Chapter 1

INTRODUCTION

The purpose of the present chapter is to provide primitive motivation and background for the remaining chapters. A complex-valued function \( f \) defined on some open set \( G \) contained in the complex plane \( \mathbb{C} \) is differentiable at a point \( z_0 \in G \) if its derivative \( f'(z_0) \) exists. Such a function \( f \) is analytic at \( z_0 \) if it is differentiable at every point in some neighborhood of \( z_0 \). If \( f \) is analytic at \( z_0 \), then \( f \) has derivatives of all orders at \( z_0 \) and has a Taylor series expansion

\[
 f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!},
\]

convergent in some open disc centered at \( z_0 \). A function \( f \) is analytic in a domain \( D \) (open connected set) contained in \( \mathbb{C} \) if it is analytic at every point in \( D \). An analytic function is also called holomorphic or regular function. A function \( f \) in a domain \( D \) is said to be univalent if it does not take the same value twice: \( f(z_1) \neq f(z_2) \) for all pairs of distinct points \( z_1 \) and \( z_2 \) in \( D \). It immediately follows from the definition that for a univalent function \( f \) in \( D \), \( f'(z) \neq 0 \) for all \( z \) in \( D \). However, non-vanishing of the derivative of a function everywhere in a domain \( D \) does not ensure univalence of that function there. For instance, for the function \( f(z) = (z-1)^n \) with \( n \geq 3 \), \( f'(z) \) is nonzero throughout the open unit disc \( E = \{ z \in \mathbb{C} : |z| < 1 \} \), yet it is not univalent in \( E \).

In 1851, Riemann articulated a splendid result that every simply connected domain \( D \neq \mathbb{C} \) can be mapped conformally onto the unit disc \( E \). This theorem was first stated by Riemann in his dissertation, but it was proved by Carathéodory [23] and Koebe [49] in 1912 using the methods of the theory of univalent functions. According to this theo-
rem, any univalent function in $D$ is associated with a univalent function in the unit disc $E$ and thus, the properties of the univalent function defined on the unit disc $E$ can be easily translated into the properties of the original function defined on the simply connected domain $D$. Therefore, we can confine to the univalent functions defined on the unit disc $E$.

Let the class of functions analytic in the unit disc $E$ be denoted by $\mathcal{H}$. For a positive integer $n$ and $a \in \mathbb{C}$, a subclass $\mathcal{H}[a,n]$ of $\mathcal{H}$ is defined in the following manner:

$$\mathcal{H}[a,n] = \{ f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}.$$  

Further, for $n \in \mathbb{N}$, let $\mathcal{A}_n$ denote the class of functions $f$ in $\mathcal{H}$ such that $f(0) = 0$ and $f'(0) = 1$, that is

$$\mathcal{A}_n = \{ f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots \}.$$

We shall denote $\mathcal{A}_1$ by $\mathcal{A}$. Denote by $S$, the subclass of $\mathcal{A}$ consisting of all univalent functions $f$ in $E$. Thus, a function $f$ in $S$ has the power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  \hspace{1cm} (1.0.1)

The theory of univalent functions is one of the most pristine topics in geometric function theory. In the beginning of the twentieth century, the theory of univalent functions began to take shape when the fascinating covering theorem due to Koebe came in light. In the year 1907, Koebe [48] proved the existence of an absolute constant $k > 0$ such that the disc $|w| < k$ is contained in the range of every function $f \in S$. But, this interesting result did not find many applications until Bieberbach [17] in 1916 proved that $k = 1/4$. Interestingly, the one-quarter disc is the largest disc that is contained in $K_0(E)$, where $K_0(z) = z(1-z)^{-2}$ is the Koebe function, which is a member of the class $S$. In the same paper, Bieberbach also proved that $|a_2| \leq 2$ for every function $f$, $f(z) = z + a_2 z^2 + \cdots$, in the class $S$. Since the equality in this result holds for the Koebe function and its every rotation, it was natural to suspect that this function $K_0$ maximizes $|a_n|$ for every $n \in \mathbb{N}$. This led Bieberbach [17], in 1916, to propose the famous Bieberbach Conjecture: “for every $f \in S$, given by (1.0.1), $|a_n| \leq n$ for every $n$”. Its simple elegance attracted the efforts of many mathematicians over the course of sixty-eight years, and a number of different techniques in Complex Analysis were devised, to obtain partial solutions. The
Bieberbach conjecture became one of the most famous unsolved problems of mathematics. Bieberbach [17] proved his conjecture for \( n = 2 \), using the area principle established by Gronwall [38]. The inequality \(|a_2| \leq 2\) led easily to the sharp forms of Koebe’s distortion and covering theorems. Loewner [61] brought a new level of sophistication to the subject with his proof of the Bieberbach conjecture for the third coefficient. The conjecture was finally proved in its full generality by Branges [19] in 1985, by making use of special functions. Detailed account of the work so far can be found in the books of Duren [29], Goluzin [36], Goodman [37], Graham and Kohr [39], Hayman [40], Jenkins [45], Nehari [73] and Pommerenke [92], and in the survey articles of Duren [30] and Hayman [41].

In the next section we briefly describe some subclasses of analytic functions which will be required in the present work.

1.1 Certain Subclasses of Analytic Functions

The fundamental basis for discussing new subclasses lies in the fact that through them certain classes of analytic and univalent functions may be associated with some special properties, not commonly associable with certain other classes. The class of functions analytic in the open unit disc \( E \) is denoted by \( \mathcal{H} \). As defined earlier, let \( \mathcal{A} \) be the subclass of \( \mathcal{H} \) containing functions normalized by the conditions \( f(0) = f'(0) - 1 = 0 \) and \( S \) be the subclass of \( \mathcal{A} \) containing univalent functions defined on the open unit disc \( E \).

