CHAPTER 4

CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS WITH MISSING COEFFICIENTS

4.1 Introduction

This chapter is devoted to study analytic univalent functions with missing coefficients. Inspired by Frasin [42, 44] and Srivastava et al. [131], in the first section of this chapter, a new subclass of starlike functions is defined and studied with regard to coefficient estimates, extreme points and other interesting properties. Further, motivated by Bharathi et al. [21], Murugusundaramoorthy and Magesh [92], another subclass of $k$–uniformly starlike and $k$–uniformly convex functions is defined in the second section of this chapter and the coefficient estimates, results on modified Hadamard product and other properties are discussed.

4.2 The Subclass of Starlike and Convex functions

Let $A(j)$ denote the family of functions of the form:

$$f(z) = z + \sum_{n=j+1}^{\infty} a_n z^n$$  \hspace{1cm} (4.1)

which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and let $A(1) := A$.

Also, denote by $T(j)$ the subclass of $A(j)$ consisting of functions of the form:

$$f(z) = z - \sum_{n=j+1}^{\infty} a_n z^n, \quad a_n \geq 0$$  \hspace{1cm} (4.2)

and let $T(1) := T$.

We now define the following class of functions.

Definition 4.2.1. For $0 < \beta \leq 1$, $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$ and $0 \leq \gamma \leq 1$, we let $S_{\alpha,\beta}^{\lambda,\gamma}(j)$ denote the subclass of $A(j)$ consisting of functions $f(z)$ of the form (4.1) and satisfying the analytic criterion

$$\frac{F_\lambda - 1}{\gamma F_\lambda + 1 - (1 + \gamma) \alpha} < \beta, \quad z \in U,$$  \hspace{1cm} (4.3)
where, for convenience,
\[ F_{\lambda} = \frac{zf(z) + (2\lambda^2 - \lambda)z^2 f''(z)}{4(\lambda - \lambda^2)z + (2\beta^2 - \lambda)zf(z) + (2\lambda^2 - 3\lambda + 1)f(z)} = \frac{\Phi_{n,\lambda}(z)}{\Psi_{n,\lambda}(z)}. \]

We also let \( T_{\alpha,\beta}^{\lambda}(j) = S_{\alpha,\beta}^{\lambda,\gamma}(j) \to T \).

We note that the class \( T_{\alpha,\beta}^{\lambda}(j) \) unifies many classical and recently studied subclasses of \( A(j) \). In particular

1. \( T_{\alpha,\beta}^{0,1}(1) = S^{\lambda}(\alpha) \) and \( T_{\alpha,\beta}^{1,1}(1) = K(\alpha) \) (Silverman [120])
2. \( T_{\alpha,\beta}^{0,1}(j) = S^{\lambda}(\alpha, j) \) and \( T_{\alpha,\beta}^{1,1}(j) = K(\alpha, j) \) (Srivastava et al [131])
3. \( T_{\alpha,\beta}^{1,2,1}(1) = P(\alpha) \) (Al-Amiri [6] and Sarangi and Uralgaddi [118])
4. \( T_{\alpha,\beta}^{0,1}(1) = S^{\lambda}(\alpha, \beta) \) and \( T_{\alpha,\beta}^{1,1}(1) = K(\alpha, \beta) \) (Gupta and Jain [53])
5. \( T_{\alpha,\beta}^{1,1}(j) = B(\lambda, \alpha, j) \) (Frasin [42, 44])

4.3 Properties of The Class \( T_{\alpha,\beta}^{\lambda}(j) \)

We begin this section by obtaining the necessary and sufficient conditions for functions \( f \in T_{\alpha,\beta}^{\lambda}(j) \).

**Theorem 4.3.1.** A function \( f(z) \) of the form (4.2) is in the class \( T_{\alpha,\beta}^{\lambda}(j) \) if
\[
\sum_{n=j+1}^\infty R(\phi_n, \psi_n, \lambda, n)a_n \leq \beta(1 - \alpha)(1 + \gamma),
\]
(4.4)

where
\[
R(\phi_n, \psi_n, \lambda, n) = (1 + \beta\gamma)\phi_n + (\beta[1 - (1 + \gamma)\alpha] - 1) \psi_n
\]
(4.5)

and
\[
\phi_n = (2\lambda^2 - \lambda)n^2 + (1 + \lambda - 2\lambda^2)n, \quad \psi_n = (2\lambda^2 - \lambda)n + (1 + 2\lambda^2 - 3\lambda).
\]
(4.6)
Proof. For \(|z| = 1\), we have \(\frac{F_{\lambda} - 1}{\gamma F_{\lambda} + 1 - (1 + \gamma)\alpha} < \beta\), \(z \in \U\). It is suffices to show that

\[
\begin{align*}
\frac{F_{\lambda} - 1}{\gamma F_{\lambda} + 1 - (1 + \gamma)\alpha} &= \frac{\Phi_{n,\lambda}(z) - \Psi_{n,\lambda}(z)}{\beta} / \gamma \Phi_{n,\lambda}(z) + [1 - (1 + \gamma)\alpha] \Psi_{n,\lambda}(z) \\
&= \sum_{n=j+1}^{\infty} \phi_n a_n z^{n-1} - \beta (1 - \alpha)(1 + \gamma) - \sum_{n=j+1}^{\infty} [\gamma \phi_n + [1 - (1 + \gamma)\alpha] \psi_n] a_n z^{n-1} \\
&\leq \sum_{n=j+1}^{\infty} [\phi_n - \psi_n] a_n - \beta (1 - \alpha)(1 + \gamma) - \sum_{n=j+1}^{\infty} [\gamma \phi_n + [1 - (1 + \gamma)\alpha] \psi_n] a_n \\
&\leq \sum_{n=j+1}^{\infty} [(1 + \beta \gamma) \phi_n + (\beta[1 - (1 + \gamma)\alpha] - 1) \psi_n] a_n - \beta(1 + \gamma)(1 - \alpha) \\
&\leq 0, \text{ by hypothesis.}
\end{align*}
\]

