Chapter 4

UNIVALENCE OF A GENERAL INTEGRAL OPERATOR AND APPLICATIONS *

4.1 Introduction

Let \( I_f : \mathcal{A} \to \mathcal{A} \) be an operator. Suppose \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are two subclasses of \( \mathcal{A} \). We say that a class \( \mathcal{F}_1 \) is \( \mathcal{F}_2 \)-admissible with respect to the operator \( I_f \) if

\[
f \in \mathcal{F}_1 \Rightarrow I_f \in \mathcal{F}_2.
\]

If \( \mathcal{F}_1 \) is \( \mathcal{F}_2 \)-admissible then we call \( I_f \), a class preserving operator. There are several theorems which are related to this definition, and most of them deal with certain well-known operators in geometric function theory, for example the Bernardi operator and its various generalizations (see [4, 68, 70, 75, 76]).

In 1994, Fournier and Ruscheweyh [34] introduced a linear integral operator

\[
V_{\lambda}[f](z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} \, dt,
\]

where \( f \in \mathcal{A} \) and \( \lambda \) is a non-negative real-valued integrable (weight) function with the normalization \( \int_0^1 \lambda(t) \, dt = 1 \). This operator contains some well-known operators such as Libera, Bernardi and Komatu as its special cases. Fournier and Ruscheweyh [34] obtained

*The contents of this chapter have appeared in Verma et al. [122].
conditions under which \( V_\lambda[f] \) is univalent in \( E \) for a function \( f \) in the class \( P(\beta) \), where
\[
P(\beta) = \left\{ f \in A : \exists \phi \in \mathbb{R} | \text{Re} \left( e^{i\phi} (f'(z) - \beta) \right) > 0, \text{ for } \beta < 1 \text{ and } z \in E \right\}.
\]

In another remarkable paper, Barnard et al. [14] obtained conditions such that \( V_\lambda[f] \in P_1(\beta) \) whenever \( f \) is in the class
\[
P_1(\beta) = \left\{ f \in A : \exists \phi \in \mathbb{R} | \text{Re} \left( e^{i\phi} \left( f'(z) \right) + \gamma f''(z) - \beta \right) > 0 \right\},
\]
with \( \beta < 1, \gamma \geq 0 \). Note that for \( 0 \leq \beta < 1 \), the functions in \( P_1(\beta) \) satisfy the condition \( \text{Re} f'(z) > \beta \) in \( E \) and thus are close-to-convex and hence, univalent in \( E \) [74, 130]. In 2008, Ponnusamy and Rønning [99] discussed the univalence of \( V_\lambda[f] \) for the functions in the class
\[
R_\gamma(\beta) = \left\{ f \in A : \exists \phi \in \mathbb{R} | \text{Re} \left( e^{i\phi} \left( f'(z) + \gamma f''(z) - \beta \right) \right) > 0, \text{ for } \beta < 1 \right\}.
\]

Recently, Ali et al. [5] defined a class \( W_\beta(\alpha, \gamma) \). We say that \( f \in A \) is in \( W_\beta(\alpha, \gamma) \) if for all \( z \in E \),
\[
\text{Re} \left( e^{i\phi} \left( (1 - \alpha + 2\gamma) f'(z) + (\alpha - 2\gamma) f''(z) + \gamma f'''(z) - \beta \right) \right) > 0
\]
for some \( \phi \in \mathbb{R} \), where \( \alpha \geq \gamma \geq 0 \) and \( \beta < 1 \). In this paper, they obtained sufficient conditions on \( \beta \) so that the integral operator \( V_\lambda[f] \) maps function \( f \in W_\beta(\alpha, \gamma) \) into the class of starlike functions. By allowing the parameters \( \alpha, \gamma \) and \( \beta \) to vary suitably in the class \( W_\beta(\alpha, \gamma) \), we comprehend a large number of known classes e.g.
\[
W_\beta(1, 0) = P(\beta), \quad W_\beta(\alpha, 0) = P_\alpha(\beta), \quad W_\beta(1 + 2\gamma, \gamma) = R_\gamma(\beta).
\]

In Section 4.3 of the present chapter, we shall mainly tackle the following problems:

(i) For given \( \delta < 1 \), to find sharp values of \( \beta \) such that \( V_\lambda[f] \in W_\delta(1, 0) \) whenever \( f \in W_\beta(\alpha, \gamma) \).

(ii) For given \( \delta < 1 \), to find sharp values of \( \beta = \beta(\delta) \) such that \( V_\lambda[f] \in W_\delta(\alpha, \gamma) \) whenever \( f \in W_\beta(\alpha, \gamma) \).
Section 4.4 is devoted to several applications of results obtained, for specific choices of the admissible function $\lambda$. In particular, the smallest value $\beta < 1$ is obtained that ensures a function $f$, satisfying

$$\text{Re} \left( f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \right) > \beta$$

in $E$, to be in a subclass of normalized univalent functions $S$.

### 4.2 Preliminaries

As we shall need the generalized hypergeometric function $pF_q$ in the proof of one of our results, so we recall its definition here. Let $\alpha_j (j = 1, 2, \ldots, p)$ and $\beta_j (j = 1, 2, \ldots, q)$ be complex numbers with $\beta_j \neq 0, -1, -2, \ldots (j = 1, 2, \ldots, q)$. Then the generalized hypergeometric function $pF_q$ is defined by

$$pF_q(z) = pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n \beta_1^n \cdots \beta_q^n}{(\beta_1)_n \cdots (\beta_q)_n n! \, n!} \, z^n \quad (p \leq q + 1), \quad (4.2.1)$$

where $(a)_n$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0; \\ a(a+1) \cdots (a+n-1), & n \in \mathbb{N}. \end{cases} \quad (4.2.2)$$

In particular, $2F_1$ is called the Gaussian hypergeometric function.

We shall also need the following lemma of Ponnusamy [97] to prove our results.