The Class of Functions With Positive Real Part

Let \( \mathcal{P} \) denote the class of functions \( p(z) = 1 + p_1z + p_2z^2 + \cdots \), analytic in \( E \), such that \( \text{Re} p(z) > 0 \) for \( z \in E \). This class is usually called the Carathéodory class. The function \( (1+z)/(1-z) \) is a distinguished member of the class \( \mathcal{P} \) and is a conformal map of \( E \) onto the right-half plane \( \{w \in \mathbb{C} : \text{Re} w > 0\} \). The class \( \mathcal{P} \) plays an important role in the study of univalent functions, since many extremal problems in the classes of univalent functions can be formulated in terms of members of \( \mathcal{P} \).
Many researchers have made ground-breaking discoveries connecting analysis and geometry. They have succeeded not only in describing those geometries in compact mathematical terms, but have also introduced some special subclasses of univalent functions which are defined by natural geometric conditions and are completely characterized by simple analytic inequalities. Among those, the classes of convex and starlike functions play a significant role in the study of univalent functions because of their simple geometric properties.

The Class of Starlike Functions

Let $D$ be a set in $\mathbb{C}$. $D$ is starlike with respect to a fixed point $w_0 \in D$ if the line segment joining $w_0$ to every other point $w \in D$ lies entirely in $D$, that is, $w_0 + t(w - w_0) \in D$ whenever $w \in D$ and $0 \leq t \leq 1$. A function $f$ in $\mathcal{A}$ is said to be starlike with respect to the origin (or simply starlike) in $E$ if it is univalent in $E$ and the domain $f(E)$ is starlike with respect to the origin. Analytically, a function $f \in \mathcal{A}$ is said to be starlike if and only if

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0, \quad z \in E. \quad (1.1.1)$$

The geometrical interpretation of the condition (1.1.1) is that for each fixed $r$, $0 < r < 1$, the function $\arg f(re^{i\theta})$ strictly increases with $\theta$, $0 \leq \theta < 2\pi$. The subclass of $S$, consisting of starlike functions with respect to the origin, is denoted by $S^*$. Let $S^*(\alpha)$, $0 \leq \alpha < 1$, denote the subclass of $\mathcal{A}$ whose members $f$ satisfy the condition

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in E. \quad (1.1.2)$$

Members of the class $S^*(\alpha)$ are called starlike functions of order $\alpha$. For each $\alpha$, $0 \leq \alpha < 1$, $S^*(\alpha)$ is a subclass of $S^*$. In general, for $\alpha < 0$, the functions in $S^*(\alpha)$ need not be univalent in $E$. It readily follows from definition that $S^*(\beta) \subseteq S^*(\alpha)$ for $\alpha \leq \beta$. Moreover, $S^*(0) \equiv S^*$. The concept of order of starlikeness is due to Robertson [105].

The Class of Strongly Starlike Functions of order $\alpha$

A function $f \in \mathcal{A}$ is said to be strongly starlike of order $\alpha$, $0 < \alpha \leq 1$, if

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\alpha \pi}{2}, \quad z \in E.$$
We denote the set of all such functions by $\tilde{S}(\alpha)$. It is to be noted that $\tilde{S}(1) \equiv S^\ast$. For $0 < \alpha < 1$, $\tilde{S}(\alpha)$ consists only of bounded starlike functions and therefore, the inclusion $\tilde{S}(\alpha) \subset S^\ast$ is proper. The class $\tilde{S}(\alpha)$ was introduced and studied independently by Brannan and Kirwan [20] and Stankiewicz [119].

The Class of Convex Functions

A set $\omega$ in $\mathbb{C}$ is said to be convex if the closed line segment joining any two points of $\omega$ lies entirely in $\omega$, that is, $\lambda w_1 + (1 - \lambda)w_2 \in \omega$ whenever $w_1, w_2 \in \omega$ and $0 \leq \lambda \leq 1$. In other words, $\omega$ is convex if and only if $\omega$ is starlike with respect to each of its points.

A function $f \in \mathcal{A}$ is said to be convex in $E$ if it is univalent in $E$ and $f(E)$ is a convex domain. Analytically, a function $f \in \mathcal{A}$ is convex if and only if $f'(z) \neq 0$ and

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in E. \quad (1.1.3)$$

The normalized class of convex functions is denoted by $K$ and consists of all $f$ in $S$ for which $f(E)$ is a convex domain. Geometrically, the condition (1.1.3) means that $w = f(re^{i\theta})$ maps each circle $|z| = r < 1$ onto a simple closed contour whose tangent rotates monotonically as $\theta$ increases in the counter clockwise direction.

A special subclass of $K$ is the class of convex functions of order $\beta$, $0 \leq \beta < 1$, and is defined as

$$K(\beta) = \left\{ f \in \mathcal{A} : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta, \quad z \in E \right\}.$$

It can be noted that

$$f \in K(\beta) \iff zf' \in S^\ast(\beta) \quad (0 \leq \beta < 1). \quad (1.1.4)$$

For $\beta = 0$, this was first observed by Alexander [3] in 1915 and is popularly known as Alexander’s theorem. Also, it is well-known that if $f \in K$, then $f \in S^\ast(1/2)$ (see Marx [62] and Strohhäcker [120]).

The Class of Close-to-convex Functions

In 1952, another interesting subclass of $S$ which contains $S^\ast$ and has a simple geometric description was introduced by Kaplan [46]. A domain $D$ in $\mathbb{C}$ is close-to-convex if its compliment in $\mathbb{C}$ can be written as union of nonintersecting half lines.
A function \( f \in \mathcal{A} \) is said to be close-to-convex in \( E \) if there exists a real number \( \alpha, -\pi/2 \leq \alpha < \pi/2 \), and a convex function \( g \) (not necessarily normalized) such that

\[
\text{Re} \left( \frac{e^{i\alpha f'(z)}}{g'(z)} \right) > 0, \quad z \in E.
\] (1.1.5)

The class of functions close-to-convex in \( E \) is denoted by \( C \). It is well-known that \( K \subset S^* \subset C \subset S \). Kaplan [46] and Sakaguchi [111] showed that the condition

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > -1/2, \quad z \in E,
\]

ensures close-to-convexity of a function \( f \) in \( E \).