Thus by Maximum Modulus Theorem and (4.3), \(f \in T_{\alpha,\beta}^{\lambda}(\U)\). Conversely, we assume that the function \(f\) is in the class \(T_{\alpha,\beta}^{\lambda}(\U)\). Then we have,

\[
\begin{align*}
\frac{F_{\lambda} - 1}{\gamma F_{\lambda} + 1 - (1 + \gamma)\alpha} &= \frac{\Phi_{n,\lambda}(z)}{\Psi_{n,\lambda}(z)} - 1 \\
&= \frac{\Phi_{n,\lambda}(z)}{\Psi_{n,\lambda}(z)} - 1 \\
&= \gamma \psi_{n,\lambda}(z) + 1 - (1 + \gamma)\alpha \\
&= \sum_{n=j+1}^{\infty} [\phi_n - \psi_n] a_n z^{n-1} - \sum_{n=j+1}^{\infty} [\gamma \phi_n + [1 - (1 + \gamma)\alpha] \psi_n] a_n z^{n-1} \\
&< \beta.
\end{align*}
\]

Since \(R(z) \leq |z|\) for all \(z\), we have

\[
R \frac{- \sum_{n=j+1}^{\infty} [\phi_n - \psi_n] a_n z^{n-1}}{(1 - \alpha)(1 + \gamma) - \sum_{n=j+1}^{\infty} [\gamma \phi_n + [1 - (1 + \gamma)\alpha] \psi_n] a_n z^{n-1}} < \beta, z \in \U.
\]

Now choose values of \(z\) on the real axis so that \(F_{\lambda}\) is real. Upon clearing the denominator in above inequality and letting \(z \to 1\) through real values, we obtain,

\[
\sum_{n=j+1}^{\infty} [\phi_n - \psi_n] a_n \leq \beta(1 - \alpha)(1 + \gamma) - \beta \sum_{n=j+1}^{\infty} [\gamma \phi_n + [1 - (1 + \gamma)\alpha] \psi_n] a_n
\]

which leads us readily to the inequality

\[
\sum_{n=j+1}^{\infty} [(1 + \beta \gamma) \phi_n + (\beta[1 - (1 + \gamma)\alpha] - 1) \psi_n] a_n \leq \beta(1 - \alpha)(1 + \gamma)
\]
and hence the proof is complete.

Finally, the function $f(z)$ given by

$$f(z) = z - \frac{\beta(1 - \alpha)(1 + \gamma)}{R(\phi_n, \psi_n, \lambda, n)} z^n, \quad n \geq j + 1, \quad j \in \mathbb{N}, \quad (4.7)$$

where $R(\phi_n, \psi_n, \lambda, n)$ as given in (4.5), is an external function for the assertion of Theorem 4.3.1.

**Corollary 4.3.2.** Let the function $f(z)$ defined by (4.2) be in the class $T_{\alpha, \beta}^j$. Then

$$a_n \leq \frac{\beta(1 - \alpha)(1 + \gamma)}{R(\phi_n, \psi_n, \lambda, n)}, \quad n \geq j + 1, \quad j \in \mathbb{N}. \quad (4.8)$$

The equality in (4.8) is attained for the function $f(z)$ given by (4.7).

Next, we obtain growth and distortion bounds for functions in the class $T_{\alpha, \beta}^j$.

**Theorem 4.3.3.** Let the function $f(z)$ defined by (4.2) be in the class $T_{\alpha, \beta}^j$. Then for $|z| < r = 1$

$$r - \frac{\beta(1 - \alpha)(1 + \gamma)}{R(\phi_{j+1}, \psi_{j+1}, \lambda, j + 1)} r^{j+1} \leq |f(z)| \leq r + \frac{\beta(1 - \alpha)(1 + \gamma)}{R(\phi_{j+1}, \psi_{j+1}, \lambda, j + 1)} r^{j+1} \quad (4.9)$$

The result (4.9) is attained for the function $f(z)$ given by (4.7) for $z = \pm r$.

**Proof.** Note that

$$R(\phi_{j+1}, \psi_{j+1}, \lambda, j + 1) \sum_{n=j+1}^{\infty} a_n \leq \sum_{n=j+1}^{\infty} R(\phi_n, \psi_n, \lambda, n) a_n \leq \beta(1 - \alpha)(1 + \gamma),$$

this last inequality follows from Theorem 4.3.1. Thus

$$|f(z)| \geq |z| - \sum_{n=j+1}^{\infty} a_n |z|^n \geq r - r^{j+1} \sum_{n=j+1}^{\infty} a_n \geq r - \frac{\beta(1 - \alpha)(1 + \gamma)}{R(\phi_{j+1}, \psi_{j+1}, \lambda, j + 1)} r^{j+1}. \quad (4.9)$$

Similarly,

$$|f(z)| \leq |z| + \sum_{n=j+1}^{\infty} a_n |z|^n \leq r + r^{j+1} \sum_{n=j+1}^{\infty} a_n \leq r + \frac{\beta(1 - \alpha)(1 + \gamma)}{R(\phi_{j+1}, \psi_{j+1}, \lambda, j + 1)} r^{j+1}. \quad (4.9)$$

This completes the proof. \qed
Theorem 4.3.4. Let the function \( f(z) \) defined by (4.2) be in the class \( T_{\alpha, \beta}^\lambda(J) \).

Then for \( |z| < r = 1 \)

\[
1 - \frac{(j + 1)(\beta(1 - \alpha)(1 + \gamma))}{R(\phi_{j+1}, \psi_{j+1}, \lambda, j+1)} r^j \leq |f(z)| \leq 1 + \frac{(j + 1)(\beta(1 - \alpha)(1 + \gamma))}{R(\phi_{j+1}, \psi_{j+1}, \lambda, j+1)} r^j.
\]

(4.10)

Proof. We have

\[
|f(z)| \geq 1 - n a_n / z^{n-1} \geq 1 - r^j \sum_{n=j+1}^\infty n a_n \tag{4.11}
\]

and

\[
|f(z)| \leq 1 + n a_n / z^{n-1} \leq 1 + r^j \sum_{n=j+1}^\infty n a_n. \tag{4.12}
\]

In view of Theorem 4.3.1,

\[
R(\phi_{j+1}, \psi_{j+1}, \lambda, j+1) \geq \sum_{n=j+1}^\infty n a_n \leq \sum_{n=j+1}^\infty R(\phi_n, \psi_n, \lambda, n) a_n \leq \beta(1 - \alpha)(1 + \gamma),
\]

