, \quad g(z) = \sum_{n=1}^{\infty} \frac{b_n}{|b_n|} z^n.$$

Recently Clunie and Sheil-Small [33], investigated the class $S_H$ as well as its geometric subclasses and its properties. Since then, there have been several studies related to the class $S_H$ and its subclasses. Detailed treatment and for additional information about harmonic univalent functions are available in the one may refer [3, 20, 40] and [124].

1.6 Multivalent Functions

Let $f$ be a function in the open unit disk $U$. If the equation $f(z) = w$ has at most $p$--solutions in $U$ and there exists some $w$ for which this equation has exactly $p$ solutions, then $f$ is said to be $p$--valent in $U$. 
Let $A_p(k)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+k}^{\infty} a_n z^n,$$  

which are analytic and $p$-valent in the open disc $U = \{z : |z| < 1\}$. Let $T_p(k)$ be the subclass of $A_p(k)$ of consisting functions of the form:

$$f(z) = z^p - \sum_{n=k+1}^{\infty} a_n z^n, \quad a_n \geq 0; \quad p, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$  

A function $f \in T_p(k)$ is said to be $p$-valently starlike of order $\alpha$, if it satisfies the following inequality:

$$R^* \frac{zf'(z)}{f(z)} > \alpha, \quad z \in U; \quad 0 \leq \alpha < p, p \in \mathbb{N}.$$  

We denote by $T_p^*(k, \alpha)$ the class of all $p$-valently starlike functions of order $\alpha$. Also, a function $f \in T_p(k)$ is said to be $p$-valently convex of order $\alpha$ if it satisfies the following inequality:

$$R^* \left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in U; \quad 0 \leq \alpha < p, p \in \mathbb{N}.$$  

We denote by $C_p(k, \alpha)$ the class of all $p$-valently convex functions of order $\alpha$. We note that (see for example [39, 49])

$$f \in C_p(k, \alpha) \iff \frac{zf'(z)}{p} \in T_p^*(k, \alpha), \quad 0 \leq \alpha < p, p \in \mathbb{N}.$$  

The classes $T_p^*(k, \alpha)$ and $C_p(k, \alpha)$ are studied by Owa [101].

### 1.7 Meromorphic Functions

Let $\Sigma$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \geq 0,$$  

which are analytic in the punctured open unit disk $U^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} =: U \setminus \{0\}$. 
Let $\Sigma_S$, $\Sigma^*(\alpha)$ and $\Sigma_K(\alpha)$, $(0 \leq \alpha < 1)$ denote the subclasses of $\Sigma$ that are meromorphic univalent, meromorphically starlike functions of order $\alpha$ and meromorphically convex functions of order $\alpha$ respectively. Analytically, $f \in \Sigma^*(\alpha)$ if and only if, $f$ is of the form \eqref{eq:1.7} and satisfies

$$-R \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \ z \in U,$$

similarly, $f \in \Sigma_K(\alpha)$, if and only if, $f$ is of the form \eqref{eq:1.7} and satisfies

$$-R \left\{ 1 + \frac{zf''(z)}{f(z)} \right\} > \alpha, \ z \in U,$$

and similar other classes of meromorphically univalent functions have been extensively studied by (for example) Altintas et al. [8], Aouf [12], Mogra et al. [90], Uralegadi et al. [136, 137, 138] and others.

1.8 Linear Operators

**Definition 1.8.1.** (Convolution Product) The convolution or Hadamard product of two functions $\Phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n$ in $A$ with $\phi_n \geq 0$, $\psi_n \geq 0$, $\phi_n \geq \psi_n$ is given by $(\Phi \ast \Psi)(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n$.

**Definition 1.8.2.** [114] The Ruscheweyh derivative of $f(z)$ denoted by $D^\mu f(z) : A \rightarrow A$, defined by the Hadamard product (or convolution)

$$D^\mu f(z) := f(z) \ast \frac{z}{(1-z)^{\mu+1}}, \quad \mu \geq -1, \ z \in U \tag{1.8}$$

which implies that

$$D^\nu f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!},$$

$$D^0 f(z) = f(z), \quad D^1 f(z) = zf(z), \quad D^2 f(z) = z^2 f''(z),$$

We observe that, the power series of $D^\mu f$ for the function $f$ of the form \eqref{eq:1.1}, in view of \eqref{eq:1.8}, is given by

$$D^\mu f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+\mu)}{(n-1)!\Gamma(1+\mu)} \phi_n z^n, \ z \in U.$$
Definition 1.8.3. [116] The Sälågean derivative of a function $f(z)$, denoted by $D^m f(z)$, $m \in \mathbb{N}_0$ is defined by

$$D^m f(z) = f(z) * z + \sum_{n=2}^{\infty} n^m z^n .$$

Also, $D^0 f(z) = f(z)$, $D^1 f(z) = z f'(z)$ and $D^m f(z) = z (D^{m-1} f(z))'$.

