CHAPTER 3

CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS

3.1 Introduction

Motivated by Brannan and Taha [28] and Srivastava et al. [128], several new subclasses of bi-univalent functions have been defined and investigated by researchers [7, 45, 55, 103, 129, 139, 141]. In light of this, in the first section of this chapter two new subclasses \( \mathcal{G}_B(\alpha, \lambda) \) and \( \mathcal{J}^*_{B}(\beta, \lambda) \) of analytic and bi-univalent functions in the open unit disk \( U \) are defined. Further, inspired by the works of Xu et al. [139, 141] in the following section of this chapter an interesting general subclass \( \mathcal{M}^{\psi,\psi}_{B}(\lambda) \) of analytic and bi-univalent functions in the open unit disk \( U \) is introduced and investigated. For aforementioned classes, the estimates on the first two Taylor-Maclaurin coefficients \( |a_2| / \lambda a_3 / | \) are obtained. The results presented in this chapter would generalize and improve some recent works of Ali et al. [7], Srivastava et al. [128], Murugusundaramoorthy and Magesh [93], Xu et al. [139] and other authors.

3.2 The Class of Bi-starlike and Strongly Bi-starlike Functions

Definition 3.2.1. A function \( f(z) \) given by (1.1) is said to be in the class \( \mathcal{J}^*_{B}(\beta, \lambda) \) if the following conditions are satisfied:

\[
f \in \Sigma_B, \quad R - \frac{zf'(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} > \beta, \quad 0 \leq \beta < 1; \quad 0 \leq \lambda < 1, \quad z \in U \tag{3.1}
\]

and

\[
R - \frac{wg'(z)}{(1 - \lambda)g(w) + \lambda wg'(w)} > \beta, \quad 0 \leq \beta < 1; \quad 0 \leq \lambda < 1, \quad w \in U, \tag{3.2}
\]

where the function \( g \) is given by (1.4).
**Definition 3.2.2.** A function $f(z)$ given by (1.1) is said to be in the class $G_{\Sigma_B}(\alpha, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma_B, \quad \arg \frac{zf'(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} : < \frac{\alpha \pi}{2}, \quad 0 < \alpha \leq 1; \quad 0 \leq \lambda < 1, \quad z \in U$$

and

$$\arg \frac{wg'(z)}{(1 - \lambda)g(w) + \lambda wg'(w)} : < \frac{\alpha \pi}{2}, \quad 0 < \alpha \leq 1; \quad 0 \leq \lambda < 1, \quad w \in U,$$  

where the function $g$ is given by (1.4).

It is interest to note that, for $\lambda = 0$ the class $J_{\Sigma_B}^{\alpha} (\beta, 0) = S_{\Sigma_B}^{\beta} (\beta)$ bi-starlike functions of order $\beta$ and $G_{\Sigma_B}(\alpha, 0) = SS_{\Sigma_B}^{\alpha}(\alpha)$ of strongly bi-starlike functions of order $\alpha$.

In order to derive our main results, we recall the following lemma.

**Lemma 3.2.3.** [106] If $h \in P$ then $|c_k| \leq 2$ for each $k$, where $P$ is the family of all functions $h$ analytic in $U$ for which $R\{h(z)\} > 0$, where $h(z) = 1 + c_1z + c_2z^2 + \ldots$ for $z \in U$.

Now, we find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $f \in G_{\Sigma_B}(\alpha, \lambda)$.

**Theorem 3.2.4.** Let $f(z)$ given by (1.1) be in the class $G_{\Sigma_B}(\alpha, \lambda)$, $0 < \alpha \leq 1$ and $0 \leq \lambda < 1$. Then

$$|a_2| \leq \frac{2\alpha}{(1 - \lambda)^{\alpha - 1} + \alpha},$$  

and

$$|a_3| \leq \frac{4\alpha^2}{(1 - \lambda)^2} + \frac{\alpha}{1 - \lambda}.$$  

**Proof.** It follows from (3.3) and (3.4) that

$$\frac{zf'(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} = \left[\rho(z)\right]^\alpha,$$  

and

$$\frac{wg'(z)}{(1 - \lambda)g(w) + \lambda wg'(w)} = \left[\varrho(w)\right]^\alpha.$$
where \( p(z) \) and \( q(w) \) in \( P \) and have the forms

\[
p(z) = 1 + p_1 z + p_2 z^2 + \ldots \quad (3.9)
\]

and

\[
q(z) = 1 + q_1 w + q_2 w^2 + \ldots \quad (3.10)
\]

