Chapter 3

A NEW INTEGRAL OPERATOR AND ITS PROPERTIES *

3.1 Introduction

Let a class $U(\lambda)$ of analytic functions be defined as follows:

$U(\lambda) := \left\{ f \in \mathcal{A} : \left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda, \quad z \in E \right\},$

where $f(z) \neq 0$ for $z \in E \setminus \{0\}$ and $\lambda$ is a positive real number. In 1972, Ozaki and Nunokawa [90] proved that $U(\lambda) \subset S$, when $0 < \lambda \leq 1$, but not necessarily if $\lambda > 1$. Interestingly, the Koebe function $z/(1-z)^2$ belongs to $U(1)$ but does not belong to the class of starlike functions of order $\alpha$ ($\alpha > 0$). Similarly, the bounded analytic function $z+z^2/2$ belongs to $U(1)$ but is not in $S^*(\alpha)$ for any $\alpha > 0$. Thus, $U(1) \not\subset S^*(\alpha)$, for any $\alpha > 0$. Recently, Ponnusamy and Vasundhra [102] obtained conditions on $\lambda$ and $\alpha = \alpha(\lambda)$ such that $U(\lambda) \not\subset S^*(\alpha)$. Since then, the class $U(\lambda)$ has been studied rigorously. Further we consider some other interesting classes, besides $U(\lambda)$, studied extensively by various researchers [33, 78–87, 100]:

$\mathcal{P}(\lambda) := \left\{ f \in \mathcal{A} : \left| \left( \frac{z}{f(z)} \right)'' \right| \leq \lambda, \quad z \in E \right\},$

$\mathcal{M}(\lambda) := \left\{ f \in \mathcal{A} : \left| z^2 \left( \frac{z}{f(z)} \right)'' + f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda, \quad z \in E \right\}.$

We denote the classes $U(1)$, $\mathcal{P}(1)$ and $\mathcal{M}(1)$ by $U$, $\mathcal{P}$ and $\mathcal{M}$ respectively.

*Most of the results of this chapter will appear in S. Verma, S. Gupta and S. Singh [129].
According to Frideman [35], there are only nine functions of the class $S$ having integral coefficients in their power series expansions. If we set

$$S_Z = \left\{ f \in S : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } a_n \in \mathbb{Z} \right\},$$

then

$$S_Z = \left\{ z, \frac{z}{1 \pm z^2}, \frac{z}{1 \pm z^2}, \frac{z}{1 \pm z + z^2} \right\}.$$

Obradović and Ponnusamy [85] established the relation

$$S_Z \subset M \subset P \subset U \subset S.$$  \hspace{1cm} (3.1.1)

In a very recent paper, Obradović and Ponnusamy [86] raised an open question that whether the class $M$ is included in $S^*$? The work presented in this chapter is inspired by this open problem. According to (3.1.1), each function in $S_Z$ belongs to $M$. It is easy to see analytically that each function in $S_Z$ is starlike in $E$. Moreover, Fournier and Ponnusamy [33] showed the existence of a non-starlike function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in U$ such that

$$0 < -|a_2| + \sqrt{2 - |a_2|^2} < \sup_{|z|<1} \left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| \leq 1 - |a_2|.$$

Thus, $U \not\subset S^*$. But $M \not\subset U$ in view of (3.1.1) and so, $M$ may or may not be a subset of the class of starlike functions. With these facts in mind, in Section 3.2, we find conditions on $\lambda$ for which the functions in $M(\lambda)$ are contained in $\tilde{S}(\alpha)$. We also obtain conditions on $\lambda$ such that $M(\lambda) \not\subset K(\alpha)$.

Section 3.3 concerns a new integral operator $T_c[f]$ defined by

$$T_c[f](z) = cz \int_0^1 \int_0^1 r^{c-1} f'(rsz) \left( \frac{rsz}{f(rsz)} \right)^2 drds, \quad c > 0,$$

for $f \in A$, where $f(z) \neq 0$ for $z \in E \setminus \{0\}$. We find conditions on $\lambda$ such that the integral operator $T_c[f]$ is starlike (or convex) of order $\alpha$, whenever $f \in M(\lambda)$.

Let $F$ and $G$ be two subclasses of $A$. If for every $f \in F$, $r^{-1} f(rz) \in G$ for $r \leq r_0$, and $r_0$ is the maximum value of $r$ for which this holds, then we say that $r_0$ is the $G$-radius in $F$. There are several results of this type in the theory of univalent functions (cf. [81, 83, 104]). In Section 3.4, we determine $M$-radius in $S$. 