**Lemma 4.2.1.** [97] Let $\beta_1 < 1$, $\beta_2 < 1$ and $\eta \in \mathbb{R}$. Then, for $p$, $q$ analytic in $E$ with $p(0) = q(0) = 1$, the conditions $\text{Re} \, p(z) > \beta_1$ and $\text{Re} \, (e^{i\eta} (q(z) - \beta_2)) > 0$ imply

$$\text{Re} \, (e^{i\eta} ((p \ast q)(z) - \delta)) > 0,$$

where $1 - \delta = 2(1 - \beta_1)(1 - \beta_2)$.

Further, recall that for a set

$$\mathcal{B} \subset \mathcal{A}_0 = \left\{ g : g(z) = \frac{f(z)}{z}, \quad f \in \mathcal{A} \right\}$$

the dual set $\mathcal{B}^*$ is defined as

$$\mathcal{B}^* = \left\{ g \in \mathcal{A}_0 : (f \ast g)(z) \neq 0, \quad \forall \, z \in E \quad \text{and for all} \quad f \in \mathcal{B} \right\}.$$

We shall need a result, from the duality theory of analytic functions to prove one of our results, which we state as a theorem below:
Theorem 4.2.2. [109, p.23] Let
\[ V_\beta = \left\{ \beta + (1 - \beta) \left( \frac{1 + xz}{1 + yz} \right) : |x| = |y| = 1, \beta \in \mathbb{R}, \beta \neq 1 \right\}. \]
Then
\[ V_0^* = \left\{ g \in \mathcal{A}_0 : \text{Re} \left( g(z) - \frac{1}{2} \right) > 0, \ z \in E \right\}. \]

4.3 Main Results

Let $\alpha, \gamma$ be any two real numbers such that $\alpha \geq \gamma \geq 0$. Define two real numbers $\mu$ and $\nu$ such that $\mu, \nu \geq 0$ and satisfy
\[ \mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu \nu = \gamma. \quad (4.3.1) \]
We have
\[ \mu = \frac{(\alpha - \gamma) - \sqrt{\alpha - \gamma})^2 - 4\gamma}{2} \quad \text{and} \quad \nu = \frac{(\alpha - \gamma) + \sqrt{(\alpha - \gamma)^2 - 4\gamma}}{2}. \]
When $\gamma = 0$, then $\mu$ is chosen to be 0, in which case, $\nu = \alpha \geq 0$. When $\alpha = 1 + 2\gamma$, (4.3.1) yields $\mu + \nu = 1 + \gamma = 1 + \mu \nu$, or equivalently $(\mu - 1)(1 - \nu) = 0.$

(i) For $\gamma > 0$, choosing $\mu = 1$ gives $\nu = \gamma$.

(ii) For $\gamma = 0$, choosing $\mu = 0$ gives $\nu = \alpha = 1$.

Theorem 4.3.1. Let $\mu \geq 0, \nu \geq 0$ satisfy (4.3.1). Further, let $\delta < 1$ be given, and define $\beta = \beta(\delta, \mu, \nu)$ by
\[ \beta = \left\{ \begin{array}{l}
1 - \frac{1 - \delta}{2} \left\{ 1 - \frac{1}{\nu} \int_0^1 \lambda(t) \left( f_0^1 \frac{ds}{1 + tr^\mu} \right) dt + \left( \frac{1}{\nu} - 1 \right) \int_0^1 \lambda(t) \left( f_0^1 f_0^1 \frac{d\eta}{1 + tr^\mu} \right) dt \right\}^{-1}, \ \gamma \neq 0; \\
1 - \frac{1 - \delta}{2} \left\{ 1 - \frac{1}{\alpha} \int_0^1 \frac{\lambda(t)}{1 + rt} dt + \left( \frac{1}{\alpha} - 1 \right) \int_0^1 \lambda(t) \left( f_0^1 \frac{d\eta}{1 + tr^\mu} \right) dt \right\}^{-1}, \ \gamma = 0.
\end{array} \right. \quad (4.3.2) \]

If $f \in W_\beta(\alpha, \gamma)$, then $V_\lambda[f] \in W_\beta(1, 0) \subset S$. The value of $\beta$ is sharp.