### Generalized Hypergeometric Functions

Let \( \alpha_j \ (j = 1, 2, \ldots, p) \) and \( \beta_j \ (j = 1, 2, \ldots, q) \) be complex numbers with \( \beta_j \neq 0, -1, -2, \ldots \) \((j = 1, 2, \ldots, q)\). Then the generalized hypergeometric function \( pF_q \) is defined by

\[
pF_q(z) = pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n z^n}{(\beta_1)_n \cdots (\beta_q)_n n!} \quad (p \leq q + 1),
\] (1.1.6)

where \( (a)_n \) is the Pochhammer symbol, defined in terms of the Gamma function, by

\[
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0; \\ a(a+1) \cdots (a+n-1), & n \in \mathbb{N}. \end{cases}
\] (1.1.7)

In particular, \( 2F_1 \) is called the Gaussian hypergeometric function. We note that the series in (1.1.6) converges absolutely for \(|z| < \infty\) if \( p < q + 1 \), and for \( z \in E \) if \( p = q + 1 \).

Since the present work is mainly related to integral operators of analytic functions, therefore, some integral operators are explained below.

### 1.2 Integral Operators

Operators play an important role in the geometric function theory. The study of operators which preserve univalence in the unit disc has been the constant theme for researchers. The most basic operators are: conjugation, rotation, dilation and disc automorphism. Advancements in the subject have led to consideration of more useful operators, particularly those which are solutions of certain linear/nonlinear differential equations.
An operator $\widetilde{L}$ is said to be linear if, for every pair of functions $f$ and $g$ and scalar $t$, we have

(i) $\widetilde{L}(f + g) = \widetilde{L}(f) + \widetilde{L}(g)$

(ii) $\widetilde{L}(tf) = t\widetilde{L}(f)$

A brief outline of various linear and non-linear operators, whose properties have been investigated in this work, is provided below.

**Alexander Operator**

Earlier description of convex and starlike functions reveals an interesting close analytic characterization between them. That is,

$$f \in K \iff zf' \in S^*.$$  

This was first observed by Alexander [3] in 1915 and is now popularly known as Alexander’s theorem. In view of this, the one-to-one correspondence between $K$ and $S^*$ is given by the well-known Alexander operator defined by

$$A[f](z) = \int_0^z \frac{f(t)}{t} dt, \quad z \in E. \quad (1.2.1)$$

Note that $A[f]$ is convex if and only if $f$ is starlike. We remark that the Alexander operator (1.2.1), in general, does not take an univalent function into another univalent function [29, p.257]. The above properties of Alexander’s operator motivate to study several other generalized operators in the theory of univalent functions.

**Libera Operator**

In 1965, Libera [55] introduced an integral operator which is defined by

$$L[f](z) = \frac{2}{z} \int_0^z f(t) dt, \quad z \in E. \quad (1.2.2)$$

This operator is known as the Libera operator. The Libera operator $L[f]$ is the solution of the first-order linear differential equation $zg'(z) + g(z) = 2f(z)$. Libera [55] showed that the classes $S^*$ and $K$ are closed under this operator.
**Bernardi Operator**

In 1969, Bernardi [16] gave a generalized operator

\[ I_\gamma[f](z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t)t^{\gamma-1}dt, \quad \text{for} \quad \gamma = 1, 2, 3, \ldots, \quad z \in E, \]  

(1.2.3)

and showed that \( I_\gamma(S^*) \subset S^* \). One can notice that for \( \gamma = 1 \), Bernardi operator reduces to the Libera operator and for \( \gamma = 0 \), it reduces to the Alexander operator. Further, for \( \gamma > -1 \), the integral operator \( I_\gamma[f] \) is known as generalized Bernardi-Libera-Livingston operator (cf. [16, 55, 60]).

**Carlson-Shaffer operator**

Carlson and Shaffer [24] used the Hadamard product to define a linear operator, \( L(a, c) : \mathbb{A} \rightarrow \mathbb{A} \), by

\[ L(a, c)[f](z) = \varphi(a, c; z) \ast f(z), \quad (a, c \in \mathbb{C}) \]  

(1.2.4)

where

\[
\varphi(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n \\
= z + \sum_{n=2}^{\infty} \frac{\Gamma(c)\Gamma(a+n-1)}{\Gamma(a)\Gamma(c+n-1)} z^n, \quad c \neq 0, -1, -2, \ldots,
\]

is the incomplete beta function with \( \varphi(a, c; z) \in \mathbb{A} \). From the above equation, one can establish a relationship between hypergeometric and incomplete beta functions as

\[ \varphi(a, c; z) = z_2F_1(a, 1; c; z) \]

and hence

\[ L(a, c)[f](z) = z_2F_1(a, 1; c; z) \ast f(z). \]  

(1.2.5)

The Carlson-Shaffer operator maps \( \mathbb{A} \) onto itself with \( L(a, a) \) as the identity if \( a \neq 0, -1, -2, \ldots \) and \( L(c, a) \), for \( a \neq 0, -1, -2, \ldots \), as the continuous inverse of \( L(a, c) \), provided \( c \neq 0, -1, -2, \ldots \).
Hohlov Operator

By using the Gaussian hypergeometric function, Hohlov [43, 44] introduced a generalized convolution operator $H_{a,b,c}[f]$ as

$$H_{a,b,c}[f](z) = z_2 F_1(a,b,c;z) * f(z), \quad (1.2.6)$$

where $a, b, c \in \mathbb{C}$ and $c \neq 0, -1, -2, \ldots$. He also discussed some interesting geometric properties exhibited by this operator. The three-parameter family of operators $H_{a,b,c}[f]$ contains, as special cases, most of the known linear integral or differential operators. In particular, if $b = 1$ in (1.2.6), then $H_{a,b,c}[f]$ reduces to the operator defined in (1.2.5), implying that the Carlson-Shaffer operator is a special case of the Hohlov operator. Similarly, it is straightforward to show that the Hohlov operator is also a generalization of the Bernardi operator.