Now, equating the coefficients in (3.7) and (3.8), we get

\[
(1 - \lambda) a_2 = \alpha p_1 \quad (3.11)
\]

\[
(\lambda^2 - 1) a_2^2 + 2(1 - \lambda) a_3 = \frac{1}{2} \alpha (\alpha - 1) p_1^2 + 2 \alpha p_2 \quad (3.12)
\]

\[
-(1 - \lambda) a_2 = \alpha q_1 \quad (3.13)
\]

and

\[
(\lambda^2 - 4 \lambda + 3) a_2^2 - 2(1 - \lambda) a_3 = \frac{1}{2} \alpha (\alpha - 1) q_1^2 + 2 \alpha q_2 \quad (3.14)
\]

From (3.11) and (3.13), we get

\[
\rho_1 = -q_1 \quad (3.15)
\]

and

\[
2(1 - \lambda)^2 a_2^2 = \alpha^2 (p_1^2 + q_1^2) \quad (3.16)
\]

From (3.12), (3.14) and (3.16), we obtain

\[
a_2^2 = \frac{\alpha^2 (p_2 + q_2)}{(\alpha + 1)(1 - \lambda)^2}.
\]

Applying Lemma 3.2.3 for the coefficients \( p_2 \) and \( q_2 \), we immediately have

\[
\frac{1}{a_2} \leq \frac{2 \alpha}{(1 - \lambda) (1 + \alpha)}.
\]

This gives the bound on \( a_2 \) as asserted in (3.5).

Next, in order to find the bound on \( a_3 \), by subtracting (3.14) from (3.12), we get

\[
4(1 - \lambda) a_3 - 4(1 - \lambda) a_2^2 = \alpha (p_2 - q_2) + \frac{\alpha (\alpha - 1)}{2} (p_1^2 - q_1^2) \quad (3.17)
\]
It follows from (3.15), (3.16) and (3.17) that

$$a_3 = \frac{\alpha(p_2 - q_2)}{4(1 - \lambda)} + \frac{\alpha^2(p_1^2 + q_1^2)}{2(1 - \lambda^2)}. \quad (3.18)$$

Applying Lemma 3.2.3 once again for the coefficients $p_1, p_2$, $q_1$ and $q_2$, we readily get

$$/a_3/ \leq \frac{4\alpha^2}{(1 - \lambda)^2} + \frac{\alpha}{1 - \lambda}.$$ 

This completes the proof of Theorem 3.2.4. \qed

In the following theorem, we find the estimates on the coefficients $/a_2/$ and $/a_3/$ for functions in the class $J^{1/2}_0(\beta, \lambda)$.

**Theorem 3.2.5.** Let $f(z)$ given by (1.1) be in the class $J^{1/2}_0(\beta, \lambda)$, $0 \leq \beta < 1$ and $0 \leq \lambda < 1$. Then

$$/a_2/ \leq \frac{\tilde{\beta}(1 - \beta)}{1 - \lambda} \quad (3.19)$$

and

$$/a_3/ \leq \frac{4(1 - \beta)^2}{(1 - \lambda)^2} + \frac{(1 - \beta)}{1 - \lambda}. \quad (3.20)$$