Proof. The case $\gamma = 0$ with $\alpha > 0$ (i.e $\mu = 0$ and $\nu = \alpha$) was proved by Barnard et al. [14, Theorem 1.5]. So we assume that $\gamma > 0$. Define
\[ (1 - \alpha + 2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma zf''(z) = H(z). \]
Since \( \mu + \nu = \alpha - \gamma \) and \( \mu \nu = \gamma \), then
\[
H(z) = (1 + \gamma - (\alpha - \gamma)) \frac{f(z)}{z} + (\alpha - \gamma - \gamma)f'(z) + \gamma f''(z) = (1 + \mu \nu - \mu - \nu) \frac{f(z)}{z} + (\mu + \nu - \mu \nu)f'(z) + \mu \nu z f''(z).
\]
Writing \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), it follows that
\[
H(z) = 1 + \sum_{n=1}^{\infty} a_{n+1} (n \nu + 1)(n \mu + 1) z^n.
\]
It is a simple exercise to see that
\[
f'(z) = H(z) * 3F_2 \left(2, 1, 1; \frac{1}{\mu}, \frac{1}{\nu}, 1 + \frac{1}{\mu} + 1; z\right).
\]
Let \( F(z) = V_\lambda[f](z) \), where \( V_\lambda[f] \) is defined by (4.1.1). Then for \( \gamma \neq 0 \), we can write
\[
F'(z) = f'(z) * \int_0^1 \frac{\lambda(t)}{1-tz} dt = H(z) * 3F_2 \left(2, 1, 1; \frac{1}{\mu}, \frac{1}{\nu}, 1 + \frac{1}{\mu} + 1; z\right) * \int_0^1 \frac{\lambda(t)}{1-tz} dt = H(z) * \int_0^1 \lambda(t) 3F_2 \left(2, 1, 1; \frac{1}{\mu}, \frac{1}{\nu}, 1 + \frac{1}{\mu} + 1; tz\right) dt.
\]
Since \( f \in W_\beta^\lambda(\alpha, \gamma) \), it follows that \( \Re \{e^{i\phi}\} > 0 \) for some \( \phi \in \mathbb{R} \). Now, for each \( \gamma > 0 \), we first claim that
\[
\Re \left[ \int_0^1 \lambda(t) 3F_2 \left(2, 1, 1; \frac{1}{\mu}, \frac{1}{\nu}, 1 + \frac{1}{\mu} + 1; tz\right) dt \right] > 1 - \frac{1}{2(1-\beta)}, z \in E, \tag{4.3.3}
\]
which, by Lemma 4.2.1, implies that \( \Re e^{i\phi} (F'(z) - \delta) > 0 \) or \( F \in W_\beta^\lambda(1,0) \). Therefore, it suffices to verify inequality (4.3.3). Using the identity (which can be checked by comparing the coefficients of \( z^n \) on both sides)
\[
3F_2(2, b; c; d, e; z) = (d - 1)3F_2(1, b; c; d - 1, e; z) - (d - 2)3F_2(1, b; c; d, e; z),
\]
it follows that
\[
3F_2 \left(2, 1, 1; \frac{1}{\mu}, \frac{1}{\nu}, 1 + \frac{1}{\mu} + 1; z\right) = \frac{1}{\nu} 3F_2 \left(1, 1; \frac{1}{\mu}, \frac{1}{\nu}, 1 + 1; z\right) - \left(\frac{1}{\nu} - 1\right) 3F_2 \left(1, 1; \frac{1}{\mu}, \frac{1}{\nu}, 1 + \frac{1}{\mu} + 1; z\right)
\]
\[
= \frac{1}{\nu} \sum_{n=0}^{\infty} \frac{1}{(n \mu + 1)^2} z^n - \left(\frac{1}{\nu} - 1\right) \sum_{n=0}^{\infty} \frac{1}{(n \mu + 1)(n \nu + 1)} z^n
\]
\[
= \frac{1}{\nu} \int_0^1 \frac{ds}{1 - z^s} + \left(\frac{1}{\nu} - 1\right) \int_0^1 \int_0^1 \frac{d\eta d\zeta}{(1 - \zeta^\nu \zeta^\mu)}.
\]
Finally, where

Then, in the view of equation (4.3.2). This verifies inequality (4.3.3).

Therefore, for \( \gamma > 0 \), we have

\[
\Re \left[ \int_0^1 \lambda(t) 3F_2 \left( \frac{2,1,1}{2,1,1} ; \frac{1}{1+sz} + 1, \frac{1}{1+sz} + 1; t+sz \right) dt \right] > \frac{1}{\nu} \int_0^1 \lambda(t) \left( \int_0^1 \frac{ds}{1+ts^\mu} + \left( \frac{1}{\nu} - 1 \right) \int_0^1 \frac{d\eta d\xi}{1+\nu \eta^\mu \xi^\mu} \right) dt
\]

\[
= 1 - \frac{1 - \delta}{2(1 - \beta)},
\]

in the view of equation (4.3.2). This verifies inequality (4.3.3).

To prove the sharpness, let \( f \in \mathcal{W}_\beta(\alpha, \gamma) \) be the function determined by

\[
(1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z) = \beta + (1 - \beta) \frac{1 + z}{1 - z}.
\]

Using a series expansion, we see that we can write

\[
f(z) = z + \sum_{n=2}^{\infty} \frac{2(1 - \beta)}{(n\nu + 1 - \nu)(n\mu + 1 - \mu)} z^n.
\]

Then, \( F(z) = V_\lambda/f(z) = z + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{\psi_n}{(n\nu + 1 - \nu)(n\mu + 1 - \mu)} z^n \),

where \( \psi_n = \int_0^1 \lambda(t) t^{n-1} dt \). Equation (4.3.2) can be restated as

\[
\frac{1}{1 - \beta} = \frac{2}{1 - \delta} \left\{ 1 - \frac{1}{\nu} \int_0^1 \lambda(t) \left( \int_0^1 \frac{ds}{1+ts^\mu} \right) dt + \left( \frac{1}{\nu} - 1 \right) \int_0^1 \lambda(t) \left( \int_0^1 \frac{d\eta d\xi}{1+\nu \eta^\mu \xi^\mu} \right) dt \right\}
\]

\[
= \frac{2}{1 - \delta} \left\{ 1 + \int_0^1 \lambda(t) \left( - \frac{1}{\nu} \int_0^1 \frac{ds}{1+ts^\mu} + \left( \frac{1}{\nu} - 1 \right) \int_0^1 \frac{d\eta d\xi}{1+\nu \eta^\mu \xi^\mu} \right) dt \right\}
\]

\[
= \frac{2}{1 - \delta} \int_0^1 \lambda(t) \left\{ \sum_{n=2}^{\infty} \frac{(-1)^n t^{n-1}}{(n\nu + 1 - \nu)(n\mu + 1 - \mu)} \left( - \frac{1}{\nu} + \left( \frac{1}{\nu} - 1 \right) \frac{1}{(n\nu + 1 - \nu)} \right) \right\} dt
\]

\[
= - \frac{2}{1 - \delta} \sum_{n=2}^{\infty} \frac{(-1)^n t^{n-1} \psi_n}{(n\nu + 1 - \nu)(n\mu + 1 - \mu)} \tag{4.3.4}
\]

Finally,

\[
F'(z) = 1 + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{\psi_n}{(n\nu + 1 - \nu)(n\mu + 1 - \mu)} z^{n-1}
\]

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which for \( z = -1 \) takes the value
\[
F'(1) = 1 + 2(1 - \beta) \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n \psi_n}{(n\nu + 1 - \nu)(n\mu + 1 - \mu)} \\
= 1 + 2(1 - \beta) \left\{ \frac{1 - (1 - \delta)}{2(1 - \beta)} \right\} \quad \text{(using (4.3.4))}
\]
\[
= \delta.
\]
This shows that the result is sharp.