(4.13)

or, equivalently

\[
\sum_{n=j+1}^\infty n a_n \leq \frac{(j + 1)(\beta(1 - \alpha)(1 + \gamma))}{R(\phi_{j+1}, \psi_{j+1}, \lambda, j+1)}. \tag{4.14}
\]

A substitution of (4.14) into (4.11) and (4.12) yields the inequality (4.10). This completes the proof. \( \square \)

Theorem 4.3.5. Define \( f_j(z) = z \), and

\[
f_n(z) = z - \frac{[\beta(1 - \alpha)(1 + \gamma)]}{R(\phi_n, \psi_n, \lambda, n)} z^n, \quad n \geq j + 1.
\]

(4.15)

Then \( f(z) \) is in the class \( T_{\alpha, \beta}^\lambda(J) \) if and only if it can be expressed in the form

\[
f(z) = \sum_{n=j}^\infty \mu_n f_n(z),
\]

(4.16)

where \( \mu_n \geq 0 \) \( (n \geq j) \) and \( \sum_{n=j}^\infty \mu_n = 1 \).

Proof. Assume that

\[
f(z) = \sum_{n=j}^\infty \mu_n f_n(z) = \sum_{n=j+1}^\infty \mu_n z - \frac{[\beta(1 - \alpha)(1 + \gamma)]}{R(\phi_n, \psi_n, \lambda, n)} z^n
\]

\[
= \sum_{n=j}^\infty \mu_n z - \sum_{n=j+1}^\infty \frac{[\beta(1 - \alpha)(1 + \gamma)]}{R(\phi_n, \psi_n, \lambda, n)} \mu_n z^n.
\]
Then it follows that
\[
\sum_{n=j+1}^{\infty} \frac{[\beta(1-\alpha)(1+\gamma)]}{R(\phi_n, \psi_n, \lambda, n)} \mu_n R(\phi_n, \psi_n, \lambda, n)
= \sum_{n=j+1}^{\infty} \mu_n [\beta(1-\alpha)(1+\gamma)]
\leq (1-\mu_j) [\beta(1-\alpha)(1+\gamma)] \leq \beta(1-\alpha)(1+\gamma),
\]
so by Theorem 4.3.1, \( f \in T_{\alpha, \beta}^{J}(j) \).

Conversely, assume that the function \( f(z) \) defined by (4.2) belongs to the class \( T_{\alpha, \beta}^{J}(j) \), then
\[
a_n \leq \frac{\beta(1-\alpha)(1+\gamma)}{R(\phi_n, \psi_n, \lambda, n)}, \quad n \geq j+1.
\]
Setting \( \mu_n = \frac{R(\phi_n, \psi_n, \lambda, n)}{\beta(1-\alpha)(1+\gamma)} a_n, n \geq j+1 \) and \( \mu_j = 1 - \sum_{n=j+1}^{\infty} \mu_n, n \geq j+1 \). Then \( f(z) = \sum_{n=j}^{\infty} \mu_n f_n(z) \) and this completes the proof.

Next, we find the radii of close-to-convexity, starlikeness and convexity for the class \( T_{\alpha, \beta}^{J}(j) \).

**Theorem 4.3.6.** Let \( f \in T_{\alpha, \beta}^{J}(j) \). Then \( f(z) \) is close-to-convex of order \( \sigma (0 \leq \sigma < 1) \) in the disc \(|z| < r_1\), where
\[
r_1 := \inf \frac{(1-\sigma) R(\phi_n, \psi_n, \lambda, n)}{n(\beta(1-\alpha)(1+\gamma))}^{\frac{1}{n+1}}, \quad n \geq j+1.
\]  
(4.17)
The result is sharp, with extremal function \( f(z) \) given by (4.7).

**Proof.** Given \( f \in T(j) \) and \( f \) is close-to-convex of order \( \sigma \), we have
\[
|f'(z) - 1| < 1 - \sigma.
\]  
(4.18)
For the left hand side of (4.18) we have
\[
|f'(z) - 1| \leq \sum_{n=j+1}^{\infty} n a_n |z|^{n-1}.
\]
The last expression is less than \( 1 - \sigma \) if
\[
\sum_{n=j+1}^{\infty} \frac{n}{1-\sigma a_n |z|^{n-1}} < 1.
\]
Using the fact, that \( f \in T_{\alpha, \beta}(j) \), if and only if
\[
\sum_{n=j+1}^{\infty} \frac{R(\phi_n, \psi_n, \lambda, n)}{(\beta(1-\alpha)(1+\gamma))^{\alpha n}} a_n \leq 1.
\]
We can say (4.18) is true if
\[
\frac{\psi_n}{1-\sigma}(z^{n-1}) \leq \frac{R(\phi_n, \psi_n, \lambda, n)}{\beta(1-\alpha)(1+\gamma)},
\]
Or, equivalently,
\[
|z|^{n-1} = \frac{(1-\alpha)R(\phi_n, \psi_n, \lambda, n)}{n(\beta(1-\alpha)(1+\gamma))},
\]
which completes the proof. \(\square\)

**Theorem 4.3.7.** Let \( f \in T_{\alpha, \beta}(j) \). Then

(i) \( f \) is starlike of order \( \sigma \) \((0 \leq \sigma < 1)\) in the disc \(|z| < r_2\); where
\[
r_2 = \inf \frac{1-\sigma}{n-\sigma} \frac{n^{\frac{1}{n-1}}}{\beta(1-\alpha)(1+\gamma)}, \quad n \geq j+1, \quad (4.19)
\]

(ii) \( f \) is convex of order \( \sigma \) \((0 \leq \sigma < 1)\) in the unit disc \(|z| < r_3\), where
\[
r_3 = \inf \frac{1-\sigma}{n(n-\sigma)} \frac{n^{\frac{1}{n-1}}}{\beta(1-\alpha)(1+\gamma)}, \quad n \geq j+1. \quad (4.20)
\]

Each of these results are sharp for the extremal function \( f(z) \) given by (4.7).

