Chapter 2

SOME DIFFERENTIAL INEQUALITIES AND STARLIKENESS OF DOUBLE INTEGRALS *

2.1 Introduction

Let $\mathcal{H}$ denote the class of all analytic functions $f$ defined in the open unit disc $E$. For a positive integer $n$ and $a \in \mathbb{C}$, a subclass $\mathcal{H}[a, n]$ of $\mathcal{H}$ is defined in the following manner:

$$\mathcal{H}[a, n] = \{ f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}.$$

Further, for $n \in \mathbb{N}$, let

$$\mathcal{A}_n = \{ f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots \}.$$

The class $\mathcal{A}_1$ is simply denoted by $\mathcal{A}$. Recall that we denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting of functions univalent in $E$. A function $f \in \mathcal{A}$ is said to be starlike of order $\beta$, $0 \leq \beta < 1$, if

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \beta, \quad z \in E \quad (2.1.1)$$

*Most of the results of this chapter have appeared in Verma et al. [123, 125] and some more will appear in Verma et al. [127].
and the class of all such functions is denoted by $S^\ast(\beta)$. $S^\ast(0) = S^\ast$ is the usual class of all univalent starlike functions in $E$.

It is not always convenient to use analytic conditions (2.1.1) to check the starlikeness of a function $f \in A$. Although a large number of conditions which imply starlikeness of a function in the unit disc exist in the literature on univalent functions, but it is always interesting to find new ones. In 2003, Fournier and Mocanu [32] investigated some second-order differential operators which map $A$ into $S^\ast$ under certain conditions. Some of these results are as follows:

**Theorem 2.1.1.** Let $0 \leq \alpha < 2$. If $f \in A$ satisfies

$$
|zf''(z) - \alpha f'(z) - f(z) - \frac{f(z)}{z}| < 1 - \frac{\alpha}{2}, \quad z \in E,
$$

then $f \in S^\ast$.

**Theorem 2.1.2.** Let $0 \leq \alpha < 2$. If $f \in A$ satisfies

$$
|zf''(z) - \alpha f(z) - 1| < 1 - \frac{\alpha}{2}, \quad z \in E,
$$

then $f \in S^\ast$.

The study of such differential inequalities, which imply starlikeness, has constantly been a subject of research in geometric function theory. Many sources and references in this direction are given in [65]. In a recent paper, Miller and Mocanu [66] extended Theorem 2.1.1, stated above, for $f \in A_n$. Motivated by them, in this chapter, we have investigated starlikeness of the functions satisfying certain differential inequalities. In Section 2.3 of this chapter, we propose some differential inequalities which imply starlikeness of order $\beta$. Some results established in this section generalize the results of Fournier and Mocanu [32] and that of Miller and Mocanu [66] in the sense that they give the order of starlikeness of functions satisfying some differential inequalities.

Many researchers have investigated integral operators of the form

$$
f(z) = \int_0^1 W(r;z)dr
$$

and determined conditions on $W(r;z)$ so as to make $f$ a starlike function. For example, Miller et al. [67] discussed such problem for generalized Bernardi integral and proved
that if the kernel \( W(r,z) \) is given by
\[
W(r,z) = (1+\gamma)g(rz)r^{\gamma-1},
\]
where \( \text{Re}(\gamma) > 0 \), then for a starlike function \( g \),
\[
f(z) = \int_0^1 W(r,z)dr = (1+\gamma)\int_0^1 g(rz)r^{\gamma-1}dr = \frac{1+\gamma}{z^\gamma} \int_0^z g(t)t^{\gamma-1}dt,
\]
is also starlike. Miller and Mocanu [66] determined conditions on the kernel \( W(r,s,z) \) so that the function \( f \) defined by double integral operator of the form
\[
f(z) = \int_0^1 \int_0^1 W(r,s,z)drds,
\]
is starlike in \( E \). The Section 2.4 is devoted to the applications of results involving second-order differential inequalities to obtain conditions for the starlikeness of a class of special double integrals.

### 2.2 Preliminaries

In this section, we state some results which are needed in the subsequent work in this chapter.

**Lemma 2.2.1.** [65, p.71] Let \( h \) be a convex function with \( h(0) = a \) and let \( \text{Re}(\gamma) > 0 \). If \( p \in H[a,n] \) and
\[
p(z) + \frac{zp'(z)}{\gamma} \prec h(z),
\]
then
\[
p(z) \prec q(z) \prec h(z)
\]
where
\[
q(z) = \frac{\gamma}{nz^{\gamma}} \int_0^z h(t)t^{\gamma-1}dt.
\]
This result is sharp.

**Lemma 2.2.2.** [65, p.383] Let \( n \) be a positive integer and \( \alpha \) a real number, with \( 0 \leq \alpha < n \). Let \( q \in H \), with \( q(0) = 0 \), \( q'(0) \neq 0 \) and
\[
\text{Re} \frac{zd''(z)}{q'(z)} + 1 > \frac{\alpha}{n}.
\]
If \( p \in H[0,n] \) satisfies
\[
z p'(z) - \alpha p(z) \prec z n q'(z) - \alpha q(z) \quad \text{in } E,
\]
then \( p(z) \prec q(z) \) in \( E \) and this result is sharp.