Letting \( \gamma = 0 \) and \( \alpha = 1 \) (i.e. \( \mu = 0 \) and \( \nu = 1 \)) in Theorem 4.3.1, we obtain the following result of Fournier and Ruscheweyh [108]:

**Corollary 4.3.2.** Let \( \delta < 1 \) be given, and let \( \beta = \beta(\delta) < 1 \) be defined by
\[
\beta(\delta) = 1 - \frac{1 - \delta}{2} \left\{ 1 - \frac{1}{\frac{1}{\gamma} \int_0^1 \lambda(t) \left( \int_0^1 \frac{ds}{1 + ts} \right) dt + \left( \frac{1}{\gamma} - 1 \right) \int_0^1 \lambda(t) \left( \int_0^1 \int_0^1 \frac{d\eta d\zeta}{1 + t\eta \zeta} \right) dt} \right\}^{-1}.
\]
If \( f \in \mathcal{P}(\beta) \), then \( V_\lambda[f] \in \mathcal{P}(\delta) \subset S \). The value of \( \beta \) is sharp.

For \( \alpha = 1 + 2\gamma \) with \( \gamma > 0 \), we have \( \mu = 1 \) and \( \nu = \gamma \). Setting \( \alpha = 1 + 2\gamma \) with \( \gamma > 0 \) in Theorem 4.3.1, we get the following criterion for univalence of \( V_\lambda[f] \):

**Corollary 4.3.3.** Let \( \gamma > 0 \) and \( \delta < 1 \) be two real numbers. Define \( \beta = \beta(\delta, \gamma) < 1 \) by
\[
\beta = 1 - \frac{1 - \delta}{2} \left\{ 1 - \frac{1}{\frac{1}{\gamma} \int_0^1 \lambda(t) \left( \int_0^1 \frac{ds}{1 + ts} \right) dt + \left( \frac{1}{\gamma} - 1 \right) \int_0^1 \lambda(t) \left( \int_0^1 \int_0^1 \frac{d\eta d\zeta}{1 + t\eta \zeta} \right) dt} \right\}^{-1}.
\]
If \( f \in \mathcal{R}_\gamma(\beta) \), then \( V_\lambda[f] \in \mathcal{P}(\delta)(1, 0) \subset S \). The value of \( \beta \) is sharp.

Taking \( \delta = 0 \) in Corollary 4.3.3, we get yet another interesting result:

**Corollary 4.3.4.** Let \( \gamma > 0 \) be a real number. Define \( \beta = \beta(\gamma) < 1 \) by
\[
\beta = 1 - \frac{1}{2} \left\{ 1 - \frac{1}{\gamma} \int_0^1 \lambda(t) \left( \int_0^1 \frac{ds}{1 + ts} \right) dt + \left( \frac{1}{\gamma} - 1 \right) \int_0^1 \lambda(t) \left( \int_0^1 \int_0^1 \frac{d\eta d\zeta}{1 + t\eta \zeta} \right) dt \right\}^{-1}.
\]
(4.3.5)

If \( f \in \mathcal{R}_\gamma(\beta) \), then \( V_\lambda[f] \in \mathcal{P}(0) \subset S \). The value of \( \beta \) is sharp.

Note that the condition (4.3.5) can further be rewritten as follows:
\[
\frac{\beta}{1 - \beta} = 1 - \frac{2}{\gamma} \int_0^1 \lambda(t) \left( \int_0^1 \frac{ds}{1 + ts} \right) dt + \left( \frac{2}{\gamma} - 1 \right) \int_0^1 \lambda(t) \left( \int_0^1 \int_0^1 \frac{d\eta d\zeta}{1 + t\eta \zeta} \right) dt \\
= 1 - \frac{2}{\gamma} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} \int_0^1 \lambda(t)t^n dt + \left( \frac{2}{\gamma} - 1 \right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(n\gamma + 1)(n+1)} \int_0^1 \lambda(t)t^n dt
\]

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\[
1 - \frac{2}{\gamma} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)} \int_0^1 \lambda(t) t^n dt - 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n\gamma+1)(n+1)} \int_0^1 \lambda(t) t^n dt \\
= 1 - \frac{2}{\gamma} \sum_{n=0}^{\infty} \frac{n(-1)^n}{(n+1)(n\gamma+1)} \int_0^1 \lambda(t) t^n dt - 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n\gamma+1)(n+1)} \int_0^1 \lambda(t) t^n dt \\
= 1 - \int_0^1 \lambda(t) \left[ \sum_{n=0}^{\infty} (-1)^n \frac{2}{(n\gamma+1)} t^n \right] dt \\
= - \int_0^1 \lambda(t) \left[ 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2}{(n\gamma+1)} t^n \right] dt
\]

Thus, Corollary 4.3.4 gives the first part of Theorem 2.2 of Ponnusamy and Rønning [99].

Another question that might be asked is the following:

Let \( \delta < 1 \) be given. Find the smallest \( \beta = \beta(\delta) \) such that if \( f \in W_\beta(\alpha, \gamma) \), then \( V_\lambda[f] \in W_\delta(\alpha, \gamma) \). Our next theorem is in this direction. The proof runs on the same lines as in Theorem 2 of Fournier and Ruscheweyh [34], we include it here just for the sake of completeness.

**Theorem 4.3.5.** Let \( \delta < 1 \) and \( \alpha \geq \gamma \geq 0 \) be given real numbers. Define \( \beta = \beta(\delta) < 1 \) by

\[
\frac{\beta}{1 - \beta} = - \int_0^1 \lambda(t) \left( \frac{1 - \frac{1+\delta}{1-\delta} t}{1+t} \right) dt.
\]

(4.3.6)

If \( f \in W_\beta(\alpha, \gamma) \), then \( V_\lambda[f] \in W_\delta(\alpha, \gamma) \). The value of \( \beta \) is sharp.