Komatu Operator

Komatu [50] introduced the integral operator $L^p_c : \mathcal{A} \to \mathcal{A}$ by

$$L^p_c[f](z) = \frac{(1 + c)^p}{\Gamma(p)} \int_0^1 \left( \log \left( \frac{1}{t} \right) \right)^{p-1} t^{c-1} f(tz) dt \quad (c > -1, p \geq 0). \quad (1.2.7)$$

It is easy to note that the operator $L^p_c[f]$ defined by (1.2.7) can be expressed by the series expansion as follows:

$$L^p_c[f](z) = z + \sum_{n=2}^{\infty} \left( \frac{c+1}{c+n} \right)^p a_n z^n, \quad z \in E. \quad (1.2.8)$$

Convolution Operator

In 1998, Ponnusamy [97] introduced and studied the convolution operator, $G_{a,b} : \mathcal{A} \to \mathcal{A}$, by

$$G_{a,b}[f](z) = \left( z + \sum_{n=2}^{\infty} \frac{(a+1)(b+1)}{(a+n)(b+n)} z^n \right) * f(z), \quad z \in E, \quad (1.2.9)$$

where $a$ and $b$ are two complex numbers such that both $a$ and $b$ assume no negative integral values. It is obvious to note that the function $G_{a,b}[f](z) = G(z)$ satisfies the differential equation

$$z^2 G''(z) + (a+b+1)z G'(z) + abG(z) = (a+1)(b+1)f(z).$$

Convolution operator $G_{a,b}[f]$ has been studied by a number of authors (see [5, 7, 10, 13, 98]).
Linear Integral Operator

In 1994, Fournier and Ruscheweyh [34] introduced the linear integral operator

\[ V_\lambda[f](z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} \, dt, \quad z \in E, \]  

(1.2.10)

for \( f \in A \). Here, \( \lambda \) is a non-negative real-valued integrable function satisfying the condition \( \int_0^1 \lambda(t) \, dt = 1 \). Note that for special choices of \( \lambda \), the operator \( V_\lambda[f] \) contains some well-known operators, such as Alexander, Libera, Bernardi, Carlson-Shaffer, Hohlov, Komatu and Convolution, as its special cases.

Further, two generalizations of the linear operator \( V_\lambda[f] \) are as follows:

(i) In 1995, Ali and Singh [8] generalized the operator \( V_\lambda[f] \) and defined the operator \( V_{\lambda,\rho}[f] \) as follows:

\[ V_{\lambda,\rho}[f](z) = \rho z + (1 - \rho)V_\lambda[f](z) = z \left( \int_0^1 \lambda(t) \frac{1 - \rho(tz)}{1 - tz} \, dt \right) \ast f(z), \quad \rho < 1. \]

Clearly, for \( \rho = 0 \), the integral operator \( V_{\lambda,\rho}[f] \) reduces to \( V_\lambda[f] \).

(ii) In 2008, Aghalary et al. [1] defined a weighted integral operator \( V_{\lambda}^{\alpha}[f] \) for \( f \in A \) and \( \alpha > 0 \) as follows:

\[ V_{\lambda}^{\alpha}[f](z) = \left( \int_0^1 \lambda(t) \left( \frac{f(tz)}{t} \right)^\alpha \, dt \right)^{\frac{1}{\alpha}}, \]

where powers are chosen so as to get the principal branch of \( V_{\lambda}^{\alpha}[f] \) and \( \lambda \) is a non-negative real-valued integrable function satisfying the condition \( \int_0^1 \lambda(t) \, dt = 1 \). It is important to note that \( V_{\lambda}^{\alpha}[f] \) reduces to the linear integral operator \( V_\lambda[f] \) for \( \alpha = 1 \).

1.3 Techniques Used

In this section, we give a brief outline of the techniques which have been used in this thesis.

Differential Subordination

The principle of subordination owes its origin to Lindelöf [56], but the basic theory was developed later on by Littlewood [57, 58] and Rogosinski [106, 107]. This principle enables us to derive information about an analytic function \( f \), if certain geometric details
of the conformal map associated with this function are known.

Let the functions \( f \) and \( g \) be analytic in \( E \). We say that \( f \) is subordinate to \( g \) (in symbols, \( f(z) \prec g(z) \) or simply \( f \prec g \)) in \( E \) if there exists a function \( \phi \) analytic in \( E \) with \(|\phi(z)| \leq |z| < 1\) and \( \phi(0) = 0 \) such that

\[
f(z) = g(\phi(z)),
\]

(1.3.1)
in \( E \). The case of subordination \( f(z) \prec g(z) \) in \( E \), between a pair of analytic functions, becomes very important when the superordinate function \( g \) is univalent in \( E \). In this case, \( f(z) \prec g(z) \) in \( E \) is equivalent to \( f(0) = g(0) \) and \( f(E) \subset g(E) \). It readily follows that if \( f(z) \prec g(z) \) holds in any circle about origin, it also holds in any smaller concentric circle.

Differential inequalities play a very important role in the theory of univalent functions, as a lot of information about the behavior of a function can be derived from a differential inequality involving that function. Following are two examples:

(i) If \( f \) is analytic in the unit disc \( E \), then \( \text{Re} f'(z) > 0 \) in \( E \) implies that \( f \) is close-to-convex and hence univalent in \( E \) (see Noshiro [74] and Warchawski [130]).