**Proof.** (i) Given \( f \in T(j) \), and \( f \) is starlike of order \( \sigma \), we have
\[
\frac{zf'(z)}{f(z)} - 1 > 1 - \sigma. \quad (4.21)
\]
For the left hand side of (4.21) we have
\[
\frac{zf'(z)}{f(z)} - 1 \leq \sum_{n=j+1}^{\infty} \frac{(n-1)a_n}{z^{n-1}} \leq \frac{\sum_{n=j+1}^{\infty} a_n}{1 - \sum_{n=j+1}^{\infty} a_n/z^{n-1}}.
\]
The last expression is less than \( 1 - \sigma \) if
\[
\frac{\sum_{n=j+1}^{\infty} a_n}{1 - \sigma a_n/z^{n-1}} < 1.
\]
Using the fact, that \( f \in T_{\alpha,\beta}^{\lambda,\gamma}(j) \) if and only if
\[
\frac{1}{n+1} \beta(1 - \alpha)(1 + \gamma) a^n \leq 1.
\]
We can say (4.21) is true if
\[
\frac{q - \sigma}{1 - \sigma} / z^{n-1} < \frac{R(\phi_n, \psi_n, \lambda, n)}{\beta(1 - \alpha)(1 + \gamma)}.
\]
Or, equivalently,
\[
/z^{n-1} = 
\frac{1 - \sigma}{n - \sigma} \frac{R(\phi_n, \psi_n, \lambda, n)}{\beta(1 - \alpha)(1 + \gamma)}
\]
which yields the starlikeness of the family.

(ii) Using the fact that \( f \) is convex if and only if \( zf' \) is starlike, we can prove (ii), on lines similar to the proof of (i). \( \square \)

Let the functions \( f_i(z)(i = 1, 2) \) be defined by
\[
f_i(z) = z - \frac{\infty}{n=0} a_{i,n} z^n, \quad a_{i,n} \geq 0; \quad j \in \mathbb{N},
\]
then we define the modified Hadamard product of \( f_1(z) \) and \( f_2(z) \) by
\[
(f_1 \ast f_2)(z) = z - \frac{\infty}{n=0} a_{1,n} a_{2,n} z^n.
\]
Now, we prove the following.

**Theorem 4.3.8.** Let each of the functions \( f_i(z)(i = 1, 2) \) defined by (4.22) be in the class \( T_{\alpha,\beta}^{\lambda,\gamma}(j) \). Then \( (f_1 \ast f_2) \in T_{\delta_1,\beta}^{\lambda,\gamma}(j) \), for
\[
\delta_1 = \frac{R(\phi_n, \psi_n, \lambda, n)^2 - [(1 + \beta \gamma) \phi_n + (\beta - 1) \psi_n] \beta(1 - \alpha)^2(1 + \gamma)}{R(\phi_n, \psi_n, \lambda, n)^2 - \psi_n \beta(1 - \alpha)(1 + \gamma)^2}.
\]
The result is sharp.

**Proof.** We need to prove the largest \( \delta_1 \) such that
\[
\frac{\infty}{n=0} [(1 + \beta \gamma) \phi_n + (\beta - 1)(1 + \gamma) \delta_1 - 1] \psi_n \beta(1 - \delta_1)(1 + \gamma) a_{1,n} a_{2,n} \leq 1.
\]
From Theorem 4.3.1, we have
\[
\frac{\infty R(\phi_n, \psi_n, \lambda, n)}{n=j+1 \beta(1-\alpha)(1+\gamma) \alpha_{n,1}^{a_{n,1}}} \leq 1
\]
and
\[
\frac{\infty R(\phi_n, \psi_n, \lambda, n)}{n=j+1 \beta(1-\alpha)(1+\gamma) \alpha_{n,2}^{a_{n,2}}} \leq 1,
\]
by the Cauchy-Schwarz inequality, we have
\[
\frac{\infty R(\phi_n, \psi_n, \lambda, n)}{n=j+1 \beta(1-\alpha)(1+\gamma) \alpha_{n,1}^{a_{n,1}} \alpha_{n,2}^{a_{n,2}}} \leq 1.
\]
(4.26)
Thus it is sufficient to show that, for \( n \geq j + 1 \)
\[
\frac{\left[(1 + \beta y) \phi_n + (\beta [1 - (1 + y) \delta]) - 1 \right] \psi_n}{\beta(1 - \delta)(1 + \gamma) \alpha_{n,1}^{a_{n,1}} \alpha_{n,2}^{a_{n,2}}} \leq \frac{R(\phi_n, \psi_n, \lambda, n)}{\beta(1 - \alpha)(1 + \gamma) \alpha_{n,1}^{a_{n,1}} \alpha_{n,2}^{a_{n,2}},}
\]
(4.27)
that is
\[
\frac{\beta(1 - \alpha)(1 + \gamma)}{R(\phi_n, \psi_n, \lambda, n)} \leq \frac{\left[(1 + \beta y) \phi_n + (\beta [1 - (1 + y) \delta]) - 1 \right] \psi_n}{\beta(1 - \delta)(1 + \gamma)},
\]
(4.28)
Note that
\[
\frac{\beta(1 - \alpha)(1 + \gamma)}{R(\phi_n, \psi_n, \lambda, n)} \leq \frac{\beta(1 - \delta)(1 + \gamma)}{R(\phi_n, \psi_n, \lambda, n)} \leq \frac{\beta(1 - \delta)(1 + \gamma)}{\beta(1 - \alpha)(1 + \gamma)}
\]
Consequently, we need only to prove that
\[
\frac{\left[(1 + \beta y) \phi_n + (\beta [1 - (1 + y) \delta]) - 1 \right] \psi_n}{\beta(1 - \delta)(1 + \gamma)} \leq \frac{\beta(1 - \alpha)(1 + \gamma)}{\beta(1 - \alpha)(1 + \gamma)}
\]
(4.30)
or equivalently
\[
\delta_1 \leq \frac{R(\phi_n, \psi_n, \lambda, n)^2 - [(1 + \beta y) \phi_n + (\beta - 1) \psi_n] \beta(1 - \alpha)^2(1 + \gamma)}{R(\phi_n, \psi_n, \lambda, n)^2 - \psi_n \beta(1 - \alpha)(1 + \gamma)^2} = \Delta(n).
\]
(4.31)
Since \( \Delta(n) \) is an increasing function of \( n \) \((n \geq j + 1)\), letting \( n = j + 1 \) in (4.31) we obtain
\[
\delta_1 \leq \Delta(j + 1) = \frac{R(\phi_{j+1}, \psi_{j+1}, \lambda_{j+1})^2 - [(1 + \beta y) \phi_{j+1} + (\beta - 1) \psi_{j+1}] \beta(1 - \alpha)^2(1 + \gamma)}{R(\phi_{j+1}, \psi_{j+1}, \lambda_{j+1})^2 - \psi_{j+1} \beta(1 - \alpha)(1 + \gamma)^2}
\]
(4.32)
which proves the main assertion of Theorem 4.3.8. The result is sharp for the functions defined by (4.7).
Theorem 4.3.9. Let the function $f_i(z)(i = 1, 2)$ defined by (4.22) be in the class $T_{\alpha,\beta}^{\lambda,\psi}(f)$. If the sequence $\{\mathcal{R}(\phi_n, \psi_n, \lambda, \alpha)\}$ is non-decreasing. Then the function