**Lemma 2.2.3.** [65, p.76] Let \( h \) be a starlike function in \( E \) with \( h(0) = 0 \). If \( p \in H[a,n] \) satisfies
\[
z p'(z) \prec h(z) \quad \text{in } E,
\]
then \( p(z) \prec q(z) = a + \frac{1}{n} \int_0^z \frac{h(t)}{t} dt \) and this result is sharp.

### 2.3 Differential Inequalities Implying Starlikeness

We begin with the following theorem:

**Theorem 2.3.1.** For \( n \in \mathbb{N} \), let \( f \in A_n \) and let \( \alpha \) and \( \beta \) be real numbers such that \( 0 \leq \alpha < n+1 \) and \( 0 \leq \beta < 1 \). If for all \( z \in E \),
\[
| z f''(z) - \alpha \left( f'(z) - \frac{f(z)}{z} \right) | < \frac{n(n+1-\alpha)(1-\beta)}{n+1-\beta} z,
\]
then \( f \) is starlike of order \( \beta \) in \( E \).

**Proof.** Rewriting inequality (2.3.1) in terms of subordination, we get
\[
z f''(z) - \alpha \left( f'(z) - \frac{f(z)}{z} \right) \prec \frac{n(n+1-\alpha)(1-\beta)}{n+1-\beta} z,
\]
where \( f(z) = z + a_n z^{n+1} + a_{n+2} z^{n+2} + \cdots \in A_n \). If we set
\[
P(z) = f'(z) - \frac{f(z)}{z} = na_{n+1} z^n + (n+1) a_{n+2} z^{n+1} + \cdots,
\]
then \( P \in H[0,n] \) and subordination (2.3.2) becomes
\[
(1-\alpha)P(z) + z P'(z) \prec \frac{n(n+1-\alpha)(1-\beta)}{n+1-\beta} z. \tag{2.3.3}
\]
In order to prove our result, we need to consider the following two cases:

**Case I.** When \( 0 \leq \alpha < 1 \), i.e., \( 0 < 1 - \alpha \leq 1 \). Then, the differential subordination (2.3.3) can be written as
\[
P(z) + \frac{z P'(z)}{1-\alpha} \prec \frac{n(n+1-\alpha)(1-\beta)}{(1-\alpha)(n+1-\beta)} z = h(z) \quad \text{(say)}.
\]
It can be easily seen that $h$ is convex and $h(0) = P(0)$. So, applying Lemma 2.2.1 (with $\gamma = 1 - \alpha$), we obtain

$$P(z) < \frac{1 - \alpha}{nz^{-\alpha}} \int_0^z \left\{ \frac{n(n+1-\alpha)(1-\beta)}{(n+1-\alpha)(1-\beta)} t \right\} \frac{1}{n} \, dt$$

or equivalently

$$f'(z) - \frac{f(z)}{z} < \frac{n(1-\beta)}{n+1-\beta} z, \quad z \in E. \quad (2.3.4)$$

**Case II.** When $1 \leq \alpha < n + 1$. In this case, differential subordination (2.3.3) can be written as

$$zp'(z) - (\alpha - 1)P(z) < nzQ'(z) - (\alpha - 1)Q(z), \quad (2.3.5)$$

where $Q(z) = \frac{n(1-\beta)}{n+1-\beta} z$, $Q(0) = 0$, $Q'(0) \neq 0$ and satisfies in $E$

$$\Re \left( 1 + \frac{zQ''(z)}{Q'(z)} \right) > \frac{\alpha - 1}{n}.$$

Since $\alpha < n + 1$, so, in view of Lemma 2.2.2, subordination (2.3.5) gives $P < Q$ in $E$ or

$$f'(z) - \frac{f(z)}{z} < \frac{n(1-\beta)}{n+1-\beta} z \quad \text{in } E. \quad (2.3.6)$$

Thus, in both the cases, we arrive at the same conclusion. Now, if we write

$$p(z) = \frac{f(z)}{z} = 1 + a_{n+1}z^n + a_{n+2}z^{n+1} + \cdots,$$

then, $p \in \mathcal{H}[1,n]$ and subordination (2.3.6) becomes

$$zp'(z) < \frac{n(1-\beta)}{n+1-\beta} z = h_1(z) \quad \text{(say)}.$$

The function $h_1$ satisfies the conditions of Lemma 2.2.3. Thus, we obtain

$$p(z) < 1 + \frac{1}{n} \int_0^z \frac{n(1-\beta)}{n+1-\beta} \, dt$$

or

$$\frac{f(z)}{z} < 1 + \frac{(1-\beta)}{n+1-\beta} z. \quad (2.3.7)$$

It follows from subordination (2.3.6) that

$$\left| f'(z) - \frac{f(z)}{z} \right| < \frac{n(1-\beta)}{n+1-\beta}, \quad z \in E,$$
while from the subordination (2.3.7), we have

\[
\left| \frac{f(z)}{z} \right| > \frac{n}{n+1-\beta}, \quad z \in E.
\]

Combining the above two inequalities, we get

\[
\frac{n}{n+1-\beta} \left| \frac{zf''(z)}{f(z)} - 1 \right| < \left| \frac{f(z)}{z} \right| \left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| f'(z) - \frac{f(z)}{z} \right| < \frac{n(1-\beta)}{(n+1-\beta)},
\]

which implies that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < (1-\beta).
\]

Thus,

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 1 - (1-\beta) = \beta.
\]

This proves that \( f \) is starlike of order \( \beta \) in \( E \).