Definition 1.8.4. Let the function $\varphi(a, c, z)$ be given by

$$\varphi(a, c, z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \quad c = 0, -1, -2, \ldots; z \in \mathbb{U},$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1 & \text{for } n = 0 \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & \text{for } n = 1, 2, 3 \ldots \end{cases}$$

Corresponding to the function $\varphi(a, c, z)$, Carlson and Shaffer [29] introduced a linear operator $L(a, c)$, which is defined by the following Hadamard product (or convolution):

$$L(a, c) f(z) := \varphi(a, c, z) * f(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} a_n z^{n+1}.$$ 

We note that

$$L(a, a) f(z) = f(z), \quad L(2, 1) f(z) = z f'(z), \quad L(\mu + 1, 1) f(z) = D^\mu f(z),$$

where $D^\mu f(z)$ is the Ruscheweyh derivative of $f(z)$ [114].

For complex parameters $\alpha_i, \ldots, \alpha_i$ and $\beta_i, \ldots, \beta_m$ ($\beta_j = 0, -1, \ldots; j = 1, 2, \ldots, m$) the generalized hypergeometric function $\mathcal{F}_m(z)$ is defined by

$$\mathcal{F}_m(z) = \mathcal{F}_m(\alpha_i, \ldots, \alpha_i; \beta_i, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_i)_n \cdots (\alpha_i)_n}{(\beta_i)_n \cdots (\beta_m)_n} \frac{z^n}{n!},$$

where $l \leq m + 1; l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in \mathbb{U}$ and $(\alpha)_n$ is the Pochhammer symbol. Recently, Dziok and Srivastava [41] defined a linear operator

$$\mathcal{H}_m(\alpha_i, \ldots, \alpha_i; \beta_i, \ldots, \beta_m) : S \rightarrow S.$$
such that

\[ H_m[\alpha_1]f(z) = H_m(\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_m)f(z) = z \frac{\partial}{\partial z} F_m(\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_m; z) = f(z) \]

\[ H_m[\alpha_1]f(z) = \sum_{n=2}^{\infty} \Gamma(n) a_n z^n, \tag{1.9} \]

where \( \Gamma(n) = \frac{(\alpha)_n \cdots (\alpha)_{n-1}}{(\beta)_n \cdots (\beta)_{n-1}} \frac{1}{(n-1)!} \).

It is easy to verify from (1.9) that

\[ z[H_m[\alpha_1]f(z)]^i = \alpha_1 H_m[\alpha_1 + 1]f(z) - (\alpha_1 - 1) H_m[\alpha_1]f(z). \]

We note that, many subclasses of analytic functions, associated with the Dziok-Srivastava operator \( H_m[\alpha_1]f(z) \) and many special cases, were investigated recently by Dziok and Srivastava [41] and Liu [80]. Also we note that the following special cases of the Dziok-Srivastava linear operator.

**Remark 1.8.5.** For \( f \in A \),

\[ H_1^2(\alpha, 1; \beta)f(z) = L(\alpha, \beta)f(z) \]

which was considered by Carlson and Shaffer [29].

**Remark 1.8.6.** For \( f \in A \),

\[ H_1^2(\mu + 1, 1; 1)f(z) = \frac{z}{(1-z)^{\mu+1}} f(z) = D^\mu f(z) \]

the symbol \( D^\mu f(z) \) was introduced by Ruscheweyh [114] and is called Ruscheweyh derivative of \( f(z) \) of \( \mu \)th order.