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3.2 Main Results

Define

\[ B_n = \left\{ w : w \text{ is analytic, } |w(z)| \leq 1 \text{ in } E \text{ and } w^{(k)}(0) = 0, \ k = 0, 1, \ldots, n \right\}, \]

where \( w^{(0)}(0) = w(0) \). Functions in \( B_0 \) are called Schwarz functions. Moreover if \( w \in B_n \), then \( w(z) = w_{n+1} z^{n+1} + w_{n+2} z^{n+2} + \cdots, w_{n+1} \neq 0 \).

We begin with the following theorem:

**Theorem 3.2.1.** For \( \alpha \in (0, 1] \), if \( \lambda > 0 \) satisfies the condition

\[
\lambda < \frac{1}{2} \left[ -a + \tan \left( \frac{\pi \alpha}{4} \right) \sqrt{4 \cos^2 \left( \frac{\pi \alpha}{4} \right) - a^2} \right],
\]

where \( a = \frac{|f''(0)|}{2} \leq 1 \), then \( M(\lambda) \subset \tilde{S}(\alpha) \).

**Proof.** Suppose \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \) belongs to the class \( M(\lambda) \). Thus there exists a function \( w \in B_1 \) such that

\[
z^2 \left( \frac{z}{f(z)} \right)'' + f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 = \lambda w(z),
\]

where \( w(0) = w'(0) = 0 \).

Set

\[
p(z) = f'(z) \left( \frac{z}{f(z)} \right)^2 - 1
\]

\[ = -z \left( \frac{z}{f(z)} \right)' + \frac{z}{f(z)} - 1.\]

Then, \( p \) is analytic in \( E \) with \( p(0) = p'(0) = 0 \). We can write (3.2.2) as

\[-zp'(z) + p(z) = \lambda w(z).\]

Integrating above equation w.r.t. \( z \) implies that

\[ p(z) = -\lambda \int_0^1 \frac{w(tz)}{t^2} dt. \]

(3.2.3)

Equivalently,

\[-z \left( \frac{z}{f(z)} \right)' + \frac{z}{f(z)} - 1 = -\lambda \int_0^1 \frac{w(tz)}{t^2} dt
\]

or \[-z \left( \frac{z}{f(z)} - 1 + a_2 z \right)' + \left( \frac{z}{f(z)} - 1 + a_2 z \right) = -\lambda \int_0^1 \frac{w(tz)}{t^2} dt.\]
Upon integration, it follows that
\[
\frac{z}{f(z)} = 1 - a_2 z + \lambda \int_0^1 \int_0^1 \frac{w(stz)}{s^2 t^2} d s d t = 1 - a_2 z + \lambda \int_0^1 t^{-2} \log \left( \frac{1}{t} \right) w(tz) d t. \tag{3.2.4}
\]

From (3.2.3), \( f'(z)(z/f(z))^2 \) ranges over the disc
\[
|\omega - 1| \leq \lambda \int_0^1 \frac{|w(tz)|}{t^2} d t \leq \lambda |z|^2 < \lambda, \tag{3.2.5}
\]

while on the other hand, from (3.2.4), \( z/f(z) \) ranges over the disc
\[
|\omega - 1| \leq |a_2| |z| + \lambda \int_0^1 t^{-2} \log \left( \frac{1}{t} \right) |w(tz)| d t \leq a |z| + \lambda |z|^2 \int_0^1 \log \left( \frac{1}{t} \right) d t < a + \lambda, \tag{3.2.6}
\]

where \( a = |a_2| \). Further, since
\[
\arg \left( \frac{z f'(z)}{f(z)} \right) = \arg \left( \left( \frac{z}{f(z)} \right)^2 f'(z) \right) - \arg \left( \frac{z}{f(z)} \right),
\]
so, in view of (3.2.5) and (3.2.6), we get
\[
\left| \arg \left( \frac{z f'(z)}{f(z)} \right) \right| \leq \left| \arg \left( \left( \frac{z}{f(z)} \right)^2 f'(z) \right) \right| + \left| \arg \left( \frac{z}{f(z)} \right) \right| \leq \arcsin(\lambda) + \arcsin(a + \lambda).
\]

Note that \( a \leq 1 \) and in view of (3.2.1), it is easy to see that \( a + \lambda < 1 \). Using the formula:
\[
\arcsin(x) + \arcsin(y) = \arcsin \left( x \sqrt{1 - y^2} + y \sqrt{1 - x^2} \right), \quad |x| \leq 1, \quad |y| \leq 1;
\]
we have
\[
\left| \arg \left( \frac{z f'(z)}{f(z)} \right) \right| \leq \arcsin \left[ \lambda \sqrt{1 - (a + \lambda)^2} + (a + \lambda) \sqrt{1 - \lambda^2} \right].
\]