(ii) If \( \alpha \) is real and \( p \) is analytic in \( E \), then \( \text{Re} \left( p(z) + \frac{\alpha z p'(z)}{p(z)} \right) > 0 \) in \( E \), implies that \( \text{Re} \, p(z) > 0 \) in \( E \) (Miller et al. [68]).

Differential inequalities can also be expressed in terms of subordination. For example, \( \text{Re} f'(z) > 0 \) in \( E \) is equivalent to \( f'(z) \prec (1+z)/(1-z) \) in \( E \).

Although there are many scattered results of the type stated above, yet a systematic study of differential inequalities began with the paper “Generating functions for some classes of univalent functions”, by Lewandowski et al. [54] in 1976. The study of differential subordination, which is the generalized form of differential inequalities, began with the publication of a remarkable article “Differential Subordination and Univalent Functions” by Miller and Mocanu [63] in 1981.
Let \( \Psi : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) be an analytic function, \( p \) be an analytic function in \( E \), with

\[
(p(z), zp'(z)) \in \mathbb{C} \times \mathbb{C} \quad \text{for all} \quad z \in E
\]

and let \( h \) be univalent in \( E \). Then the function \( p \) is said to satisfy first order differential
subordination if

\[ \Psi(p(z), zp'(z)) \prec h(z), \quad \Psi(p(0), 0) = h(0). \quad (1.3.2) \]

A univalent function \( q \) is called a dominant of the differential subordination \((1.3.2)\) if \( p(0) = q(0) \) and \( p \prec q \) for all \( p \) satisfying \((1.3.2)\). A dominant \( \tilde{q} \) that satisfies \( \tilde{q} \prec q \) for all dominants \( q \) of \((1.3.2)\), is said to be the best dominant of \((1.3.2)\). The best dominant is unique up to the rotation of \( E \).

A comprehensive analysis of the theory of differential subordination and its applications is available in the book entitled “Differential Subordinations - Theory and Applications”, authored by Miller and Mocanu [65]. The text book entitled “Differential Subordinations and Superordinations - Recent Results”, authored by Bulboaca [21], is also a fruitful source in the theory of differential subordination for Ph. D. students as well as for research mathematicians.

**Hadamard Product**

If \( f \) and \( g \) are analytic functions on \( E \) with \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \), then the Hadamard product (or convolution) of \( f \) and \( g \), denoted by \( f \ast g \), is an analytic on \( E \) given by

\[ (f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in E. \quad (1.3.3) \]

The convolution has the algebraic properties of ordinary multiplication. The concept of convolution arose from the integral

\[ h(r^2 e^{i\theta}) = (f \ast g)(r^2 e^{i\theta}) = \frac{1}{2\pi} \int_{0}^{2\pi} f(r e^{i(\theta-t)}) g(r e^{it}) dt, \quad r < 1 \]

and has proved very resourceful in dealing with certain problems of the theory of analytic and univalent functions. This is so because many times a transformation of \( f \) is expressible as convolution of \( f \) with some other analytic function, sometimes with predetermined behavior. It is natural, therefore, to investigate the convolution properties of many classes of functions.

In 1958, Pólya and Schoenberg [91] have conjectured that \( f \ast g \) is in \( K \) whenever both \( f \) and \( g \) are in \( K \). They showed that the conjecture was true in several special cases. This conjecture was finally proved to be true by Ruscheweyh and Shell-Small [110] in 1973.
From the definition of Hadamard product, it is easy to see that for any two functions $f$, $g \in \mathcal{A}$

$$f(z) = f(z) \frac{z}{1-z},$$
$$zf'(z) = f(z) \frac{z}{(1-z)^2},$$
$$z(f * g)'(z) = z f'(z) g(z) = f(z) z g'(z).$$

The book entitled “Convolutions in Geometric Function Theory” by Ruscheweyh [109] gives a detailed treatment of this elegant theory.

**Duality Theory**

In convolution theory, the concept of duality is central. For a set

$$\mathcal{B} \subset \mathcal{A}_0 = \left\{ g : g(z) = f(z) \frac{z}{z}, \quad f \in \mathcal{A} \right\},$$

we define

$$\mathcal{B}^* = \left\{ g \in \mathcal{A}_0 : (f * g)(z) \neq 0, \quad z \in E \text{ and for all } f \in \mathcal{B} \right\}.$$

The set $\mathcal{B}^*$ is called the dual of $\mathcal{B}$. Further, the second dual of $\mathcal{B}$ is defined as $\mathcal{B}^{**} = (\mathcal{B}^*)^*$. This is a useful information since in many cases of interest $\mathcal{B}^{**}$ is much larger than $\mathcal{B}$. Thus by investigating the smaller set we can get results about the larger set. One such pair of the sets is described in the theorem below.

**Theorem 1.3.1.** [108] Let

$$\mathcal{V}_\beta = \left\{ (1 - \beta) \left( \frac{1 + xz}{1 + yz} \right) + \beta : |x| = 1, |y| = 1, \quad \beta \in \mathbb{R}, \quad \beta \neq 1 \right\}.$$

Then,

$$\mathcal{V}_\beta^{**} = \left\{ g \in \mathcal{A}_0 : \exists \phi \in \mathbb{R} \text{ such that } \Re \left( e^{i\phi} (g(z) - \beta) \right) > 0, \quad z \in E \right\}.$$ 

A subset $\mathcal{B} \subset \mathcal{A}_0$ is said to be complete if it has the following property:

$$f \in \mathcal{B} \Rightarrow f(xz) \in \mathcal{B}, \quad \text{for all } |x| \leq 1.$$
Theorem 1.3.2 (Duality Principle). [108] Let $\mathcal{B} \subset \mathcal{A}_0$ be compact and complete.