$$h(z) = z - \sum_{n=0}^{\infty} \left( a_{n,1}^2 + a_{n,2}^2 \right) z^n$$

belongs to the class $T_{\delta_2,\psi}(f)$, where

$$\delta_2 = \frac{R(\phi_n, \psi_n, \lambda, \alpha)^2 - 2[(1 + \beta)\phi_n + (\beta - 1)\psi_n](\beta(1 - \alpha)(1 + \beta))^2}{\psi_n^2}.$$

Proof. By virtue of Theorem 4.3.1, we have for $f_j \in T_{\alpha,\beta}^{\lambda,\psi}(f)$, we have

$$\sum_{n=j+1}^{\infty} \frac{R(\phi_n, \psi_n, \lambda, \alpha)}{\beta(1 - \alpha)(1 + \beta)} a_{n,1}^2 \leq \sum_{n=j+1}^{\infty} \frac{R(\phi_n, \psi_n, \lambda, \alpha)}{\beta(1 - \alpha)(1 + \beta)} a_{n,1}^2 \leq 1$$

and

$$\sum_{n=j+1}^{\infty} \frac{R(\phi_n, \psi_n, \lambda, \alpha)}{\beta(1 - \alpha)(1 + \beta)} a_{n,2}^2 \leq \sum_{n=j+1}^{\infty} \frac{R(\phi_n, \psi_n, \lambda, \alpha)}{\beta(1 - \alpha)(1 + \beta)} a_{n,2}^2 \leq 1. \quad (4.35)$$

It follows from (4.34) and (4.35) that

$$\sum_{n=j+1}^{\infty} \frac{1}{2} \frac{R(\phi_n, \psi_n, \lambda, \alpha)}{\beta(1 - \alpha)(1 + \beta)} (a_{n,1}^2 + a_{n,2}^2) \leq 1. \quad (4.36)$$

Therefore we need to find the largest $\delta_2$, such that

$$\frac{[(1 + \beta)\phi_n + (\beta[1 - (1 + \beta)] - 1)\psi_n]}{\beta(1 - \beta_2)(1 + \beta)} \leq \frac{1}{2} \frac{R(\phi_n, \psi_n, \lambda, \alpha)}{\beta(1 - \alpha)(1 + \beta)}, \quad n \geq j + 1$$

that is

$$\delta_2 \leq \frac{R(\phi_n, \psi_n, \lambda, \alpha)^2 - 2[(1 + \beta)\phi_n + (\beta - 1)\psi_n](\beta(1 - \alpha)(1 + \beta))^2}{\psi_n^2} = \psi(n).$$

Since $\psi(n)$ is an increasing function of $n, (n \geq j + 1)$, we readily have

$$\delta_2 \leq \psi(j + 1) = \frac{R(\phi_{j+1}, \psi_{j+1}, \lambda, j+1)^2 - 2[(1 + \beta)\phi_{j+1} + (\beta - 1)\psi_{j+1}](\beta(1 - \alpha)(1 + \beta))^2}{\psi_{j+1}^2}$$

which completes the proof. \qed
4.4 The Class of Uniformly Starlike and Corresponding Class of Uniformly Convex Functions

Motivated by the works of Bharati et al [21], Frasin [42, 44], Murugusundaramoorthy and Magesh [92] and others [50, 51, 85, 113, 107, 135, 134], we define the following class:

**Definition 4.4.1.** For $\beta \geq 0$, $-1 \leq \alpha < 1$ and $0 \leq \lambda < 1$, we let $U_{\alpha,\beta}^1(j)$ denote the subclass of $S$ consisting of functions $f(z)$ of the form (4.1) and satisfying the analytic criterion

\[
\Re \left\{ \frac{zf'(z) + (2\lambda^2 - \lambda)z^2 f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \right\} - \alpha > \beta, \quad z \in U.
\]

We also let $UT_{\alpha,\beta}^1(j) = U_{\alpha,\beta}^1(j)$ $T$ where $T$, the subclass of $S$ consisting of functions of the form (4.2).

We note that, by specializing the parameters $j$, $\lambda$, $\alpha$, and $\beta$, we obtain the following subclasses studied by various authors.

1. $UT_{0,0}^1(1) = S^*(\alpha)$ and $UT_{0,0}^1(1) = K(\alpha)$ (Silverman [120])

2. $UT_{0,0}^1(j) = S^*(\alpha, j)$ and $UT_{0,0}^1(j) = K(\alpha, j)$ (Srivastava et al. [131])

3. $UT_{0,0}^1(1) = P(\alpha)$ (Al-Amiri [6], Gupta and Jain [53] and Sarangi and Uraleegaddi [118])

4. $UT_{0,0}^1(j) = B_2(\lambda, \alpha, j)$ (Frasin [42, 44])

5. $UT_{0,0}^1(1) = TR(\alpha, \beta)$ (Rosy [107] and Stephen and Subramanian [133])

6. $UT_{0,0}^1(1) = TS(\alpha, \beta)$ and $UT_{0,0}^1(1) = UCV(\alpha, \beta)$ (Bharati et al. [21])

7. $UT_{0,0}^1(1) = TS(\beta)$ (Subramanian et al. [135]) and $UT_{0,0}^1(1) = UCV(\beta)$ (Subramanian et al. [134])
4.5 Characterization Properties

We begin this section, by obtaining the following necessary and sufficient conditions for functions \( f(z) \) in the class \( UT_{\alpha,\beta}(j) \).