**Proof.** Let

\[
F(z) = V_\lambda[f](z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt.
\]

Clearly,

\[
F'(z) = \int_0^1 \frac{\lambda(t)}{1-tz} dt \ast f'(z).
\]

(4.3.7)

Since, \( f \in W_\beta(\alpha, \gamma) \), so with

\[
g(z) = \frac{(1-\alpha+2\gamma)f(z)}{\beta} + (\alpha-2\gamma)f'(z) + \gamma zf''(z) - \beta
\]

(4.3.8)

we have

\[
\text{Re} \left[ e^{i\phi} g(z) \right] > 0,
\]

where \( \phi \in \mathbb{R} \). For \( \gamma \neq \frac{\alpha}{2} \),

\[
f'(z) = \frac{1}{\alpha-2\gamma} (\beta + (1-\beta) g(z) - \frac{1-\alpha+2\gamma}{\alpha} f(z) - \frac{\gamma}{\alpha} \frac{zf''(z)}{z}).
\]
Putting this value in (4.3.7),
\[ F'(z) = \int_0^1 \frac{\lambda(t)}{1-tz} dt \ast \left( \frac{1}{\alpha - 2\gamma} (\beta + (1-\beta)g(z)) \right) - \frac{1-\alpha + 2\gamma f(z)}{\alpha - 2\gamma} \frac{\gamma}{z} - \frac{\gamma}{\alpha - 2\gamma} z f''(z). \]

Equivalently,
\[ F'(z) = \frac{1}{\alpha - 2\gamma} g(z) \ast \left[ \beta + (1-\beta) \int_0^1 \frac{\lambda(t)}{1-tz} dt \right] - \frac{1-\alpha + 2\gamma F(z)}{\alpha - 2\gamma} \frac{\gamma}{z} - \frac{\gamma}{\alpha - 2\gamma} z f''(z). \]

Thus
\[ (1 - \alpha + 2\gamma) \frac{F(z)}{z} + (\alpha - 2\gamma) F'(z) + \gamma z f''(z) = g(z) \ast \left[ \beta + (1-\beta) \int_0^1 \frac{\lambda(t)}{1-tz} dt \right], \tag{4.3.9} \]

In the case when \( \gamma = \frac{\alpha}{2} \),
\[ g(z) = \frac{f(z) + \gamma z f''(z) - \beta}{1-\beta}. \]

Since
\[ \frac{f(z)}{z} = \beta + (1-\beta)g(z) - \gamma z f''(z), \]

so,
\[ \frac{F(z)}{z} + \gamma z f''(z) = g(z) \ast \left[ \beta + (1-\beta) \int_0^1 \frac{\lambda(t)}{1-tz} dt \right], \]

which is the same as (4.3.9) with \( \gamma = \frac{\alpha}{2} \).

Further \( F \in \mathcal{W}_0(\alpha, \gamma) \) if and only if \( G(z) := (F(z) - \delta z)/(1 - \delta) \in \mathcal{W}_0(\alpha, \gamma) \). Now using equation (4.3.9), we obtain
\[ (1 - \alpha + 2\gamma) \frac{G(z)}{z} + (\alpha - 2\gamma) G'(z) + \gamma G''(z) = g(z) \ast \left[ \frac{\beta - \delta}{1-\delta} + \frac{1-\beta}{1-\delta} \int_0^1 \frac{\lambda(t)}{1-tz} dt \right]. \tag{4.3.10} \]

Since, \( \text{Re} e^{i\phi} g(z) > 0 \) for some \( \phi \in \mathbb{R} \), it follows from the Theorem 4.2.2 that
\[ g(z) \ast \left[ \frac{\beta - \delta}{1-\delta} + \frac{1-\beta}{1-\delta} \int_0^1 \frac{\lambda(t)}{1-tz} dt \right] \neq 0 \tag{4.3.11} \]

if, and only if,
\[ \text{Re} \left[ \frac{\beta - \delta}{1-\delta} + \frac{1-\beta}{1-\delta} \int_0^1 \frac{\lambda(t)}{1-tz} dt \right] > \frac{1}{2}. \]

Using \( \text{Re} \frac{1}{1-tz} > \frac{1}{1+t} \), we get
\[ \text{Re} \left[ \frac{\beta - \delta}{1-\delta} + \frac{1-\beta}{1-\delta} \int_0^1 \frac{\lambda(t)}{1-tz} dt \right] > \frac{1-\beta}{1-\beta} \left[ \frac{\beta - \delta}{1-\delta} + \int_0^1 \frac{\lambda(t)}{1+t} dt \right]. \]
By using equation (4.3.6), we have

\[
\frac{\beta - \left(\frac{1+\delta}{2}\right)}{1-\beta} = -\int_0^1 \frac{\lambda(t)}{1+t} dt.
\]

Thus,

\[
\frac{\beta - \delta}{1-\beta} + \int_0^1 \frac{\lambda(t)}{1+t} dt = 1/2 \frac{1-\delta}{1-\beta},
\]

which implies that

\[
\text{Re} \left[ \frac{\beta - \delta}{1-\delta} + \frac{1-\beta}{1-\delta} \int_0^1 \frac{\lambda(t)}{1-tz} dt \right] > \frac{1-\beta}{1-\delta} \left[ \frac{\beta - \delta}{1-\beta} + \int_0^1 \frac{\lambda(t)}{1+t} dt \right] = 1/2.
\]

Thus, in the view of equation (4.3.10) and (4.3.11), we have

\[
(1-\alpha+2\gamma) \frac{G(z)}{z} + (\alpha - \gamma) G'(z) + \gamma z G''(z) \neq 0,
\]

which means that \((1-\alpha+2\gamma) \frac{G(z)}{z} + (\alpha - \gamma) G'(z) + \gamma z G''(z)\) is contained in a half plane not containing the origin. So, \(G \in W_0(\alpha, \gamma)\) and hence \(F \in W_0(\alpha, \gamma)\).