(i) If $\Gamma$ is a continuous linear functional on $\mathcal{H}$, then

$$
\Gamma(\mathcal{B}) = \Gamma(\mathcal{B}^{**}), \quad \overline{co}(\mathcal{B}) = \overline{co}(\mathcal{B}^{**}).
$$

Here $\overline{co}$ denotes the closed convex hull of a set.

(ii) If $\Gamma_1$ and $\Gamma_2$ are two continuous linear functionals on $\mathcal{H}$ with $0 \notin \Gamma_2$, then for every $g \in \mathcal{B}^{**}$ we can find $v \in \mathcal{B}$ such that

$$
\frac{\Gamma_1(g)}{\Gamma_2(g)} = \frac{\Gamma_1(v)}{\Gamma_2(v)}.
$$

Thus duality principle states that, under certain conditions on $\mathcal{B}$, the range of a continuous linear functional on $\mathcal{B}$ equals the range of the same linear functional on $\mathcal{B}^{**}$. Note that in Theorem 1.3.1, the sets $\mathcal{V}_\beta$ and $\mathcal{V}_\beta^{**}$ are compact and complete. The basic reference to the duality theory is the book by Ruscheweyh [109].

1.4 Survey of Literature

Univalent conformal mappings of the unit disc $E$ form a non-linear compact set $S$ which lies in the linear topological space $\mathcal{A}$ of all normalized analytic functions defined on $E$. The impossibility of equipping $S$ with a linear structure induced from $\mathcal{A}$ (a linear combination of univalent functions may not be univalent) makes it difficult to construct variations of univalent functions, to solve extremal problems, to obtain integral representations and to describe extreme points. Therefore choosing objects and structures in $S$ which have linear nature still remains an important and non-trivial problem.

Many papers in the theory of univalent functions are devoted to questions of construction of linear integral (or integro-differential) operators which map the class $S$ and its subclasses into themselves or to some other subclasses of $S$ (see [4, 6, 42, 53, 66, 67, 69, 70, 75, 76, 93, 114–116]). One of the first papers in this direction is due to Biernacki [18]. Using the observation of Alexander [3] that the operator $A[f]$ transfers starlike functions into convex functions, he claimed that the operator $A[f]$ maps the class $S$ into itself. However, Krzyz and Lewandowski [52] gave an example of an univalent function which
is transferred, by the Alexander operator, into a non-univalent function.

Singh and Singh [114] showed that the Alexander operator $A[f]$ is starlike if $f \in \mathcal{A}$ with $\text{Re} f'(z) > 0$ in $E$. In a later paper, Singh and Singh [116] showed that $\text{Re} f'(z) > -\frac{1}{4}$ in $E$ is sufficient to ensure the starlikeness of $A[f]$. Ali [4] improved this result further by showing that $\text{Re} f'(z) > -0.273$ is sufficient to ensure the starlikeness of $A[f]$. Ali [4] applied the method of differential subordination which did not give very sharp result here. In 1994, Fournier and Ruscheweyh [34] applied the duality principle and settled this problem by obtaining the sharp estimate. In fact they showed that $A[f]$ is starlike even when $\text{Re} f'(z) > -0.629$, $z \in E$.

In 1965, Libera [55] introduced the Libera operator $L[f]$ and showed that if $f$ is starlike (or convex) then so is $L[f]$. In 1969, Bernardi [16] gave a generalization of the Libera operator, known as Bernardi operator $I_\gamma[f]$. He proved that the classes $S^*$ and $K$ are closed under this operator. In this remarkable paper, he suggested to investigate functions $F$ defined by

$$F(z) = \int_0^1 f(tz)W(t)dt.$$ 

He posed the problem: For what classes of weight functions $W$ will a property $T$, such as starlikeness, convexity or close-to-convexity, possessed by $f$ also be possessed by $F$? This motivated many researchers to study several other generalized operators in the theory of univalent functions. Since then, many new integral operators have been introduced by various researchers e.g. Ali and Singh [8], Carlson and Shaffer [24], Hohlov [43, 44], Komatu [50] and Ponnusamy [97]. In 1994, Fournier and Ruscheweyh [34] introduced the linear integral operator $V_\lambda[f]$ for $f \in \mathcal{A}$. For special choices of $\lambda$, we note that the operator $V_\lambda[f]$ contains some well-known operators such as Alexander, Libera, Bernardi, Carlson-Shaffer, Hohlov and Komatu. In fact, one can find a large number of articles dealing with integral operators between classes of analytic functions. For a detailed information, we refer to the survey articles by Miller and Mocanu [64], Srivastava [118] and references therein. Many interesting subclasses of analytic functions, associated with operator $V_\lambda[f]$ and its many special cases, were investigated recently by Ali and Singh [8], Ali et al. [5], Balasubramanian et al. [10–12], Barnard et al. [14], Choi and Kim [26], Fournier and Ruscheweyh [34], Kim and Rønning [47] and Ponnusamy and Rønning [98, 99] and others (see [7, 31, 103, 117]).
1.5 Synopsis of the Present Work

In the present work, an attempt has been made to explore various integral operators which map the class $\mathcal{A}$ of analytic functions and its subclasses into the classes of univalent, starlike or convex functions. Certain new sufficient conditions for univalence, starlikeness and convexity of some integral operators on some classes of analytic functions have been obtained. In the process, certain known results have been either strengthened or improved. The thesis has been divided into seven chapters including the present chapter which is introductory in nature and contains background material and pre-requisites for the other chapters. A brief chapter-wise description is as under.

Chapter 2

It has always been of interest to find new differential inequalities which can be used as criteria for starlikeness and also to find applications of such differential inequalities to the theory of univalent functions. In 2003, Fournier and Mocanu [32] investigated some second-order differential operators which map $\mathcal{A}$ into $S^*$ under certain conditions. We state below the results proved by them.