**Theorem 4.5.1.** A function \( f(z) \) of the form (4.1) is in \( UT_{\alpha,\beta}(j) \) if

\[
\sum_{n=j+1}^{\infty} [M_n(1 + \beta) - (\alpha + \beta)F_n]/a_n \leq 1 - \alpha,
\]

where

\[
M_n = (2\lambda^2 - \lambda)n^2 + (1 + \lambda - 2\lambda^2)n, \quad F_n = (2\lambda^2 - \lambda)n + (1 + 2\lambda^2 - 3\lambda).
\](4.38)

**Proof.** It suffices to show that

\[
\beta: z^\beta f(z) + (2\lambda^2 - \lambda)z^2 f''(z) \leq 1:
\]

\[
\Re - \frac{z^\beta f(z) + (2\lambda^2 - \lambda)z^2 f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)z^2 f'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} - 1 \leq 1 - \alpha.
\]

We have

\[
\beta: \frac{z^\beta f(z) + (2\lambda^2 - \lambda)z^2 f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)z^2 f'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \leq 1:
\]

\[
\Re - \frac{z^\beta f(z) + (2\lambda^2 - \lambda)z^2 f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)z^2 f'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \leq 1:
\]

\[
(1 + \beta) \sum_{n=j+1}^{\infty} (M_n - F_n)/a_n \leq 1 - \alpha,
\]

This last expression is bounded above by \( (1 - \alpha) \) if

\[
\sum_{n=j+1}^{\infty} [M_n(1 + \beta) - (\alpha + \beta)F_n]/a_n \leq 1 - \alpha,
\]

and hence the proof is complete. \( \square \)

**Theorem 4.5.2.** A necessary and sufficient condition for \( f(z) \) of the form (4.2) to be in the class \( UT_{\alpha,\beta}(j) \), is that

\[
\sum_{n=j+1}^{\infty} [M_n(1 + \beta) - (\alpha + \beta)F_n] a_n \leq 1 - \alpha.
\]
Proof. In view of Theorem 4.5.1, we need only to prove the necessity. If \( f \in UT_{\alpha,\beta}(j) \) and \( z \) is real then

\[
1 - \sum_{n=j+1}^{\infty} M_n a_n z^{n-1} - \alpha \geq \beta \sum_{n=j+1}^{\infty} (M_n - F_n) a_n z^{n-1}.
\]

Letting \( z \to 1 \) along the real axis, we obtain the desired inequality

\[
\sum_{n=j+1}^{\infty} [M_n(1 + \beta) - (\alpha + \beta)F_n] a_n \leq 1 - \alpha.
\]

Finally, the function \( f(z) \) given by

\[
f(z) = z - \frac{1 - \alpha}{[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}] z^{j+1}}, \tag{4.39}
\]

where \( M_{j+1} \) and \( F_{j+1} \) as written in (4.38), is extremal for the function. \( \square \)

**Corollary 4.5.3.** Let the function \( f(z) \) defined by (4.2) be in the class \( UT_{\alpha,\beta}(j) \). Then

\[
a_n \leq \frac{1 - \alpha}{[M_n(1 + \beta) - (\alpha + \beta)F_n]}, \quad n \geq j + 1. \tag{4.40}
\]

This equality in (4.40) is attained for the function \( f(z) \) given by (4.39).

Next in the following theorems we find, growth and distortion bounds for the class \( UT_{\alpha,\beta}(j) \).

**Theorem 4.5.4.** Let the function \( f(z) \) defined by (4.2) be in the class \( UT_{\alpha,\beta}(j) \). Then for \( |z| < r = 1 \)

\[
r - \frac{1 - \alpha}{[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}]} z^{j+1} \leq |f(z)| \leq r + \frac{1 - \alpha}{[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}]} z^{j+1}. \tag{4.41}
\]

The result (4.41) is attained for the function \( f(z) \) given by (4.39) for \( z = \pm r \).

**Proof.** Note that

\[
[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}] z^{j+1} a_n \leq \frac{1}{z^{j+1}} [M_n(1 + \beta) - (\alpha + \beta)F_n] a_n \leq 1 - \alpha,
\]
this last inequality follows from Theorem 4.5.2. Thus

\[
|f(z)| = |z| - \sum_{n=j+1}^{\infty} a_n/z^n \geq r - r^{j+1} \sum_{n=j+1}^{\infty} a_n \geq r - \frac{1 - \alpha}{[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}]} r^{j+1}.
\]

Similarly,

\[
|f(z)| = |z| + \sum_{n=j+1}^{\infty} a_n/z^n \leq r + r^{j+1} \sum_{n=j+1}^{\infty} a_n \leq r + \frac{1 - \alpha}{[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}]} r^{j+1}.
\]

This completes the proof.

**Theorem 4.5.5.** Let the function \( f(z) \) defined by (4.2) be in the class \( \mathcal{U}_{T, \beta}(j) \).

Then for \( |z| < r = 1 \)

\[
r - \frac{(j + 1)(1 - \alpha)}{[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}]} \leq |\hat{f}(z)| \leq r + \frac{(j + 1)(1 - \alpha)}{[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}]}.
\]  

(4.42)

**Proof.** We have

\[
|\hat{f}(z)| \geq 1 - \sum_{n=j+1}^{\infty} na_n/z^{n-1} \geq 1 - r^j \sum_{n=j+1}^{\infty} na_n
\]  

(4.43)

and

\[
|\hat{f}(z)| \leq 1 + \sum_{n=j+1}^{\infty} na_n/z^{n-1} \leq 1 + r^j \sum_{n=j+1}^{\infty} na_n.
\]  

(4.44)

In view of Theorem 4.5.2,

\[
[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}] \sum_{n=j+1}^{\infty} na_n \leq \sum_{n=j+1}^{\infty} [M_n(1 + \beta) - (\alpha + \beta)F_n]a_n \leq 1 - \alpha,
\]

or, equivalently

\[
\sum_{n=j+1}^{\infty} na_n \leq \frac{(j + 1)(1 - \alpha)}{[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}]}.  
\]

(4.45)

A substitution of (4.45) into (4.43) and (4.44) yields the inequality (4.42). This completes the proof.