**Remark 1.8.7.** For \( f \in A \),

\[ H_1^2(c + 1, 1; c + 2)f(z) = \zeta + \frac{1}{(z^c)} \int_0^z t^{c-1} f(t) dt = J_c f(z) \]

where \( c > -1 \). The operator \( J_c \) is the Bernardi integral operator. In particular, the operator \( J_1 \) is the Liber-Livingston integral operator.
Remark 1.8.8. For \( f \in A \),

\[
H_f^2(2, 1; 2 - \lambda) f(z) = \Gamma(2 - \lambda)z^\lambda D^\lambda f(z) = \Omega^\lambda f(z), \quad \lambda \not\in \mathbb{N} \setminus \{1\}.
\]

The operator \( \Omega^\lambda \) was introduced by Srivastava and Owa [130] and \( \Omega^\lambda \) is also called Srivastava-Owa fractional derivative operator, where \( D^\lambda f(z) \) denotes the fractional derivative of \( f(z) \) of order \( \lambda \), studied by Owa [98].

Let us now consider the meromorphic analogue of the above operators. Let \( \Sigma \) denote the class of all analytic functions of the form (1.7). Let \( f, g \in \Sigma \), where \( f \) is given by (1.7) and \( g \) is defined by

\[
g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n.
\]

Then the Hadamard product (or convolution) \( f \ast g \) of the functions \( f \) and \( g \) is defined by

\[
(f \ast g)(z) := \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n =: (g \ast f)(z).
\]

Corresponding to the function \( F(\alpha_1, ..., \alpha_i; \beta_1, ..., \beta_m; z) \), defined by

\[
F(\alpha_1, ..., \alpha_i; \beta_1, ..., \beta_m; z) = z^{-1} \frac{1}{\Gamma_m(\alpha_1, \alpha_2, ..., \alpha_i; \beta_1, \beta_2, ..., \beta_m)} f(z) = z^{-1} \frac{1}{\Gamma_m(\alpha_1, \alpha_2, ..., \alpha_i; \beta_1, \beta_2, ..., \beta_m)} f(z),
\]

we consider a linear operator \( HL_m(\alpha_1, ..., \alpha_i; \beta_1, ..., \beta_m) : \Sigma \to \Sigma \) which is defined by the following Hadamard product (or convolution):

\[
HL_m(\alpha_1, ..., \alpha_i; \beta_1, ..., \beta_m) f(z) := z^{-1} \prod_{m=1}^{1} \frac{1}{1} \prod_{m}^{2} \frac{1}{2} \prod_{m}^{n} \frac{1}{m} F(\alpha, \alpha, ..., \alpha; \beta, \beta, ..., \beta; z) \ast f(z).
\]

We observe that, for a function \( f(z) \) of the form (1.7), we have

\[
HL_m(\alpha_1, ..., \alpha_i; \beta_1, ..., \beta_m) f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_{n+1}...((\alpha_i)_{n+1}}{(\beta_1)_{n+1}...((\beta_m)_{n+1} (n+1)! a_n z^n}.
\]

For notational simplicity, we use \( HL_m[\alpha_1] f(z) \) for \( HL_m(\alpha_1, ..., \alpha_i; \beta_1, ..., \beta_m) f(z) \) as a shorter notation and

\[
\Gamma_n(\alpha_1) = \frac{(\alpha_1)_{n+1}...((\alpha_i)_{n+1}}{(\beta_1)_{n+1}...((\beta_m)_{n+1} (n+1)!}.
\]
unless otherwise stated in the sequel. The linear operator $H^{l,m}[\alpha]$ was earlier defined for multivalent functions by Dziok and Srivastava [41] and investigated by Liu and Srivastava [82, 83, 84] and Aouf [13] (see, for details, Dziok and Srivastava [41]), which contains such well-known operators as the Hohllov linear operator, Saitohs generalized linear operator, the Carlson- Shaffer linear operator, the Ruscheweyh derivative operator as well as its generalized version, the Bernardi- Libera -Livingston operator and the Srivastava - Owa fractional derivative operator.