Let $0 \leq \alpha < 2$. If $f \in \mathcal{A}$ satisfies any one of the following differential inequalities:

(i) $|zf''(z) - \alpha (f'(z) - 1)| \leq 1 - \alpha$,

(ii) $|zf''(z) - \alpha \left( f'(z) - \frac{f(z)}{z} \right) | \leq 1 - \frac{\alpha}{2}$,

(iii) $|zf''(z) - \alpha \left( \frac{f(z)}{z} - 1 \right) | \leq 1 - \frac{\alpha}{2}$,

for $z \in E$, then $f \in S^*$.

In 2008, Miller and Mocanu [66] used the technique of differential subordination to extend some of the results mentioned above for $f \in \mathcal{A}_n$. In this chapter, we investigate some differential inequalities with the intention to generalize and unify some known results. The technique of differential subordination, to investigate the order of starlikeness of the functions satisfying certain differential inequalities, has been used. Each of the results is sharp in the sense that we have obtained an extremal function corresponding to each differential inequality. Some of our results generalize the results of Fournier and Mocanu [32] and also that of Miller and Mocanu [66].
In the later part of this chapter, we determine conditions on the kernel \( W(r,s,z) \) so as to investigate starlikeness properties of functions \( f \) defined by the double integral operator of the form

\[
f(z) = \int_0^1 \int_0^1 W(r,s,z) dr ds.
\]

In the process, we obtain several weight functions \( W(r,s,z) \) for which the double integral operator is starlike.

**Chapter 3**

Let the class of analytic functions \( \mathcal{U}(\lambda) \) be defined as follows:

\[
\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda, \ z \in E \right\},
\]

where \( f(z) \neq 0 \) for \( z \in E \setminus \{0\} \) and \( \lambda > 0 \). In 1972, Ozaki and Nunokawa [90] proved that \( \mathcal{U}(\lambda) \subset S \), where \( 0 < \lambda \leq 1 \). Interestingly, the Koebe function \( z/(1-z)^2 \) and the bounded analytic function \( z + z^2/2 \) belong to \( \mathcal{U}(1) \) but do not belong to the class of starlike functions of order \( \alpha, \alpha > 0 \). Thus, \( \mathcal{U}(1) \not\subset S^*(\alpha) \), for any \( \alpha > 0 \). Recently, Ponnusamy and Vasundhara [102] obtained conditions on \( \lambda \) and \( \alpha = \alpha(\lambda) \) such that \( \mathcal{U}(\lambda) \not\subset S^*(\alpha) \). Since then, the class \( \mathcal{U}(\lambda) \) and some other interesting classes have been studied extensively (see [33, 78–87, 100]). Let

\[
\mathcal{P}(\lambda) = \left\{ f \in \mathcal{A} : \left| \frac{z}{f(z)} \right|^\prime \leq \lambda, \ z \in E \right\},
\]

\[
\mathcal{M}(\lambda) = \left\{ f \in \mathcal{A} : \left| z^2 \left( \frac{z}{f(z)} \right)^\prime + f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda, \ z \in E \right\},
\]

where \( f(z) \neq 0 \) for \( z \in E \setminus \{0\} \). We denote the classes \( \mathcal{U}(1), \mathcal{P}(1) \) and \( \mathcal{M}(1) \) by \( \mathcal{U}, \mathcal{P} \) and \( \mathcal{M} \), respectively. In a very recent paper, Obradović and Ponnusamy [86] conjectured that \( \mathcal{M} \) is not included in \( S^* \). This conjecture provided us motivation to find conditions on \( \lambda \) for which the functions in \( \mathcal{M}(\lambda) \) are strongly starlike of order \( \alpha \). Using properties of convolution, we obtain conditions on \( \lambda \) such that \( \mathcal{M}(\lambda) \not\subset K(\alpha) \).

Additionally, we discuss the starlikeness and convexity of a new integral operator \( \mathcal{T}_c[f] \) defined by

\[
\mathcal{T}_c[f](z) = cz \int_0^1 \int_0^1 r^{c-1} f'(rsz) \left( \frac{rsz}{f(rsz)} \right)^2 dr ds, \quad \text{for } c > 0,
\]

for \( f \in \mathcal{M}(\lambda) \), where \( f(z) \neq 0 \) for \( z \in E \setminus \{0\} \). Some examples have also been presented in support of our results obtained here.
Chapter 4

In 1994, Fournier and Ruscheweyh [34] introduced a linear integral operator

\[ V_\lambda[f](z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \quad z \in E, \]

where \( f \in \mathcal{A} \) and \( \lambda \) is a non-negative real-valued integrable function with the normalization \( \int_0^1 \lambda(t) dt = 1 \). For special choices of \( \lambda \), we note that the operator \( V_\lambda[f] \) contains some well-known operators, such as Alexander, Libera, Bernardi, Carlson-Shaffer, Hohlov and Komatu, as its special cases. Recently, Ali et al. [5] defined the class \( \mathcal{W}_\beta(\alpha; \gamma) \) as follows:

\[ \mathcal{W}_\beta(\alpha; \gamma) = \left\{ f \in \mathcal{A} : \Re \left( e^{i\phi} \left( (1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma zf''(z) - \beta \right) \right) > 0 \} \]

for some \( \phi \in \mathbb{R} \) and \( z \in E \), where \( \alpha \geq \gamma \geq 0 \) and \( \beta < 1 \) are real numbers. In this chapter two problems have been addressed. First, to find sharp values of \( \beta \) such that \( V_\lambda[f] \) is close-to-convex and thus univalent, whenever \( f \in \mathcal{W}_\beta(\alpha; \gamma) \). Second, to find sharp values of \( \beta = \beta(\delta) \) such that \( V_\lambda[f] \in \mathcal{W}_\delta(\alpha; \gamma) \) whenever \( f \in \mathcal{W}_\beta(\alpha; \gamma) \) and \( \delta < 1 \). The technique of duality as well as general properties of convolution have been used to prove our results. Some results are extended for the operator \( V_{\lambda, \rho}[f] \), which was defined by Ali and Singh [8] as follows:

\[ V_{\lambda, \rho}[f](z) = z \left( \int_0^1 \lambda(t) \frac{1 - \rho tz}{1 - tz} dt \right) * f(z), \quad \rho < 1. \]

Clearly, for \( \rho = 0 \), the integral operator \( V_{\lambda, \rho}[f] \) reduces to \( V_\lambda[f] \). Our results unify a number of known results in literature.