Therefore, \( f \in \tilde{S}_\alpha \) whenever
\[
\arcsin \left[ \lambda \sqrt{1 - (a + \lambda)^2} + (a + \lambda) \sqrt{1 - \lambda^2} \right] \leq \frac{\pi \alpha}{2},
\]
or equivalently, whenever
\[
\lambda \sqrt{1 - (a + \lambda)^2} \leq \sin \left( \frac{\pi \alpha}{2} \right) - (a + \lambda) \sqrt{1 - \lambda^2}.
\]

Squaring both sides of the above inequality:
\[
\lambda^2 \left( 1 - (a + \lambda)^2 \right) \leq \left( \sin \left( \frac{\pi \alpha}{2} \right) - (a + \lambda) \sqrt{1 - \lambda^2} \right)^2,
\]

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\( \lambda^2 - \lambda^2(a + \lambda)^2 \leq \sin^2 \left( \frac{\pi \alpha}{2} \right) + (a + \lambda)^2(1 - \lambda^2) - 2 \sin \left( \frac{\pi \alpha}{2} \right)(a + \lambda)\sqrt{1 - \lambda^2}. \)

Rearranging the terms in above inequality, we have
\[
\sin^2 \left( \frac{\pi \alpha}{2} \right) (1 - \lambda^2) - 2 \sin \left( \frac{\pi \alpha}{2} \right)(a + \lambda)\sqrt{1 - \lambda^2} + (a + \lambda)^2 \geq \lambda^2 - \lambda^2 \sin^2 \left( \frac{\pi \alpha}{2} \right),
\]
or,
\[
\left[ \sin \left( \frac{\pi \alpha}{2} \right) \sqrt{1 - \lambda^2} - (a + \lambda) \right]^2 \geq \lambda^2 \cos^2 \left( \frac{\pi \alpha}{2} \right).
\]
Taking square-root, we have
\[
\sin \left( \frac{\pi \alpha}{2} \right) \sqrt{1 - \lambda^2} - (a + \lambda) \geq \lambda \cos \left( \frac{\pi \alpha}{2} \right). \tag{3.2.7}
\]

Here, we have excluded the case, when
\[
\sin \left( \frac{\pi \alpha}{2} \right) \sqrt{1 - \lambda^2} - (a + \lambda) \leq -\lambda \cos \left( \frac{\pi \alpha}{2} \right),
\]
as it leads to the imaginary values for \( \lambda \). Again, squaring inequality (3.2.7), we have
\[
\sin^2 \left( \frac{\pi \alpha}{2} \right) (1 - \lambda^2) \geq a^2 + \lambda^2 \left( 1 + \cos \left( \frac{\pi \alpha}{2} \right) \right)^2 + 2a\lambda \left( 1 + \cos \left( \frac{\pi \alpha}{2} \right) \right),
\]
or equivalently
\[
2\lambda^2 + 2a\lambda - \frac{\sin^2 \left( \frac{\pi \alpha}{2} \right) - a^2}{1 + \cos \left( \frac{\pi \alpha}{2} \right)} \leq 0,
\]
which holds in view of (3.2.1).

Setting \( \alpha = 1 \) in Theorem 3.2.1, we obtain the following result:

**Corollary 3.2.2.** If \( f \in \mathcal{M} (\lambda) \), \( a = |f''(0)|/2 \leq 1 \) and \( 0 < \lambda \leq \frac{\sqrt{2} - a - a}{2} \), then, \( f \in S^* \).

Choosing \( a = 0 \) in Corollary 3.2.2, we get the following criterion for starlikeness:

**Corollary 3.2.3.** If \( f \in \mathcal{M} (\lambda) \) such that \( f''(0) = 0 \) and \( 0 < \lambda \leq \frac{1}{\sqrt{2}} \), then \( f \in S^* \).

In our next result, we will find conditions on \( \lambda \) such that \( \mathcal{M} (\lambda) \not\subseteq K (\alpha) \).

**Theorem 3.2.4.** Let \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{M} (\lambda) \) be such that \( a_2 = 0 \) and let \( 0 \leq \alpha < 1. \) Then,
\[
\left| \frac{zf''(z)}{f'(z)} \right| < 1 - \alpha, \tag{3.2.8}
\]
for \( z \in E \) holds whenever \( 0 < \lambda \leq (1 - \alpha)/(7 - \alpha) \).
Proof. For \( f \in \mathcal{M}(\lambda) \), proceeding as in Theorem 3.2.1, we have

\[
\frac{w}{f(z)} = 1 + \lambda \int_0^1 t^{-2} \log \left( \frac{1}{t} \right) w(tz) dt,
\]

(3.2.11)

for some function \( w \in \mathcal{B}_1 \) with \( w(0) = w'(0) = 0 \). It is easy to verify that

\[
\left( \frac{z}{f(z)} \right)^2 f'(z) \left\{ \frac{z f''(z)}{f'(z)} + 2 - 2 \frac{z f'(z)}{f(z)} \right\} = z \left\{ \left( \frac{z}{f(z)} \right)^2 f'(z) \right\}
\]

\[
= -z^2 \left( \frac{z}{f(z)} \right)''.
\]