1.9 **Scope of This Thesis**

We begin our study in Chapter 2 of this thesis, by defining a subclass $W_{\mu,\beta}^{\gamma,\eta}(A, B, \gamma, \lambda)$ of $T$ consisting of functions of the form (1.5) satisfying the analytic condition

\[ \frac{zF_\lambda'(z)}{F_\lambda(z)} - 1 > 0, \quad z \in \mathbb{U}, \]

where

\[ \frac{zF_\lambda'(z)}{F_\lambda(z)} = \frac{zR_\mu^n f(z) + \lambda z^2 R_\mu'' f(z)}{(1 - \lambda) R_\mu^n f(z) + \lambda z (R_\mu^n f(z))} \]

Further, we obtain the necessary and sufficient conditions, distortion bounds, extreme points, closure properties and the radii of close to convexity, starlikeness and convexity of the aforementioned class. Also, we study the neighborhood results, partial sums, integral means inequality, closure properties under integral transform and Holder’s inequality result for the same class. In the following section of this chapter, we extend the study to discuss some of the interesting characteristic properties of functions $f \in W_{\mu,\beta}^{\gamma,\eta}(A, B, \gamma, \lambda)$ by fixing the second coefficient.

Chapter 3 is devoted to study, the class of bi-univalent functions. Motivated by the works of Ali et al. [7], Frasin and Aouf [45] and Srivastava et al. [128], in the first section of this chapter, we introduce the following subclasses of bi-starlike and bi-strongly starlike functions.
(i) For a function \( f(z) \) given by (1.1) is said to be in \( J_{\beta}^\psi (\beta, \lambda) \) \((0 \leq \beta < 1; 0 \leq \lambda < 1) \) if

\[
f \in \Sigma_B, \quad R \left( \frac{zf(z)}{(1-\lambda)f(z) + \lambda f'(z)} \right) > \beta, \quad z \in U
\]

and

\[
R \left( \frac{wg(w)}{(1-\lambda)g(w) + \lambda wg'(w)} \right) > \beta, \quad w \in U,
\]

where the function \( g \) is given by (1.4).

(ii) For a function \( f(z) \) given by (1.1) is said to be in \( G_{\alpha} \) \( (\alpha, \lambda) \) \((0 < \alpha \leq 1; 0 \leq \lambda < 1) \) if

\[
f \in \Sigma_B, \quad \arg \left( \frac{zf(z)}{(1-\lambda)f(z) + \lambda f'(z)} \right) :< \frac{\alpha \pi}{2}, \quad z \in U
\]

and

\[
\arg \left( \frac{wg(w)}{(1-\lambda)g(w) + \lambda wg'(w)} \right) :< \frac{\alpha \pi}{2}, \quad w \in U,
\]

where the function \( g \) is given by (1.4).

Further, inspired by the works of Xu et al. [139], another subclass of the function class \( \Sigma_B \) is defined below:

Let \( f \in A \) and the functions \( \varphi, \psi : U \rightarrow \mathbb{C} \) so constrained that

\[
\min \{ R(\varphi(z)), R(\psi(z)) \} > 0, \quad z \in U
\]

and \( \varphi(0) = \psi(0) = 1 \). We say that \( f \in M_{\beta, \psi}^\alpha (\lambda) \) if the following conditions are satisfied

\[
f \in \Sigma_B, \quad \frac{zf(z)}{(1-\lambda)z + \lambda f(z)} \in \varphi(U) \quad \text{and} \quad \frac{wg(w)}{(1-\lambda)w + \lambda g(w)} \in \psi(U), \quad z, w \in U,
\]

where \( 0 \leq \lambda \leq 1 \) and the function \( g \) is the extension of \( f^{-1} \) to \( U \), given by (1.4).

For the aforementioned classes of this chapter, we obtain estimates on the first two coefficients of Taylor-Maclaurin series of the form (1.1). It is interesting to note that, our results improve and generalize several well known results in [7, 28, 45, 128] and [139].
In Chapter 4 of this thesis, we define a subclass $S_{\alpha,\beta}^j(j)$ of $S(j)$ consisting of functions with $j$ missing coefficients and satisfying the analytic criterion

$$ \frac{F_\lambda - 1}{yF_\lambda + 1 - (1 + y)\alpha^j} < \beta, \quad z \in U,$$

where

$$ F_\lambda = \frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)}, $$

and $0 < \beta \leq 1$, $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$ and $0 \leq y \leq 1$.

We also let $T_{\alpha,\beta}^{j,\gamma}(j) = S_{\alpha,\beta}^j(j)^T_j T(j)$.