In the last section of this chapter, we obtain some applications of the results obtained here for some suitable choices of the admissible function \( \lambda \).

Chapter 5

This chapter is devoted to the question of obtaining condition on the function \( \lambda \) such that the operator \( V_\lambda[f] \) maps the class \( \mathcal{W}_\beta(\alpha; \gamma) \) into a class of starlike or convex functions of order \( \delta \), for a given \( \delta \in [0, 1/2] \). For special choices of \( \lambda \), these problems were studied in a number of earlier papers by various authors, for example, see [4, 70, 75, 76, 93, 114–116]. The technique of differential subordination was used earlier to tackle such problems, but, the duality methodology seems to work best in the sense that it gives sharp estimates.
in situations where it can be applied. We obtain necessary and sufficient conditions such that \(V_\lambda[f]\) is starlike of order \(\delta\) \((0 \leq \delta \leq 1/2)\) whenever \(f \in \mathcal{W}_\beta(\alpha, \gamma)\). As applications, various well-known integral operators are considered and conditions for starlikeness of these integral operators for \(f \in \mathcal{W}_\beta(\alpha, \gamma)\) are obtained.

In 1995, Ali and Singh [8] discussed the convexity properties of the integral operator \(V_\lambda[f]\) for functions \(f\) in the class \(P(\beta)\). Since then, several researchers have investigated the convexity of the operator whenever \(f\) belongs to various subclasses of \(A\) (see [5, 7, 8, 10–14, 26, 28, 98]). We have obtained necessary and sufficient condition such that \(V_\lambda[f]\) is convex of order \(\delta\) \((0 \leq \delta \leq 1/2)\) whenever \(f \in \mathcal{W}_\beta(\alpha, \gamma)\). Some of the results are also extended for the generalized operator \(V_\lambda, \rho[f]\).

Chapter 6

Define the class \(P^\alpha_\gamma(\beta)\) of normalized analytic functions \(f \in A\) such that
\[
\text{Re}\left\{e^{i\eta}\left(1 - \gamma\left(\frac{f(z)}{z}\right)^\alpha + \gamma(zf'(z))\left(\frac{f(z)}{z}\right)^\alpha - \beta\right)\right\} > 0 \quad (z \in E, \eta \in \mathbb{R}),
\]
where \(\alpha \geq 0, \gamma \geq 0\) and \(\beta < 1\) are real numbers. Here power is chosen so as to get the principal branch of the concerned function. Several subclasses of the class \(P^\alpha_\gamma(\beta)\) have been studied by various researchers, for example, see [8, 25–27, 34, 47, 59, 113]. For a real-valued non-negative function \(\lambda\) with the normalization \(\int_0^1 \lambda(t)dt = 1\) and \(\mu > 0\), Aghalary et al. [1] defined an integral operator for normalized analytic functions as under:
\[
V^\mu_\lambda[f](z) = \left(\int_0^1 \lambda(t)\left(\frac{f(tz)}{t}\right)^\mu dt\right)^{\frac{1}{\mu}}, \quad z \in E,
\]
where the powers are chosen so as to get the principal branch. \(V^\mu_\lambda[f]\) contains various well-known operators such as Libera, Bernardi and Komatu for particular choices of \(\lambda\) and \(\mu\). Aghalary et al. [1] applied the theory of convolution, to prove the univalence of the integral operator \(V^\mu_\lambda[f]\) over functions \(f\) in \(P^\alpha_\gamma(\beta)\). Later on, Ebadian et al. [31] used the duality technique to find the conditions such that \(V^\alpha_\lambda[f]\) maps \(P^\alpha_\gamma(\beta)\) into the class of starlike functions.

Motivated by the above work, we define a subclass \(S_\delta(\alpha)\) of analytic functions as follows:
\[
S_\delta(\alpha) = \left\{f \in A : \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) + (\alpha - 1)\left(\frac{zf'(z)}{f(z)}\right) > \delta, \quad z \in E\right\},
\]
for $\delta < \alpha \leq 1 + \delta$ and $0 \leq \delta < 1$. Note that $S_\delta(1) = K(\delta)$.

The aim of the present chapter is to solve the problem of finding the sharp estimate of the parameter $\beta$ that ensures $V^\alpha_\lambda[f]$ to be in the class $S_\delta(\alpha)$ for $f \in P^\gamma(\beta)$. Some applications are presented in the chapter as well.

**Chapter 7**

This closing chapter contains brief summary of the present work and some directions for further research in this area are also included.

Major part of the thesis has been published/accepted for publication/communicated for publication (see [122–129]). The thesis is appended with a list of 130 references which have been quoted/referred in the present work. The list, however, is not claimed to be exhaustive. Sections, subsections, theorems, remarks and equations are numbered consecutively along with the chapter number. For example, Section 2.1 means Section 1 of Chapter 2, Subsection 6.4.1 means Subsection 1 of Section 4 in Chapter 6 and Theorem 6.4.1 means Theorem 1 of Section 4 in Chapter 6.