Below, we give an example in support of Theorem 2.3.1.

**Example 2.3.2.** Consider the function \( f(z) = z + \frac{(1-\beta)}{n+1-\beta}z^{n+1} \), \( 0 \leq \beta < 1 \). Now,

\[
\left| zf''(z) - \alpha \left( f'(z) - \frac{f(z)}{z} \right) \right| = \left| \frac{n(n+1-\alpha)(1-\beta)}{n+1-\beta}z^{n} \right| = \left| \frac{n(n+1-\alpha)(1-\beta)}{n+1-\beta} \right| |z|^n < \frac{n(n+1-\alpha)(1-\beta)}{n+1-\beta}, \quad z \in E.
\]

Thus, \( f \) satisfies the criterion of Theorem 2.3.1. Further, we have

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) = \text{Re} \left( \frac{1 + \frac{(n+1)(1-\beta)}{(n+1-\beta)}z^n}{1 + \frac{(1-\beta)}{(n+1-\beta)}z^n} \right) > 1 - \frac{(n+1)(1-\beta)}{(n+1-\beta)} = \beta, \quad z \in E.
\]

Letting \( \beta = 0 \) in Theorem 2.3.1, we obtain the following result of Miller and Mocanu [66]:

**Corollary 2.3.3.** Let \( f \in A_n \) and \( 0 \leq \alpha < n+1 \). If for all \( z \in E \)

\[
\left| zf''(z) - \alpha \left( f'(z) - \frac{f(z)}{z} \right) \right| < \frac{n(n+1-\alpha)}{n+1},
\]

then \( f \in S^* \).
Remark 2.3.4. For \( n = 1 \), Corollary 2.3.3 reduces to Theorem 2.1.1 of Fournier and Mo-
canu [32] stated in Section 2.1.

Setting \( \alpha = 0 \) and \( n = 1 \) in Corollary 2.3.3, we obtain the following result of Obradović [77]:

**Corollary 2.3.5.** Let \( f \in \mathcal{A} \) be such that \( |zf''(z)| < 1 \) in \( E \). Then \( f \in S^* \).

In the next result, we present another differential inequality involving a function \( f \in \mathcal{A}_n \) and its second derivative which guarantees that \( f \) is starlike function of order \( \beta \).

**Theorem 2.3.6.** Let \( f \in \mathcal{A}_n \) for some \( n \in \mathbb{N} \) and let \( 0 \leq \alpha < n + 1 \) and \( 0 \leq \beta < 1 \). If for all \( z \in E \)

\[
|zf''(z) - \alpha \left( \frac{f(z)}{z} - 1 \right)| < \frac{(1 - \beta)(n(n + 1) - \alpha)}{n + 1 - \beta}, \tag{2.3.8}
\]

then \( f \) is starlike of order \( \beta \).

**Proof.** In terms of subordination, inequality (2.3.8) can be written as

\[
zf''(z) - \alpha \left( \frac{f(z)}{z} - 1 \right) \prec \frac{(1 - \beta)(n(n + 1) - \alpha)}{n + 1 - \beta}, \tag{2.3.9}
\]

where \( f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots \in \mathcal{A}_n \). For \( \gamma > 0 \), if we write

\[
P(z) = f'(z) - \gamma f(z) = (1 - \gamma) + (n + 1 - \gamma)a_{n+1}z^n + \cdots,
\]

then \( P \in \mathcal{H}[-1, \gamma] \), where \( \gamma(\gamma - 1) = \alpha \). Since \( 0 \leq \alpha < n + 1 \), therefore \( 0 \leq \gamma(\gamma - 1) < n + 1 \). For \( \gamma(\gamma - 1) \geq 0 \), we have \( \gamma \geq 1 \). For \( \gamma(\gamma - 1) < n + 1 \), \( (\gamma - \gamma_1)(\gamma - \gamma_2) < 0 \), or

\[
\gamma_1 < \gamma < \gamma_2 < n + 1, \quad \text{where} \quad \gamma_1 = \frac{1 - \sqrt{4n + 5}}{2} \quad \text{and} \quad \gamma_2 = \frac{1 + \sqrt{4n + 5}}{2}.
\]

Combining, we have \( 1 \leq \gamma < n + 1 \). The subordination (2.3.9) becomes

\[
\gamma P'(z) + z P'(z) \prec -(\gamma - 1) + \frac{(1 - \beta)(n(n + 1) - \alpha)}{n + 1 - \beta} z
\]

or

\[
P(z) + \frac{zP'(z)}{\gamma} \prec -(\gamma - 1) + \frac{(n + \gamma)(n - \gamma + 1)(1 - \beta)}{(n + 1 - \beta)\gamma} z = h(z) \quad \text{(say)}.
\]
Clearly, \( h \) is convex and \( h(0) = P(0) \). Applying Lemma 2.2.1, we obtain
\[
P(z) \prec \frac{\gamma}{n\gamma z^n} \int_0^z \left\{ - (\gamma - 1) + \frac{(n + \gamma)(n - \gamma + 1)(1 - \beta)}{(n + 1 - \beta)\gamma} t \right\} t^{\gamma - 1} dt
\]
or equivalently
\[
f'(z) - \gamma \frac{f(z)}{z} \prec - (\gamma - 1) + \frac{(1 - \beta)(n - \gamma + 1)}{n + 1 - \beta} z.
\] (2.3.10)