**Proof.** It follows from (3.1) and (3.2) that there exists $p, q \in P$ such that

$$\frac{zf(z)}{(1 - \lambda)f(z) + \lambda zf(z)} = \beta + (1 - \beta)p(z) \quad (3.21)$$

and

$$\frac{wg(z)}{(1 - \lambda)g(w) + \lambda wg(w)} = \beta + (1 - \beta)q(w), \quad (3.22)$$

where $p(z)$ and $q(w)$ have the forms (3.9) and (3.10), respectively. Equating coefficients in (3.21) and (3.22), we get

$$(1 - \lambda)a_2 = (1 - \beta)p_1 \quad (3.23)$$

$$(\lambda^2 - 1)a_2^2 + 2(1 - \lambda)a_3 = (1 - \beta)p_2 \quad (3.24)$$

$$-(1 - \lambda)a_2 = (1 - \beta)q_1 \quad (3.25)$$
and
\[ (\lambda^2 - 4\lambda + 3)a_2^2 - 2(1 - \lambda)a_3 = (1 - \beta)q_2. \] (3.26)
From (3.23) and (3.25), we get
\[ p_1 = -q_1 \] (3.27)
and
\[ 2(1 - \lambda)^2a_2^2 = (1 - \beta)^2(p_1^2 + q_1^2). \] (3.28)
Also, from (3.24), (3.26) and (3.28), we obtain
\[ a_2^2 = \frac{(1 - \beta)(p_2 + q_2)}{2(1 - \lambda)^2}. \]
Applying Lemma 3.2.3 for the coefficients \( p_2 \) and \( q_2 \), we immediately have
\[ |a_2| \leq \frac{2(1 - \beta)}{1 - \lambda}. \]
This gives the bound on \( |a_2| \) as asserted in (3.19).

Next, in order to find the bound on \( |a_3| \), by subtracting (3.26) from (3.24), we get
\[ 4(1 - \lambda)a_3 - 4(1 - \lambda)a_2^2 = (1 - \beta)(p_2 - q_2). \] (3.29)
It follows from (3.27), (3.28) and (3.29) that
\[ a_3 = \frac{(1 - \beta)(p_2 - q_2)}{4(1 - \lambda)} + \frac{(1 - \beta)^2(p_1^2 + q_1^2)}{2(1 - \lambda)^2}. \] (3.30)
Applying Lemma 3.2.3 once again for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), we readily get
\[ |a_3| \leq \frac{4(1 - \beta)^2}{(1 - \lambda)^2} + \frac{1 - \beta}{1 - \lambda}. \]
This completes the proof of Theorem 3.2.5.

Taking \( \lambda = 0 \) in Theorems 3.2.4 and 3.2.5 one can get the following corollaries.
Corollary 3.2.6. Let \( f(z) \) given by (1.1) be in the class \( SS_{\xi,\chi}(\alpha) \) and \( 0 < \alpha \leq 1 \). Then
\[
|a_2| \leq \frac{2\alpha}{\alpha + 1}
\]  
(3.31)
and
\[
|a_3| \leq 4\alpha^2 + \alpha.
\]  
(3.32)

Corollary 3.2.7. Let \( f(z) \) given by (1.1) be in the class \( S_{\xi,\chi}(\beta) \) and \( 0 \leq \beta < 1 \). Then
\[
|a_2| \leq \sqrt{2 - 2\beta} \quad \text{and} \quad |a_3| \leq 4(1 - \beta)^2 + (1 - \beta).
\]  
(3.33)

3.3 Coefficient Bounds for The General Class of Bi-starlike Functions

This section is motivated and stimulated especially by the works of Murugusundaramoorthy and Magesh [93] and Xu et al. [139, 141]. Here we propose to investigate the bi-univalent function class \( \mathcal{M}_{\xi,\chi}^{\psi}(\lambda) \).

Now, we define \( SS_{\xi,\chi}(\alpha, \lambda) \) of functions \( f \in \mathcal{A} \) satisfying the following conditions
\[
f \in \Sigma_{\chi}, \quad \arg - \frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} < \frac{\alpha \pi}{2} \quad \text{and} \quad \arg - \frac{wg'(w)}{(1-\lambda)w + \lambda g(w)} < \frac{\alpha \pi}{2}
\]
for some \( \alpha (0 < \alpha \leq 1) \), where \( z, w \in U \), and \( g(w) \) is the extension of \( f^{-1}(w) \) to \( U \). Similarly, we say that a function \( f \in \mathcal{A} \) belongs to the class \( \mathcal{M}_{\xi,\chi}(\beta, \lambda) \) if \( f(z) \) satisfies the following inequalities
\[
f \in \Sigma_{\chi}, \quad R - \frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} > \beta \quad \text{and} \quad R - \frac{wg'(w)}{(1-\lambda)w + \lambda g(w)} > \beta
\]
for some \( \beta (0 \leq \beta < 1) \), where \( z, w \in U \), and \( g(w) \) is the extension of \( f^{-1}(w) \) to \( U \). The classes \( SS_{\xi,\chi}(\alpha, \lambda) \) and \( \mathcal{M}_{\xi,\chi}(\beta, \lambda) \) were introduced by Murugusundaramoorthi and Magesh [93].