**Theorem 4.5.6.** Let \( f_j(z) = z \), and

\[
f_n(z) = z - \frac{1 - \alpha}{[M_n(1 + \beta) - (\alpha + \beta)F_n]} z^n, \quad n \geq j + 1.
\]
for $0 \leq \lambda \leq 1, \beta \geq 0, -1 \leq \alpha < 1$. Then $f(z)$ is in the class $UT_{\alpha, \beta}(j)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=j}^{\infty} \mu_n f_n(z), \quad (4.46)$$

where $\mu_n \geq 0$ ($n \geq j$) and $\mu_n \geq 1$.

**Proof.** Assume that

$$f(z) = \sum_{n=j}^{\infty} \mu_n z^n - \frac{1 - \alpha}{[M_n(1 + \beta) - (\alpha + \beta)F_n]} z^n,$$

Then it follows that

$$\sum_{n=j}^{\infty} \frac{1 - \alpha}{[M_n(1 + \beta) - (\alpha + \beta)F_n]} \mu_n \left[ M_n(1 + \beta) - (\alpha + \beta)F_n \right] \mu_n z^n = \sum_{n=j}^{\infty} \mu_n z^n,$$

so by Theorem 4.5.2, $f \in UT_{\alpha, \beta}(j)$. Conversely, assume that the function $f(z)$ defined by (4.2) belongs to the class $UT_{\alpha, \beta}(j)$, then

$$a_n \leq \frac{1 - \alpha}{[M_n(1 + \beta) - (\alpha + \beta)F_n]} a_n (n \geq j + 1)$$

Setting $\mu_n = \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]}{1 - \alpha} a_n (n \geq j + 1)$ and $\mu_j = 1 - \sum_{n=j+1}^{\infty} \mu_n$, we have,

$$f(z) = z - \sum_{n=j+1}^{\infty} a_n z^n$$

Then (4.46) gives

$$f(z) = f(z) \mu_j + \sum_{n=j+1}^{\infty} \mu_n f_n(z),$$

and hence the proof is complete. □
Now, we obtain the radii of close-to-convexity, starlikeness and convexity for the class $UT^\lambda_{\beta}(f)$.

**Theorem 4.5.7.** Let $f \in UT^\lambda_{\beta}(j)$. Then $f(z)$ is close-to-convex of order $\sigma (0 \leq \sigma < 1)$ in the disc $|z| < r_1$, where

$$r_1 := \inf \frac{(1 - \sigma)(M_n(1 + \beta) - (\alpha + \beta)F_n)}{n(1 - \alpha)} \frac{1}{n!}, \quad n \geq j + 1.$$  

The result is sharp, with extremal function $f(z)$ given by (4.39).

**Proof.** Given $f \in T$ and $f$ is close-to-convex of order $\sigma$, we have

$$|f'(z) - 1| < 1 - \sigma.$$  

(4.47)

For the left hand side of (4.47), we have

$$|f'(z) - 1| \leq \sum_{n=j+1}^{\infty} n a_n / z^{n-1}.$$  

The last expression is less than $1 - \sigma$ if

$$\sum_{n=j+1}^{\infty} \frac{n}{1 - \sigma} a_n / z^{n-1} < 1.$$  

Using the fact, that $f \in UT^\lambda_{\beta}(j)$, if and only if

$$\sum_{n=j+1}^{\infty} \frac{M_n(1 + \beta) - (\alpha + \beta)F_n}{n(1 - \alpha)} a_n \leq 1.$$  

(4.47)

We can say (4.47) is true if

$$\sum_{n=j+1}^{\infty} \frac{M_n(1 + \beta) - (\alpha + \beta)F_n}{n(1 - \alpha)} a_n \leq 1.$$  

(4.47)

which completes the proof. \qed

**Theorem 4.5.8.** Let $f \in UT^\lambda_{\beta}(j)$. Then
(i) \( f \) is starlike of order \( \sigma (0 \leq \sigma < 1) \) in the disc \(|z| < r_2\); where
\[
r_2 = \inf_+ \frac{1 - \sigma}{n - \sigma} \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]}{(1 - \alpha)}^{\frac{n+1}{n}}, \quad n \geq j + 1,
\]

(ii) \( f \) is convex of order \( \sigma (0 \leq \sigma < 1) \) in the unit disc \(|z| < r_3\), where
\[
r_3 = \inf_+ \frac{1 - \sigma}{n(n - \sigma)} \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]}{(1 - \alpha)}^{\frac{n+1}{n-1}}, \quad n \geq j + 1.
\]

Each of these results are sharp for the extremal function \( f(z) \) given by (4.39).

**Proof.** (i) Given \( f \in T \) and \( f \) is starlike of order \( \sigma \), we have
\[
\frac{zf'(z)}{f(z)} - 1 < 1 - \sigma.
\]

For the left hand side of (4.48) we have
\[
\frac{zf'(z)}{f(z)} - 1 \leq \frac{\sum_{n=j+1}^{\infty} (n-1)a_n |z|^{n-1}}{1 - \sum_{n=j+1}^{\infty} a_n |z|^{n-1}}.
\]

The last expression is less than \( 1 - \sigma \) if
\[
\sum_{n=j+1}^{\infty} \frac{n - \sigma}{1 - \sigma} a_n |z|^{n-1} < 1.
\]

Using the fact that \( f \in UT_{\alpha,\beta}(f) \), if and only if
\[
\sum_{n=j+1}^{\infty} \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]}{(1 - \alpha)} a_n \leq 1.
\]

We can say (4.48) is true if
\[
\frac{\beta - \sigma}{1 - \sigma} |z|^{n-1} < \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]}{(1 - \alpha)}.
\]

Or, equivalently,
\[
|z|^{n-1} = \frac{1 - \sigma}{n - \sigma} \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]}{(1 - \alpha)}
\]

which yields the starlikeness of the family.

(ii) Using the fact that \( f \) is convex if and only if \( zf' \) is starlike, we can prove

(ii), on lines similar to the proof of (i). \( \square \)
Let the functions \( f_i(z)(i = 1, 2) \) be defined by
\[
f_i(z) = z - \sum_{n=j+1}^{\infty} a_{n,i} z^n, \quad a_{n,i} \geq 0; j \in \mathbb{N}, \tag{4.49}
\]
then we define the modified Hadamard product of \( f_1(z) \) and \( f_2(z) \) by
\[
(f_1 \ast f_2)(z) = z - \sum_{n=j+1}^{\infty} a_{n,1} a_{n,2} z^n.
\]

Now, we prove the following.