Further, motivated by the works of Bharati et al [21], Frasin [42], Murugusundaramoorthy and Magesh [92] and others, in the following section of this chapter we define, another subclass $U_{\alpha,\beta}^j(j)$ of $S(j)$ as follows:  

For $\beta \geq 0$, $-1 \leq \alpha < 1$ and $0 \leq \lambda < 1$, we let $U_{\alpha,\beta}^j(j)$ denote the subclass of $S(j)$ consisting of functions $f(z)$ with $j$ missing coefficients and satisfying the analytic criterion

$$ R^\alpha \frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} - 1 \geq \beta^\alpha. \quad \frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} - 1: \quad z \in U. $$

We also let $U_{\alpha,\beta}^j(j) = U_{\alpha,\beta}^j(j)^T_j T(j)$. For aforementioned classes, we obtain coefficient inequalities, growth and distortion theorems, the radii of close-to-convexity, starlikeness, convexity and results on modified hadamard product for functions belonging to those subclasses are discussed extensively.

Chapter 5 is dedicated to discuss a subclass of multivalent functions. In this chapter we define a new subclass $S_{\mu,\alpha}^j(\alpha)$ of $A_p(k)$ associated with the generalized fractional calculus operator satisfying the condition

$$ R^\alpha \frac{z(M_{0,z}^{\mu,\eta}(f(z))) + \lambda z^2(M_{0,z}^{\mu,\eta}(f(z)))''}{(1 - \lambda)M_{0,z}^{\mu,\eta}(f(z)) + \lambda z(M_{0,z}^{\mu,\eta}(f(z)))'} > \alpha, \quad 0 \leq \lambda \leq 1; 0 \leq \alpha < 1. $$

Further, we obtain the coefficient estimates, distortion bounds, extreme points, neighborhood properties, inclusion results, results on modified hadamard
product. Also we discuss the radii of starlikeness and convexity, results on quasi Hadamard product and class preserving integral operator for functions in aforementioned class.

In Chapter 6 of this thesis, we introduce the following two new subclasses of meromorphically univalent functions associated with the generalized Liu-Srivastava operator:

1. For \( \alpha > 1 \) and \( \beta \leq 1 \), let \( \Sigma^{\lambda,k}_{p}(\alpha, \beta) \) denote a subclass of \( \Sigma_{p} \) satisfying the condition
\[
R zD_{\lambda,k}^{l,m}f(z) - \alpha z^{2}(D_{\lambda,k}^{l,m}f(z))^{-1} > \beta, \quad z \in \mathbb{U}^*.
\]

2. For \( \alpha < 1 \) and \( \beta \leq \eta < 1 \), let \( \mathcal{M}(\alpha, \eta, k) \) denote a subclass of \( \Sigma_{p} \) satisfying the condition that
\[
R \frac{z(D_{\lambda,k}^{l,m}f(z))^{\lambda}}{(\eta - 1)D_{\lambda,k}^{l,m}f(z) + \eta z(D_{\lambda,k}^{l,m}f(z))^{\lambda}} > \alpha, \quad z \in \mathbb{U}^*.
\]

The characteristic properties like coefficients estimates, extreme points, growth and distortion bounds, radii of meromorphically starlikeness and partial sums are discussed for \( \Sigma^{\lambda,k}_{p}(\alpha, \beta) \) and further the study is extended for the function class \( \mathcal{M}(\alpha, \eta, k) \) by fixing the second coefficient.

Chapter 7 deals with the study on a subclass of harmonic univalent functions. Motivated by the works of Ahuja [3], Ahuja et al. [4], Frasin [43], Magesh and Mayilvaganan [87], Magesh and Porwal [88], Rosy et al. [108] and Yalcin and Öztürk [142], a new subclass \( G_{H}(\Phi, \Psi; \beta, \gamma, b, t) \) of harmonic convex functions of complex order is defined. Also, the coefficient estimate, distortion bounds, extreme points and closure properties are discussed extensively. We remark that the results so obtained for this general family can be viewed as extensions and generalizations for various subclasses of \( S_{H} \). Furthermore, inspired by the works of Bostancı and Öztürk [24] and Rosy et al. [109], in the second section of this chapter, a new subclass \( \mathcal{M}_{H}(\gamma, k, m) \) of harmonic meromorphic univalent functions is considered. Further, we obtain the sufficient coefficient conditions for the
harmonic meromorphic functions to be in the class $M_H(\gamma, k, m)$. We then show that the sufficient condition is also necessary for $M_{\tilde{H}}(\gamma, k, m)$. Moreover, distortion bounds, extreme points, convolution conditions and convex combinations for functions in $M_{\tilde{H}}(\gamma, k, m)$ are studied extensively.