Definition 3.3.1. Let \( f \in \mathcal{A} \) and the functions \( \varphi, \psi : U \to \mathbb{C} \) be so constrained that
\[
\min \{R(\varphi(z)), R(\psi(z))\} > 0, \quad z \in U
\]
and \( \varphi(0) = \psi(0) = 1 \). We say that \( f \in \mathcal{M}_{\Sigma_B}^{\psi}(\lambda) \) if the following conditions are satisfied

\[
f \in \Sigma_B, \quad \frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} \in \varphi(U), \quad z \in U \tag{3.34}
\]

and

\[
\frac{wg'(w)}{(1-\lambda)w + \lambda g(w)} \in \psi(U), \quad w \in U, \tag{3.35}
\]

where \( 0 \leq \lambda \leq 1 \) and the function \( g(w) \) is the extension of \( f^{-1}(w) \) to \( U \).

We note that by specializing the functions \( \varphi \) and \( \psi \) we get interesting known and new subclasses of the analytic function class \( A \). For example, if we set

\[
\varphi(z) = \psi(z) = \frac{1 + z^{-\alpha}}{1 - z}, \quad 0 < \alpha \leq 1, \quad z \in U
\]

and

\[
\varphi(z) = \psi(z) = \frac{1 + (1 - 2\beta)z}{1 - z}, \quad 0 \leq \beta < 1, \quad z \in U
\]

then the class \( \mathcal{M}_{\Sigma_B}^{\psi}(\lambda) \) reduces to \( SS_{\Sigma_B}^{\psi}(\alpha, \lambda) \) and \( \mathcal{M}_{\Sigma_B}(\beta, \lambda) \) respectively.

In the next theorem, we find the general estimates for the coefficients \( |a_2| \) and \( |a_3| \) for functions in the class \( \mathcal{M}_{\Sigma_B}^{\psi}(\lambda) \).

**Theorem 3.3.2.** Let \( f(z) \) be of the form (1.1). If \( f \in \mathcal{M}_{\Sigma_B}^{\psi}(\lambda) \), then

\[
|a_2| \leq \frac{\varphi^\prime(0) + \psi^\prime(0)}{4(\lambda^2 - 3\lambda + 3)} \tag{3.36}
\]

and

\[
|a_3| \leq \frac{(\lambda^2 - 4\lambda + 6)\varphi^\prime(0) + (\lambda^2 - 2\lambda)\psi^\prime(0)}{(12 - 4\lambda)(\lambda^2 - 3\lambda + 3)}. \tag{3.37}
\]

**Proof.** Since \( f \in \mathcal{M}_{\Sigma_B}^{\psi}(\lambda) \). From (3.34) and (3.35), we have,

\[
\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} = \varphi(z), \quad z \in U \tag{3.38}
\]

and

\[
\frac{wg'(w)}{(1-\lambda)w + \lambda g(w)} = \psi(w), \quad w \in U, \tag{3.39}
\]

where

\[
\varphi(z) = 1 + \varphi_1 z + \varphi_2 z^2 + \ldots \tag{3.40}
\]
\[ \psi(z) = 1 + \psi_1 z + \psi_2 z^2 + \ldots \] (3.41)

and satisfy the conditions of Definition 3.3.1. Now, equating the coefficients in (3.38) and (3.39), we get

\[ (2 - \lambda) a_2 = \varphi_1 \] (3.42)

\[ (\lambda^2 - 2\lambda) a_2^2 + (3 - \lambda) a_3 = \varphi_2 \] (3.43)

\[ -(2 - \lambda) a_2 = \psi_1 \] (3.44)

and

\[ (\lambda^2 - 4\lambda + 6) a_2^2 - (3 - \lambda) a_3 = \psi_2. \] (3.45)