If we write
\[
p(z) = \frac{f(z)}{z} - 1 = a_{n+1}z^n + a_{n+2}z^{n+1} + \cdots,
\]
then \( p \in \mathcal{H}[0, n] \). Writing
\[
Q(z) = \frac{1 - \beta}{n + 1 - \beta} z,
\]
we see that \( Q \) is analytic in \( E \) and \( Q(0) = 0, Q'(0) = \frac{1 - \beta}{n + 1 - \beta} \neq 0 \). Now, subordination (2.3.10) can be written as
\[
z p'(z) - (\gamma - 1) p(z) \prec \frac{(1 - \beta)(n - \gamma + 1)}{n + 1 - \beta} z = nzQ'(z) - (\gamma - 1)Q(z).
\] (2.3.11)

Since \( 0 \leq \gamma - 1 < n \) and the function \( Q \) satisfies the hypothesis of Lemma 2.2.2. Hence, we obtain \( p \prec Q \), i.e.
\[
\frac{f(z)}{z} - 1 \prec \frac{1 - \beta}{n + 1 - \beta} z.
\] (2.3.12)

It follows from subordination (2.3.10) that
\[
\left| f'(z) - \gamma \frac{f(z)}{z} \right| < (\gamma - 1) + \frac{(1 - \beta)(n - \gamma + 1)}{n + 1 - \beta} = \frac{n(\gamma - \beta)}{n - \beta + 1},
\]
while subordination (2.3.12) implies that
\[
\left| \frac{f(z)}{z} \right| > 1 - \frac{1 - \beta}{n - \beta + 1} = \frac{n}{n - \beta + 1}.
\]

Combining the above two inequalities, we get
\[
\frac{n}{n - \beta + 1} \left| \frac{zf'(z)}{f(z)} - \gamma \right| < \left| \frac{f(z)}{z} \right| \left| \frac{zf'(z)}{f(z)} - \gamma \right| = \left| f'(z) - \gamma \frac{f(z)}{z} \right| < \frac{n(\gamma - \beta)}{n - \beta + 1},
\]
which implies
\[
\left| \frac{zf'(z)}{f(z)} - \gamma \right| < \gamma - \beta, \quad \text{in } E.
\]

It proves that \( f \) is starlike of order \( \beta \). \( \square \)
Remark 2.3.7. As in Example 2.3.2, it can be shown that the function
\[ f(z) = z + \frac{(1 - \beta)}{n+1 - \beta} z^{n+1}, \quad 0 \leq \beta < 1 \]
is an extremal function for Theorem 2.3.6.

Letting \( \beta = 0 \) in Theorem 2.3.6, we obtain the following criterion for starlikeness:

**Corollary 2.3.8.** Let \( f \in \mathcal{A}_n \) and \( 0 \leq \alpha < n+1 \). If
\[
|zf''(z) - \alpha \left( \frac{f(z)}{z} - 1 \right)| < \frac{(n(n+1) - \alpha)}{n+1},
\]
then \( f \in S^* \).

Remark 2.3.9. Letting \( n = 1 \) in Corollary 2.3.8, we obtain Theorem 2.1.2 of Fournier and Mocanu [32] stated in Section 2.1.

In a very recent paper, Ali et al. [5] discussed the starlikeness of a linear integral operator over normalized analytic functions \( f \) in the class \( \mathcal{W}_\beta^\alpha(\alpha, \gamma) \). We say that \( f \in \mathcal{A} \) is in \( \mathcal{W}_\beta^\alpha(\alpha, \gamma) \) if in \( E \)
\[
\Re \left( e^{i\phi} \left( (1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma zf''(z) - \beta \right) \right) > 0,
\]
for some \( \phi \in \mathbb{R} \). Here \( \alpha, \beta \) and \( \gamma \) are real numbers such that \( \alpha \geq \gamma \geq 0 \) and \( \beta < 1 \). Motivated by the definition of the class \( \mathcal{W}_\beta^\alpha(\alpha, \gamma) \), we present a new differential inequality which generates starlike function of order \( \beta \). We follow the notations used in [5]. Let \( \mu \geq 0 \) and \( \nu \geq 0 \) satisfy
\[
\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu \nu = \gamma. \tag{2.3.13}
\]
When \( \alpha = 1 + 2\gamma \), (2.3.13) yields \( \mu + \nu = 1 + \mu \nu \) or \( (\mu - 1)(\nu - 1) = 0 \). In particular, for \( \gamma > 0 \), choosing \( \mu = 1 \) gives \( \nu = \gamma \), while in the case when \( \gamma = 0 \), we have \( \mu = 0 \) and \( \nu = 1 = \alpha \).