To prove the sharpness, let \(f \in W_0(\alpha, \gamma)\) be the function determined by

\[
(1-\alpha+2\gamma) \frac{f(z)}{z} + (\alpha - \gamma) f'(z) + \gamma z f''(z) = \beta + (1-\beta) \frac{1+z}{1-z}.
\]

Using a series expansion, we see that we can write

\[
f(z) = z + 2(1-\beta) \sum_{n=2}^\infty \frac{z^n}{(n\mu+1-\mu)(n\nu+1-\nu)}.
\]

Thus,

\[
F(z) = V_\lambda[f](z) = z + 2(1-\beta) \sum_{n=2}^\infty \frac{z^n \omega_n}{(n\mu+1-\mu)(n\nu+1-\nu)},
\]

where \(\omega_n = \int_0^1 \lambda(t)t^{n-1} dt\). Further,

\[
\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) \frac{\left(1 - \frac{1+\delta}{1-\delta} t \right)}{(1+t)} dt,
\]

gives,

\[
\frac{\beta}{1-\beta} = -1 + \int_0^1 \lambda(t) \frac{\left(1 + \frac{1+\delta}{1-\delta} t \right)}{(1+t)} dt
\]

or

\[
\frac{1}{1-\beta} = 2 \frac{1}{1-\delta} \int_0^1 \frac{t\lambda(t)}{1+t} dt = 2 \frac{1}{1-\delta} \sum_{n=2}^\infty (-1)^n \omega_n.
\]
Further, assume that
\[ H(z) = (1 - \alpha + 2\gamma) \frac{F(z)}{z} + (\alpha - \gamma) F'(z) + \gamma z F''(z). \]

Since,
\[ F(z) = z + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{\omega_n z^n}{(n\mu + 1 - \mu)(n\nu + 1 - \nu)}, \]
so,
\[ H(z) = 1 + 2(1 - \beta) \sum_{n=2}^{\infty} \omega_n z^{n-1}. \]
Therefore, for \( z = -1 \),
\[ H(-1) = 1 - 2(1 - \beta) \sum_{n=2}^{\infty} \omega_n (-1)^n = 1 - 2(1 - \beta) \frac{1 - \delta}{2(1 - \beta)} = \delta, \]
or,
\[ (1 - \alpha + 2\gamma) F(z)/z + (\alpha - \gamma) F'(z) + \gamma z F''(z) \mid_{z=-1} = \delta. \]
This shows that the result is sharp. \( \square \)

Letting \( \gamma = 0 \) with \( \alpha > 0 \) (i.e. \( \mu = 0 \) and \( \nu = \alpha \)) in Theorem 4.3.5, we obtain the following result of Kim and Rønning [47]:

**Corollary 4.3.6.** Let \( \delta < 1 \) and \( \alpha > 0 \), and define \( \beta = \beta(\delta) \) by
\[
\frac{\beta}{1 - \beta} = -\int_{0}^{1} \frac{\lambda(t)}{1 + t} \left( 1 - \frac{1 + \delta t}{1 + \delta} \right) dt. \tag{4.3.12}
\]
If \( f \in P_\alpha(\beta) \), then \( V_\lambda[f] \in P_\alpha(\delta) \). The value of \( \beta \) is sharp.

In 1995, Ali and Singh [8] introduced and studied the operator \( V_{\lambda,\rho}[f] \) defined as follows:
\[ V_{\lambda,\rho}[f](z) = \rho z + (1 - \rho) V_\lambda[f](z) = z \left( \int_{0}^{1} \lambda(t) \left( \frac{1 - \rho t z}{1 - t z} \right) dt \right) * f(z), \quad \rho < 1. \tag{4.3.13} \]
Clearly, for \( \rho = 0 \), the integral operator \( V_{\lambda,\rho}[f] \) reduces to \( V_\lambda[f] \). In our next results, we generalize Theorem 4.3.1 and Theorem 4.3.5 to obtain conditions on \( \beta \) such that \( V_{\lambda,\rho}[f] \in \mathcal{W}_\delta(1,0) \) or \( \mathcal{W}_\delta(\alpha,\gamma) \) for \( f \in \mathcal{W}_\beta(\alpha,\gamma) \).

**Theorem 4.3.7.** Let \( \mu \geq 0 \), \( \nu \geq 0 \) satisfy (4.3.1). Further, let \( \rho < 1 \) and \( \delta < 1 \) be given real numbers. Define \( \beta = \beta(\delta, \mu, \nu, \rho, \alpha) \) by
\[
\beta = \begin{cases} 
1 - \frac{1 - \delta}{2(1 - \rho)} \left\{ 1 - \frac{1}{\nu} \int_{0}^{1} \lambda(t) \left( \int_{0}^{1} \frac{ds}{1 + s^2} \right) dt + (\frac{1}{\nu} - 1) \int_{0}^{1} \lambda(t) \left( \int_{0}^{1} \frac{dt}{1 + t^2} \right) \frac{d\eta}{d\xi} \right\}^{-1}, & \gamma \neq 0; \\
1 - \frac{1 - \delta}{2(1 - \rho)} \left\{ 1 - \frac{1}{\alpha} \int_{0}^{1} \lambda(t) \frac{dt}{1 + t^2} + (\frac{1}{\alpha} - 1) \int_{0}^{1} \lambda(t) \left( \int_{0}^{1} \frac{d\eta}{d\xi} \right) dt \right\}^{-1}, & \gamma = 0.
\end{cases} \]
If \( f \in \mathcal{W}_\beta(\alpha,\gamma) \), then \( V_{\lambda,\rho}[f] \in \mathcal{W}_\delta(1,0) \subset S \). The value of \( \beta \) is sharp.
**Theorem 4.3.8.** Let \( \rho < 1 \), \( \delta < 1 \) and \( \alpha \geq \gamma \geq 0 \) be real numbers. Define \( \beta = \beta(\delta, \rho) < 1 \) by

\[
\beta \frac{1}{1 - \beta} = -\int_0^1 \lambda(t) \frac{1 - \frac{1+\delta-2\rho t}{1-\delta}}{1 + t} dt.
\]

If \( f \in W_\rho^\beta(\alpha, \gamma) \), then \( V_{\lambda, \rho}[f] \in W_\delta(\alpha, \gamma) \). The value of \( \beta \) is sharp.