From (3.42) and (3.44), we have

\[ \varphi_1 = -\psi_1 \] (3.46)

and

\[ 2(2 - \lambda)^2 a_2^2 = \varphi_1^2 + \psi_1^2. \] (3.47)

From (3.43) and (3.45), we obtain

\[ a_2^2 = \frac{\varphi_2 + \psi_2}{2(\lambda^2 - 3\lambda + 3)}. \] (3.48)

Since \( \varphi \in \varphi(U) \) and \( \psi \in \psi(U) \), we immediately have

\[ |a_2| \leq |\varphi(0)| + |\psi(0)| \leq \frac{1}{4(\lambda^2 - 3\lambda + 3)}. \]

This gives the bound on \( |a_2| \) as asserted in (3.36).

Next, in order to find the bound on \( |a_3| \), by subtracting (3.45) from (3.43), we get

\[ 2(3 - \lambda) a_3 - 2(3 - \lambda) a_2^2 = \varphi_2 - \psi_2. \] (3.49)

It follows from (3.46), (3.48) and (3.49) that

\[ a_3 = \frac{(\lambda^2 - 4\lambda + 6) \varphi_2 - (\lambda^2 - 2\lambda) \psi_2}{(6 - 2\lambda)(\lambda^2 - 3\lambda + 3)}. \] (3.50)
Since \( \varphi(z) \in \varphi(U) \) and \( \psi(z) \in \psi(U) \), we readily get
\[
|a_3| \leq \frac{(\lambda^2 - 4\lambda + 6)/\varphi''(0) + (\lambda^2 - 2\lambda)/\psi''(0)}{(12 - 4\lambda)(\lambda^2 - 3\lambda + 3)}.
\]
This completes the proof of Theorem 3.3.2.

If we choose
\[
\varphi(z) = \psi(z) = \frac{1 + z^{-\alpha}}{1 - z}, \quad 0 < \alpha \leq 1, \ z \in U
\]
in Theorem 3.3.2, we have the following corollary.

Corollary 3.3.3. Let \( f(z) \) be of the form (1.1) and in the class \( S^+_\Sigma_B(\alpha, \lambda) \), \( 0 < \alpha \leq 1 \) and \( 0 \leq \lambda \leq 1 \). Then
\[
|a_3| \leq \alpha \frac{2}{\lambda^2 - 3\lambda + 3} \quad \text{and} \quad |a_5| \leq \frac{2\alpha^2}{3 - 2\lambda}.
\]

If we set
\[
\varphi(z) = \psi(z) = \frac{1 + (1 - 2\beta)z}{1 - z}, \quad 0 \leq \beta < 1, \ z \in U
\]
in Theorem 3.3.2, we readily have the following corollary.

Corollary 3.3.4. Let be of the form (1.1) and in the class \( M_{\Sigma_B}(\beta, \lambda) \), \( 0 \leq \beta < 1 \) and \( 0 \leq \lambda \leq 1 \). Then
\[
|a_3| \leq \frac{2(1 - \beta)}{\lambda^2 - 3\lambda + 3} \quad \text{and} \quad |a_5| \leq \frac{2(1 - \beta)}{3 - \lambda}.
\]

Remark 3.3.5. For \( \lambda = 0 \) the results discussed in this section are coincidence with outcome of Xu et al.[139]. Taking \( \lambda = 1 \) in Corollaries 3.3.3 and 3.3.4, the estimates on the coefficients \( |a_2| \) and \( |a_3| \) are improvement of the estimates on the first two Taylor-Maclaurin coefficients obtained in [78, Corollaries 2.3 and 3.3]. Furthermore, various other interesting corollaries and consequences of our main result can be derived similarly by specializing \( \varphi \) and \( \psi \).

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