**Theorem 2.3.10.** Let \( \mu, \nu \) satisfy (2.3.13) such that \( \mu > 0 \) and \( \nu > \frac{2}{1 - \beta} \), where \( 0 \leq \beta < 1 \). If \( f \in \mathcal{A}_n \) satisfies
\[
|(1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma zf''(z) - 1| < \lambda, \tag{2.3.14}
\]
for \( z \in E \), then \( f \in S^*(\beta) \). Here
\[
\lambda = \frac{(1 + n\mu)(1 + n\nu)(\nu(1 - \beta) - 2)}{\nu(n + 1 - \beta)}.
\]
Proof. Differential inequality (2.3.14) can be written as follows:

\[(1 - \alpha + 2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma zf''(z) < 1 + \lambda z,\]  

(2.3.15)

where \(f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots \in \mathcal{A}_n\) and

\[\lambda = \frac{(1 + n\mu)(1 + n\nu)(\nu(1 - \beta) - 2)}{\nu(n + 1 - \beta)}.
\]

If we set

\[p(z) = (1 - \nu)\frac{f(z)}{z} + \nu f'(z) = 1 + (1 + n\nu)a_{n+1}z^n + \cdots,
\]

then \(p \in \mathcal{H}[1, n]\) and subordination (2.3.15) becomes

\[p(z) + \mu z p'(z) < 1 + \frac{(1 + n\mu)(1 + n\nu)(\nu(1 - \beta) - 2)}{\nu(n + 1 - \beta)} z = h(z) \quad \text{say}.\]  

(2.3.16)

It can be easily seen that \(h\) is convex and \(h(0) = p(0)\). So, applying Lemma 2.2.1 (with \(\gamma = 1/\mu\)), we obtain

\[p(z) < \frac{1}{n\mu z^{n\mu}} \int_0^z t^{\frac{1}{n\mu} - 1} h(t) dt, \quad z \in \mathcal{E}.
\]

Equivalently,

\[(1 - \nu)\frac{f(z)}{z} + \nu f'(z) < 1 + \frac{(1 + n\nu)(\nu(1 - \beta) - 2)}{\nu(n + 1 - \beta)} z.\]  

(2.3.17)

Now, if we set

\[q(z) = \frac{f(z)}{z} = 1 + a_{n+1}z^n + \cdots,
\]

then \(q \in \mathcal{H}[1, n]\) and subordination (2.3.17) leads to

\[q(z) + \nu z q'(z) < 1 + \frac{(1 + n\nu)(\nu(1 - \beta) - 2)}{\nu(n + 1 - \beta)} z = h_1(z) \quad \text{say}.
\]

The function \(h_1\) satisfies the conditions of Lemma 2.2.1. Thus, we obtain

\[\frac{f(z)}{z} = q(z) < \frac{1}{n\nu z^{n\nu}} \int_0^z t^{\frac{1}{n\nu} - 1} h_1(t) dt = 1 + \frac{(\nu(1 - \beta) - 2)}{\nu(n + 1 - \beta)} z.\]  

(2.3.18)

It follows from subordination (2.3.17) that for all \(z \in \mathcal{E},\)

\[\left| (1 - \nu)\frac{f(z)}{z} + \nu f'(z) \right| < 1 + \frac{(1 + n\nu)(\nu(1 - \beta) - 2)}{\nu(n + 1 - \beta)} = \frac{(n\nu + 2)(\nu(1 - \beta) - 1)}{\nu(n + 1 - \beta)},\]  

(2.3.19)
while from subordination (2.3.18), we have
\[
\left| \frac{f(z)}{z} \right| > 1 - \frac{(v(1-\beta) - 2)}{v(n+1-\beta)} = \frac{nv + 2}{v(n+1-\beta)}, \quad z \in E. \tag{2.3.20}
\]
Combining last two inequalities, we see that
\[
\frac{nv + 2}{v(n+1-\beta)} \left| \frac{zf'(z)}{f(z)} - \left(1 - \frac{1}{v}\right) \right| < \frac{1}{v} \left| vf'(z) + (1-v) \frac{f(z)}{z} \right| < \frac{1}{v} \left[ \frac{(nv + 2)(v(1-\beta) - 1)}{v(n+1-\beta)} \right],
\]
which simplifies to
\[
\left| \frac{zf'(z)}{f(z)} - \left(1 - \frac{1}{v}\right) \right| < \left(1 - \frac{1}{v} - \beta \right).
\]
Thus,
\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \left(1 - \frac{1}{v}\right) - \left(1 - \frac{1}{v} - \beta \right) = \beta,
\]
which implies that \( f \) is starlike of order \( \beta \) in \( E \).

Taking \( \alpha = 1 + 2\gamma \) with \( \gamma > 0 \) (i.e. \( \mu = 1 \) and \( v = \gamma \)) in Theorem 2.3.10, we obtain the following result:

**Corollary 2.3.11.** Let \( f \in \mathcal{A}_n \) and \( \gamma > 0 \) be a real number such that \( \gamma > \frac{2}{1-\beta} \) (0 \( \leq \beta < 1 \)).
If \( f \) satisfies
\[
\left| f'(z) + \gamma zf''(z) - 1 \right| < \frac{(n + 1)(1 + n\gamma)(\gamma(1-\beta) - 2)}{\gamma(n+1-\beta)},
\]
for \( z \in E \), then \( f \) is starlike of order \( \beta \).