Since the proofs of these theorems do not present any new features, so we skip them.

Some corollaries of Theorem 4.3.7 and Theorem 4.3.8 are interesting enough to be stated separately.

Letting \( \gamma = 0 \) and \( \alpha = 1 \) (i.e. \( \mu = 0 \) and \( \nu = 1 \)) in Theorem 4.3.7, we obtain the following result:

**Corollary 4.3.9.** Let \( \rho < 1 \) and \( \delta < 1 \) be two real numbers. Define \( \beta = \beta(\delta, \rho) < 1 \) by

\[
\beta(\delta) = 1 - \frac{1 - \delta}{2(1 - \rho)} \left\{ 1 - \int_0^1 \lambda(t) \frac{1}{1 + t} dt \right\}^{-1}.
\]

If \( f \in P(\beta) \), then \( V_{\lambda, \rho}[f] \in P(\delta) \subset S \). The value of \( \beta \) is sharp.

Setting \( \alpha = 1 + 2\gamma \) with \( \gamma > 0 \) (i.e. \( \mu = 1 \) and \( \nu = \gamma > 0 \)) in Theorem 4.3.7, we get the following result:

**Corollary 4.3.10.** Let \( \rho < 1 \), \( \gamma > 0 \) and \( \delta < 1 \) be real numbers, and define \( \beta = \beta(\delta, \gamma, \rho) < 1 \) by

\[
\beta = 1 - \frac{1 - \delta}{2(1 - \rho)} \left\{ 1 - \frac{1}{\gamma} \int_0^1 \frac{ds}{1 + ts} \left( \int_0^1 \frac{d\eta d\zeta}{1 + t\eta \zeta} dt \right) \right\}^{-1}.
\]

If \( f \in R_\gamma(\beta) \), then \( V_{\lambda, \rho}[f] \subset S \). The value of \( \beta \) is sharp.

Taking \( \gamma = 0 \) (i.e. \( \mu = 0 \) and \( \nu = \alpha \)), in Theorem 4.3.8, we obtain:

**Corollary 4.3.11.** Let \( \rho < 1 \), \( \delta < 1 \) and \( \alpha \geq 0 \) be real numbers. Define \( \beta = \beta(\delta, \rho) < 1 \) by

\[
\beta \frac{1}{1 - \beta} = -\int_0^1 \lambda(t) \frac{1 - \frac{1+\delta-2\rho t}{1-\delta}}{1 + t} dt \quad (4.3.14)
\]

If \( f \in P_\alpha(\beta) \), then \( V_{\lambda, \rho}[f] \in P_\alpha(\delta) \). The value of \( \beta \) is sharp.
4.4 Applications to Certain Integral Operators

In this section, we shall consider some particular values of $\lambda$ which will lead to various well-known integral operators. Let $\lambda$ be defined by

$$\lambda(t) = (1 + c)t^c, \quad c > -1.$$  

Then the integral operator

$$V_\lambda[f](z) = (1 + c) \int_0^1 t^{-1} f(tz) dt, \quad c > -1,$$

becomes the Bernardi integral operator $I_c[f]$. The classical Alexander operator

$$A[f](z) = \int_0^1 \frac{f(tz)}{t} dt$$

and Libera operator

$$L[f](z) = 2 \int_0^1 f(tz) dt$$

are special cases of Bernardi operator with $c = 0$ and $c = 1$ respectively. For this special case, Theorem 4.3.1 gives the following result:

**Theorem 4.4.1.** Let $\mu \geq 0$, $\nu \geq 0$ satisfy (4.3.1) and $c > -1$ be real numbers. Further, let $\delta < 1$ be given, and define $\beta < 1$ by

$$\beta = \begin{cases} 
1 - \frac{1-\delta}{2} \left\{ 1 - \sum_{n=0}^{\infty} \left( \frac{(-1)^n(n+1)(1+c)}{(n+1)(n+c+1)} \right) \right\}^{-1}, & \gamma \neq 0; \\
1 - \frac{1-\delta}{2} \left\{ 1 - \sum_{n=0}^{\infty} \left( \frac{(-1)^n(n+1)(1+c)}{(n+1)(n+c+1)} \right) \right\}^{-1}, & \gamma = 0. 
\end{cases}$$

If $f \in W_\beta(\alpha, \gamma)$, then the Bernardi operator $I_c[f] \in W_\delta(1, 0) \subset S$. The value of $\beta$ is sharp.

The class $W_\beta(\alpha, \gamma)$ is closely related to the class $R(\alpha, \gamma, h)$ consisting of all functions $f \in A$ which satisfy

$$f'(z) + \alpha zf''(z) + \gamma z^2 f'''(z) \prec h(z), \quad z \in E,$$

with $h(z) := h_\beta(z) = (1 + (1 - 2\beta)z)/(1 - z)$. We observe that $f \in R(\alpha, \gamma, h_\beta)$ if and only if $zf'$ is in $W_\beta(\alpha, \gamma)$ (with $\phi = 0$). Recently, Ali et al. [6] established that for a suitably normalized convex function $h$, the function $f \in R(\alpha, \gamma, h)$ has a double integral representation as follows:

$$f(z) = \int_0^1 \int_0^1 G(x^\mu s^\nu) t^{-\mu}s^{-\nu} dsdt$$

with $G'$ subordinate to $h$. The particular case of Theorem 4.4.1, when $c = 0$ leads to a very interesting result about functions in the class $R(\alpha, \gamma, h)$ as follows:
Theorem 4.4.2. Let $0 < \gamma \leq \alpha \leq 1 + 2\gamma$. If $F \in \mathcal{A}$ satisfies
\[
\text{Re} \left( F'(z) + \alpha z F''(z) + \gamma z^2 F'''(z) \right) > \beta
\]
in $E$, and $\beta < 1$ satisfies
\[
\frac{1}{1-\beta} = \frac{2}{1-\delta} \left( 1 - 3 F_2 \left( 1, \frac{1}{\mu} + 1, \frac{1}{v} + 1; 1 \right) \right),
\]
(4.4.1)
then $F$ is univalent in $E$. The value of $\beta$ is sharp.