**Theorem 4.5.9.** Let each of the functions \( f_i(z)(i = 1, 2) \) defined by (4.49) be in the class \( UT_{\delta, \beta}(j) \). Then \( f_1 \ast f_2 \in UT_{\delta, \beta}(j) \), for
\[
\delta_i = \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n] - [M_n(1 + \beta) - \beta F_n](1 - \alpha)^2}{[M_n(1 + \beta) - (\alpha + \beta)F_n] - F_n(1 - \alpha)^2}. \tag{4.50}
\]
The result is sharp.

**Proof.** We need to prove the largest \( \delta_i \) such that
\[
\sum_{n=j+1}^{\infty} \frac{[M_n(1 + \beta) - (\delta_i + \beta)F_n]}{1 - \delta_i} a_{n,1} a_{n,2} \leq 1.
\]
From Theorem 4.5.2, we have
\[
\sum_{n=j+1}^{\infty} \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]}{1 - \alpha} a_{n,1} \leq 1
\]
and
\[
\sum_{n=j+1}^{\infty} \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]}{1 - \alpha} a_{n,2} \leq 1,
\]
by the Cauchy-Schwarz inequality, we have
\[
\sum_{n=j+1}^{\infty} \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]}{1 - \alpha} \sqrt{a_{n,1} a_{n,2}} \leq 1.
\]
Thus it is sufficient to show that, for \( n \geq j + 1, \)
\[
\frac{[M_n(1 + \beta) - (\delta_i + \beta)F_n]}{1 - \delta_i} a_{n,1} a_{n,2} \leq \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]}{1 - \alpha} \sqrt{a_{n,1} a_{n,2}},
\]
that is
\[
\sqrt{a_{n,1} a_{n,2}} \leq \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n](1 - \delta_i)}{[M_n(1 + \beta) - (\delta_i + \beta)F_n](1 - \alpha)}, \quad n \geq j + 1.
Note that
\[ \sqrt{a_{n,1}a_{n,2}} \leq \frac{(1 - \alpha)}{[M_n(1 + \beta) - (\alpha + \beta)F_n]}, \quad n \geq j + 1. \]

Consequently, we need only to prove that
\[ \frac{(1 - \alpha)}{[M_n(1 + \beta) - (\alpha + \beta)F_n]} \leq \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n](1 - \delta_1)}{[M_n(1 + \beta) - (\delta_1 + \beta)F_n](1 - \alpha)}. \]

or equivalently
\[ \delta_1 \leq \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]^2 - [M_n(1 + \beta) - \beta F_n](1 - \alpha)^2}{[M_n(1 + \beta) - (\alpha + \beta)F_n]^2 - F_n(1 - \alpha)^2} = \Delta(n). \quad (4.51) \]

Since \( \Delta(n) \) is an increasing function of \( n(n \geq j + 1) \), letting \( n = j + 1 \) in (4.51) we obtain
\[ \delta_1 \leq \Delta(j + 1) = \frac{[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}]^2 - [M_{j+1}(1 + \beta) - \beta F_{j+1}](1 - \alpha)^2}{[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}]^2 - F_{j+1}(1 - \alpha)^2} \]

which proves the main assertion of Theorem 4.5.9. The result is sharp for the functions defined by (4.39).

\[ \square \]

**Theorem 4.5.10.** Let the function \( f_i(z)(i = 1, 2) \) defined by (4.49) be in the class \( UT_{\alpha, \beta}(j) \). If the sequence \( \{M_n(1 + \beta) - (\alpha + \beta)F_n\}^2 \) is non-decreasing. Then the function
\[ h(z) = z - \sum_{n=j+1}^{\infty} \left( \frac{a_{n,1}^2 + a_{n,2}^2}{1 - \alpha} \right) z^n \]

belongs to the class \( UT_{\delta_1, \beta}(j) \) where
\[ \delta_2 = \frac{[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}]^2 - [M_{j+1}(1 + \beta) - F_{j+1}](1 - \alpha)^2}{[M_{j+1}(1 + \beta) - (\alpha + \beta)F_{j+1}]^2 - 2F_{j+1}(1 - \alpha)^2}. \]

\[ \text{Proof.} \] By virtue of Theorem 4.5.2, we have for \( f_j \in UT_{\alpha, \beta}(j) \), \( j = 1, 2 \), we have
\[ \sum_{n=j+1}^{\infty} \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]^2}{1 - \alpha} a_{n,1}^2 \leq \sum_{n=j+1}^{\infty} \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]}{1 - \alpha} a_{n,1}^2 \leq 1 \]
\[ (4.52) \]

and
\[ \sum_{n=j+1}^{\infty} \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]^2}{1 - \alpha} a_{n,2}^2 \leq \sum_{n=j+1}^{\infty} \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]}{1 - \alpha} a_{n,2}^2 \leq 1. \]
\[ (4.53) \]
It follows from (4.52) and (4.53) that
\[
\lim_{n \to \infty} \frac{1}{\overline{2}} \cdot \frac{\left( M_n(1 + \beta) - (\alpha + \beta)F_n \right)^2}{1 - \alpha} (\hat{a}_{n,1}^2 + \hat{a}_{n,2}^2) \leq 1.
\]

Therefore we need to find the largest \( \delta_2 \), such that
\[
\frac{[M_n(1 + \beta) - (\delta_2 + \beta)F_n]}{1 - \delta_2} \leq \frac{1}{2} \cdot \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]^2}{1 - \alpha}, \quad n \geq j + 1
\]
that is
\[
\delta_2 \leq \frac{[M_n(1 + \beta) - (\alpha + \beta)F_n]^2 - 2[M_n(1 + \beta) - \beta F_n](1 - \alpha)^2}{[M_n(1 + \beta) - (\alpha + \beta)F_n]^2 - 2F_n(1 - \alpha)^2} = \Psi(n).
\]

Since \( \Psi(n) \) is an increasing function of \( n \) \((n \geq j + 1)\), we readily have
\[
\delta_2 \leq \Psi(j + 1) = \frac{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]^2 - 2[M_{j+1}(1+\beta)-\beta F_{j+1}](1-\alpha)^2}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]^2 - 2F_{j+1}(1-\alpha)^2}
\]
which completes the proof.