Letting \( \gamma \rightarrow \infty \) in Corollary 2.3.11, we obtain the following result of Kuroki and Owa [53]:

**Corollary 2.3.12.** Let \( f \in \mathcal{A}_n \) and \( 0 \leq \beta < 1 \). If \( f \) satisfies
\[
\left| zf''(z) \right| < \frac{n(n+1)(1-\beta)}{(n+1-\beta)},
\]
for \( z \in E \), then \( f \) is starlike of order \( \beta \).

**Remark 2.3.13.** We note that \( \beta = 0 \) and \( n = 1 \) in Corollary 2.3.12 leads us to the well known result of Obradović [77], already stated in Corollary 2.3.5.

Substitution \( n = 1 \) and \( \beta = 0 \) in Corollary 2.3.11 yields the following interesting criterion for starlikeness:
Corollary 2.3.14. Let \( f \in A \) and \( \gamma \) be a real number such that \( \gamma > 2 \). If \( f \) satisfies
\[
\left| zf''(z) + \frac{1}{\gamma} \left( f'(z) - 1 \right) \right| < \frac{(1+\gamma)(\gamma-2)}{\gamma^2},
\]
for \( z \in E \), then
\[
\left| \frac{zf'(z)}{f(z)} - \left( 1 - \frac{1}{\gamma} \right) \right| < \left( 1 - \frac{1}{\gamma} \right),
\]
i.e \( f \in S^* \).

Remark 2.3.15. Corollary 2.3.14 is a particular case of Theorem 1.7 in [101] with \( \alpha = \frac{1}{\gamma} \in \mathbb{R} \).

We now present the following example in support of Theorem 2.3.10.

Example 2.3.16. Consider the function
\[
f(z) = z + \frac{(v(1-\beta)-2)}{v(n+1-\beta)} z^{n+1}, \quad 0 \leq \beta < 1.
\]
Now,
\[
\left| (1 - \alpha) + 2\gamma \right| \left| \frac{zf(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma zf''(z) - 1 \right|
= \left| (1 - \alpha + 2\gamma) + (n+1)(\alpha - 2\gamma) + n(n+1)\gamma \right| \left( \frac{v(1-\beta)-2}{v(n+1-\beta)} z^n \right|
= \left| 1 + n\alpha \right| \left( 1 - \left( \frac{v(1-\beta)-2}{v(n+1-\beta)} \right) \right| \left| z^n \right|
< \left( 1 + n\alpha \right) \left( 1 + n\nu \right) \left( \frac{v(1-\beta)-2}{v(n+1-\beta)} \right) , \quad z \in E.
\]
Thus, \( f \) satisfies the criterion of Theorem 2.3.10. Further, for \( \mu > 0 \) and \( v > \frac{2}{1-\beta} \) as defined in (2.3.13), we have
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) = \Re \left( \frac{1 + \frac{(n+1)(v(1-\beta)-2)}{v(n+1-\beta)} z^n}{1 + \frac{(v(1-\beta)-2)}{v(n+1-\beta)} z^n} \right)
> \left( \frac{1 - \frac{(n+1)(v(1-\beta)-2)}{v(n+1-\beta)}}{1 - \frac{(v(1-\beta)-2)}{v(n+1-\beta)}} \right)
= \left( \frac{nv\beta + 2n + 2}{nv + 2} \right)
> \beta.
\]
2.4 Starlikeness of Double Integrals

In this section, we obtain some new criteria for starlikeness of analytic functions which can be expressed in terms of double integrals of some suitable analytic function $g$.

**Theorem 2.4.1.** For some $n \in \mathbb{N}$, let $0 \leq \alpha < n + 1$, $0 \leq \beta < 1$ and let $g \in H$ satisfy
\[
|g(z)| \leq \frac{n(n + 1 - \alpha)(1 - \beta)}{(n + 1 - \beta)}, \quad z \in E.
\]

Then, the function $f$ given by
\[
f(z) = z + z^{n+1} \int_0^1 \int_0^1 g(rsz) r^{n-\alpha}s^{n-1} dr ds \tag{2.4.1}
\]
is starlike of order $\beta$ in $E$.

**Proof.** Let $f \in A_n$ satisfy the differential equation
\[
zf''(z) - \alpha \left( f'(z) - \frac{f(z)}{z} \right) = z^n g(z), \tag{2.4.2}
\]
where $0 \leq \alpha < n + 1$. Clearly,
\[
|zf''(z) - \alpha \left( f'(z) - \frac{f(z)}{z} \right)| < \frac{n(n + 1 - \alpha)(1 - \beta)}{n + 1 - \beta}, \quad z \in E.
\]
Thus, from Theorem 2.3.1, we see that the solution $f$ of the differential equation (2.4.2) must be starlike of order $\beta$.