Proof. For given $F \in \mathcal{A}$, define $f$ such that $f = zF'$. It is evident that $f$ belongs to the class $W_\beta(\alpha, \gamma)$ (with $\phi = 0$). Clearly,
\[
F(z) = \int_0^1 \frac{f(tz)}{t} \, dt.
\]

Thus, $F$ is the Alexander operator (or $I_c[f]$ with $c = 0$). Also, note that
\[
\beta = 1 - \frac{1 - \delta}{2} \left( 1 - 3 F_2 \left( 1, \frac{1}{\mu} + 1, \frac{1}{v} + 1; 1 \right) \right)^{-1} = 1 - \frac{1 - \delta}{2} \left( 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(n\nu + 1)(n\mu + 1)} \right)^{-1}.
\]

Hence the result now follows from Theorem 4.4.1 with $c = 0$. The condition on $\alpha$ and $\gamma$ implies that $1 \leq \mu \leq \nu$. 

If we write $\gamma = 1$ and $\alpha = 3$ in equation (4.3.1), then $\mu = \nu = 1$. Making these substitution with $\delta = 0$ in (4.4.1), we get
\[
\frac{1}{1-\beta} = 2 \left( 1 - 3 F_2 \left( 1, 1, 1; 2, 2; 1 \right) \right) = 2 \left( 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} \right) = 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{(n)^2} = 2 \left( 1 - \frac{\pi^2}{12} \right),
\]
or
\[
\beta = \frac{6 - \pi^2}{12 - \pi^2} = -1.81637\ldots.
\]

For this particular case in Theorem 4.4.2, we obtain the following result, which improves a result of Ali et al. [6].

Corollary 4.4.3. Let $F \in \mathcal{A}$ satisfy
\[
\text{Re} \left( F'(z) + 3z F''(z) + z^2 F'''(z) \right) > -1.81637\ldots,
\]
in $E$. Then $F$ is univalent.
Upon setting $\lambda(t) = (1 + c)t^c$ $(c > -1)$ and $\delta = 0$ in Theorem 4.3.5, we have:

**Theorem 4.4.4.** Let $\alpha \geq \gamma \geq 0$ and $-1 < c \leq 0$ be given and the Bernardi integral $I_c[f]$ be defined by

$$I_c[f](z) = \frac{(1 + c)}{z^c} \int_0^z u^{c-1} f(u) du.$$ 

Suppose that $f \in W_0^\beta(\alpha, \gamma)$. Then $I_c[f] \in W_0^\beta(\alpha, \gamma)$, where

$$\beta = \frac{2(1 + c)\frac{3}{2}F_1(1, 2 + c; 3 + c, -1) - (2 + c)}{2(1 + c)\frac{3}{2}F_1(1, 2 + c; 3 + c, -1)}.$$ 

The constant $\beta$ is sharp.

**Remark 4.4.5.** The special case of Theorem 4.4.4 (with $\gamma = 0$) has been obtained by Aghalary et al. [1].

The case $c = 0$ in Theorem 4.4.4 yields yet another interesting result, which we state as a theorem.

**Theorem 4.4.6.** Let $\alpha \geq \gamma \geq 0$ be given. Suppose that $f \in W_0^\beta(\alpha, \gamma)$. Then, the classical Alexander operator $A[f] \in W_0^\beta(\alpha, \gamma)$, for $\beta = \frac{1 - \log(4)}{2 - \log(4)} = 0.28466$.... The constant $\beta$ is sharp.

In the last result, we present a condition which when satisfied by a function $f \in W_0^\beta(\alpha, \gamma)$, gives the region of variability of $\frac{f(z)}{z}$.

**Theorem 4.4.7.** Let $\delta < 1$ be a given real number. Let $\nu > \mu > 0$ satisfy (4.3.1). Define $\beta < 1$ such that

$$\beta = 1 - \frac{2(1 - \delta)}{\mu - \nu} \int_0^1 \frac{t(t^\frac{1}{\mu} - 1 - t^\frac{1}{\nu} - 1)}{1 + t} dt.$$ 

(4.4.2)

If $f \in W_0^\beta(\alpha, \gamma)$, then for some $\phi \in \mathbb{R}$ we have

$$\text{Re} \left( e^{i\phi} \left( \frac{f(z)}{z} - \delta \right) \right) > 0.$$ 

**Proof.** By (4.3.1)

$$\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu
\nu = \gamma.$$ 

Define

$$\lambda(t) = \frac{1}{\mu - \nu} (t^\frac{1}{\mu} - 1 - t^\frac{1}{\nu} - 1)$$ 

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and \( \psi_k = \int_0^1 \lambda(t)t^{k-1}dt \). A simple computation gives that

\[
\psi_k = \frac{1}{(1+(k-1)\mu)(1+(k-1)\nu)}.
\]

Writing \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \), we obtain

\[
F(z) = V_k[f](z) = z + \sum_{k=2}^{\infty} \frac{a_k}{(1+(k-1)\mu)(1+(k-1)\nu)} z^k.
\]

Finally we see that

\[
(1-\alpha+2\gamma) \frac{F(z)}{z} + (\alpha - 2\gamma) F'(z) + \gamma z F''(z) = 1 + \sum_{k=2}^{\infty} a_k z^{k-1}
\]

\[
= \frac{f(z)}{z}.
\]

In the view of Theorem 4.3.5, if \( f \in W_\beta(\alpha, \gamma) \), then \( F(z) = V_k[f] \in W_\delta(\alpha, \gamma) \). Finally, in the view of equation (4.4.3), we have

\[
\Re \left( e^{i\phi} \left( \frac{f(z)}{z} - \delta \right) \right) > 0.
\]

Which completes the proof. \( \square \)