Setting
\[
\phi(z) = f'(z) - \frac{f(z)}{z}
\]
in the differential equation (2.4.2), we see that $\phi \in H[0, n]$ and
\[
z\phi'(z) + (1 - \alpha)\phi(z) = z^n g(z).
\]

Solving this equation, we get
\[
\phi(z) = z^{-1+\alpha} \int_0^z r^{n-\alpha} g(r) dr = z^n \int_0^1 r^{n-\alpha} g(rz) dr.
\]

Since $\phi(z) = f'(z) - \frac{f(z)}{z}$, we have
\[
f'(z) - \frac{f(z)}{z} = z^n \int_0^1 r^{n-\alpha} g(rz) dr
\]
or
\[
\left( \frac{f(z)}{z} \right)' = z^{n-1} \int_0^1 r^{n-\alpha} g(rz) dr.
\]
Integrating, we get
\[
\frac{f(z)}{z} = 1 + \int_0^z \zeta^{n-1} \int_0^1 g(r\zeta)r^{n-\alpha}drd\zeta.
\]

Thus, putting \(\zeta = sz\), we have
\[
f(z) = \frac{z}{z} + z^{n+1} \int_0^1 \int_0^1 g(sz)r^{n-\alpha}r^{n-1}drds.
\]

This completes the proof of the theorem. \(\square\)

Taking various permissible values of \(\alpha\) and \(n\), we obtain several special cases of above result. However, we mention only one such result by taking \(\alpha = 0\) and \(n = 1\).

**Corollary 2.4.2.** Let \(0 \leq \beta < 1\) and let \(g \in \mathcal{H}\) be such that
\[
|g(z)| \leq \frac{2(1-\beta)}{2-\beta}, \quad z \in E.
\]
Then,
\[
f(z) = z + z^2 \int_0^1 \int_0^1 g(rsz)rdrds
\]
is starlike of order \(\beta\) in \(E\).

**Example 2.4.3.** As an implication of Corollary 2.4.2, if we take \(g(z) = \lambda e^z\) with
\[
|\lambda| \leq \frac{2(1-\beta)}{e(2-\beta)}, \quad 0 \leq \beta < 1.
\]
In this case,
\[
|g(z)| \leq \frac{2(1-\beta)}{2-\beta}, \quad z \in E.
\]
Thus,
\[
f(z) = z + z^2 \lambda \int_0^1 \int_0^1 re^{r^2sz}drds,
\]
which, upon integration, yields
\[
f(z) = z + \lambda \sum_{n=2}^{\infty} \frac{z^n}{n!} = (1 - \lambda)z + \lambda (e^z - 1)
\]
is starlike of order \(\beta\) in \(E\). \(\square\)

In the following result, by applying Theorem 2.3.6, we construct a function \(f\) which is starlike of order \(\beta\) in \(E\).
Theorem 2.4.4. Let \( n \in \mathbb{N}, \, 1 \leq \gamma < n+1, \, 0 \leq \beta < 1 \) and let \( g \in \mathcal{H} \) be such that
\[
|g(z)| \leq \frac{(1 - \beta)[n(n+1) - \gamma(\gamma-1)]}{n+1 - \beta}, \quad z \in E.
\]

Then, the function \( f \) given by
\[
f(z) = z + z^{n+1} \int_0^1 \int_0^1 g(rsz) r^{n+\gamma-1}s^{n-\gamma}drds
\]
(2.4.3)
is starlike of order \( \beta \) in \( E \).

Proof. Let \( f \in \mathcal{A}_n \) satisfy the differential equation
\[
zf''(z) - \gamma(\gamma - 1) \left( \frac{f(z)}{z} - 1 \right) = z^n g(z).
\]
(2.4.4)

Clearly,
\[
\left| zf''(z) - \gamma(\gamma - 1) \left( \frac{f(z)}{z} - 1 \right) \right| < \frac{(1 - \beta)[n(n+1) - \gamma(\gamma-1)]}{n+1 - \beta}, \quad z \in E.
\]

Thus, from Theorem 2.3.6, we see that the solution \( f \) of the differential equation (2.4.4) must be starlike of order \( \beta \).

Equation (2.4.4) simplifies to
\[
z^n g(z) = z^{1-\gamma} \left( z^\gamma f''(z) - \gamma(\gamma - 1) z^{\gamma-1} \left( \frac{f(z)}{z} - 1 \right) \right)
\]
\[
= z^{1-\gamma} \left( z^\gamma \left( f'(z) - \gamma \frac{f(z)}{z} \right)' + \gamma z^{\gamma-1} \left( f'(z) - \gamma \frac{f(z)}{z} \right) \right) + \gamma(\gamma - 1)
\]
\[
= z^{1-\gamma} \left( z^\gamma \left( f'(z) - \gamma \frac{f(z)}{z} \right)' \right) + \gamma(\gamma - 1).
\]

Thus,
\[
z^\gamma \left( f'(z) - \gamma \frac{f(z)}{z} \right) = \int_0^z \left( \xi^{n+\gamma-1} g(\xi) - \gamma(\gamma - 1) \xi^{\gamma-1} \right) d\xi.
\]

Substituting \( \xi = rz \) in the above integral, we get
\[f'(z) - \gamma \frac{f(z)}{z} = \int_0^1 r^{n+\gamma-1} z^n g(rz)dr - (\gamma - 1),\]

which further simplifies to
\[
\int_0^1 r^{n+\gamma-1} z^n g(rz)dr = \left( f'(z) - \gamma \frac{f(z)}{z} \right) + (\gamma - 1)
\]
\[
= z^\gamma \left( z^{-\gamma} f'(z) - \gamma z^{-1-\gamma} f(z) \right) + (\gamma - 1)
\]
\[
= z^\gamma \left( z^{1-\gamma} \left( \frac{f(z)}{z} - 1 \right)' + (1 - \gamma) z^{-\gamma} \left( \frac{f(z)}{z} - 1 \right) \right).
\]
Thus,
\[ z^{1-\gamma} \left( \frac{f(z)}{z} - 1 \right) = \int_{0}^{1} \xi^{1-\gamma} \left( \int_{0}^{1} r^{\mu+\gamma-1} \xi^\mu g(r) dr \right) d\xi. \]
Again substituting \( \xi = sz \) in the integral, we get
\[ \left( \frac{f(z)}{z} - 1 \right) = z^n \int_{0}^{1} \int_{0}^{1} g(rsz) r^{\mu+\gamma-1} s^{n-\gamma} dr ds. \]

This completes the proof. \( \square \)

Finally, in the next result, we apply Theorem 2.3.10 to get a function which is starlike of order \( \beta \).

**Theorem 2.4.5.** For \( 0 \leq \beta < 1 \), let \( \mu > 0 \) and \( \nu > \frac{2}{1-\beta} \) as defined in (2.3.13). Further, assume that \( g \in H \) satisfy
\[ |g(z)| \leq \frac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}, \quad z \in E. \] (2.4.5)

Then the function \( f \) given by
\[ f(z) = z + z^{n+1} \int_{0}^{1} \int_{0}^{1} g(rsz) r^{\mu+\gamma-1} s^{n-\gamma} dr ds \]
is starlike of order \( \beta \) in \( E \).

**Proof.** We first consider the function \( f \in A_n \) satisfying the differential equation
\[ (1-\alpha+2\gamma) \frac{f(z)}{z} + (\alpha-2\gamma) f'(z) + \gamma z f''(z) - 1 = z^n g(z). \] (2.4.6)

In view of (2.4.5), we have
\[
\left| (1-\alpha+2\gamma) \frac{f(z)}{z} + (\alpha-2\gamma) f'(z) + \gamma z f''(z) - 1 \right|
\leq |z|^n |g(z)|
\leq \frac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}, \quad z \in E.
\]

By Theorem 2.3.10, we see that the solution of differential equation (2.4.6) must be a starlike function of order \( \beta \). Further, to obtain the solution, let \( \phi(z) = (1-\nu) \frac{f(z)}{z} + \nu f'(z) \). Then \( \phi \in H[1,n] \) and (2.4.6) gives
\[ \mu z \phi'(z) + \phi(z) - 1 = z^n g(z). \]
On integration, we get
\[
\varphi(z) = 1 + \frac{1}{\mu z^\beta} \int_0^z g(\zeta) \zeta^{n+\frac{1}{p}-1} d\zeta = 1 + \frac{1}{\mu} \int_0^1 g(rz)r^{n+\frac{1}{p}-1} dr.
\]

Thus,
\[
(1 - \nu) \frac{f(z)}{z} + \nu f'(z) = 1 + \frac{1}{\mu} \int_0^1 g(rz)r^{n+\frac{1}{p}-1} dr. \tag{2.4.7}
\]

Further, setting \( \psi(z) = \frac{f(z)}{z} \), we see that \( \psi \in \mathcal{H}[1,n] \) and the differential equation (2.4.7) reduces to
\[
\nu z \psi'(z) + \psi(z) = 1 + \frac{1}{\mu} \int_0^1 g(rz)r^{n+\frac{1}{p}-1} dr.
\]

A simple calculation gives
\[
\psi(z) = 1 + \frac{1}{\mu \nu z^\beta} \int_0^z \left( \int_0^1 g(r\zeta)r^{n+\frac{1}{p}-1} dr \right) \zeta^{n+\frac{1}{p}-1} d\zeta.
\]

Since, \( \psi(z) = f(z)/z \) and a change of variable yields
\[
f(z) = z + \frac{z^{n+1}}{\mu \nu} \int_0^1 \int_0^1 g(rsz)rs^{1/\nu} dr ds.
\]

This completes the proof of the theorem. \( \square \)

Taking \( n = 1 \) and \( \alpha = 1 + 2\gamma \) with \( \gamma > 0 \) (i.e. \( \mu = 1 \) and \( \nu = \gamma \)) in Theorem 2.4.5, we get the following:

**Corollary 2.4.6.** Let \( 0 \leq \beta < 1, \nu > 2/(1 - \beta) \) and \( g \in \mathcal{H} \) satisfy
\[
|g(z)| \leq \frac{2(1 + \nu)(\nu(1 - \beta) - 2)}{\nu(2 - \beta)}, \quad z \in E.
\]

Then the function \( f \) given by
\[
f(z) = z + z^2 \left( \frac{1}{\nu} \int_0^1 \int_0^1 g(rsz)rs^{1/\nu} dr ds \right)
\]

is starlike of order \( \beta \) in \( E \).