Equivalently

\[
\frac{z f''(z)}{f'(z)} = -z^2 \left( \frac{z}{f(z)} \right)'' \left\{ \left( \frac{z}{f(z)} \right)^2 f'(z) \right\}^{-1} + 2 \frac{z f''(z)}{f(z)} - 2,
\]

(3.2.12)

In the view of (3.2.9), we can see that

\[
-z^2 \left( \frac{z}{f(z)} \right)'' \left\{ \left( \frac{z}{f(z)} \right)^2 f'(z) \right\}^{-1} = -\frac{(1 + \lambda w(z))}{\left( \frac{z}{f(z)} \right)^2 f'(z)} + 1.
\]

(3.2.13)

Substituting values from the equations (3.2.10), (3.2.11) and (3.2.13) in (3.2.12), we have

\[
\frac{z f''(z)}{f'(z)} = -\frac{(1 + \lambda w(z))}{1 - \lambda \int_0^1 \frac{w(tz)}{t^2} dt} + 2 \frac{1 - \lambda \int_0^1 \frac{w(tz)}{t^2} dt}{1 + \lambda \int_0^1 t^{-2} \log \left( \frac{1}{t} \right) w(tz) dt}
\]

\[
= -\lambda \frac{(w(z) + f_0^1 \frac{w(tz)}{t^2} dt)}{1 - \lambda \int_0^1 \frac{w(tz)}{t^2} dt} - 2 \lambda \frac{f_0^1 \frac{w(tz)}{t^2} dt + f_0^1 t^{-2} \log \left( \frac{1}{t} \right) w(tz) dt}{1 + \lambda \int_0^1 t^{-2} \log \left( \frac{1}{t} \right) w(tz) dt}.
\]

Applying triangular inequality, we have

\[
\left| \frac{z f''(z)}{f'(z)} \right| \leq \frac{\lambda \left( |w(z)| + f_0^1 \frac{|w(tz)|}{t^2} dt \right)}{1 - \lambda \int_0^1 \frac{|w(tz)|}{t^2} dt} + 2 \lambda \frac{f_0^1 \frac{|w(tz)|}{t^2} dt + f_0^1 t^{-2} \log \left( \frac{1}{t} \right) |w(tz)| dt}{1 - \lambda \int_0^1 t^{-2} \log \left( \frac{1}{t} \right) |w(tz)| dt}
\]

\[
\leq \frac{6\lambda |z|^2}{1 - \lambda |z|^2}.
\]
Note that
\[ \frac{6\lambda |z|^2}{1 - \lambda |z|^2} < 1 - \alpha \]
or
\[ |z|^2 < \frac{1 - \alpha}{\lambda(7 - \alpha)} \]
holds for all \( z \in E \), whenever \( 0 < \lambda \leq (1 - \alpha)/(7 - \alpha) \).

Letting \( \alpha = 0 \) in Theorem 3.2.4, we obtain the following criterion of convexity.

**Corollary 3.2.5.** If \( f \in M(\lambda) \) such that \( f''(0) = 0 \), then \( f \) is convex provided \( 0 < \lambda \leq 1/7 \).

**Remark 3.2.6.** Theorem 3.2.4 also gives us that if \( f \in M(\lambda) \) such that \( f''(0) = 0 \) then
\[ \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \]
in
\[ |z| < \sqrt{\frac{1 - \alpha}{\lambda(7 - \alpha)}} \]
whenever \( \frac{1 - \alpha}{7 - \alpha} < \lambda \leq 1 \).

In particular, if \( 1/7 < \lambda \leq 1 \), then the radius of convexity for the functions \( f \in M(\lambda) \) such that \( f''(0) = 0 \) is at least \( 1/\sqrt{7\lambda} \).

### 3.3 Properties of a new Integral Operator

A function \( f \in A \) is said to be bounded starlike of order \( \alpha \) (\( 0 \leq \alpha < 1 \)) if it satisfies
\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha. \]

In this section, we define a new integral operator \( T_c[f] \) as follows:
\[ T_c[f](z) = cz \int_0^1 \int_0^1 r^{c-1} f'(rsz) \left( \frac{rsz}{f'(rsz)} \right)^2 \, dr \, ds, \quad c > 0, \quad (3.3.1) \]
for \( f \in A \), where \( f(z) \neq 0 \) for \( z \in E \setminus \{0\} \). Our next result concerns the bounded starlikeness of \( T_c[f] \).

**Theorem 3.3.1.** Let \( f \in M(\lambda) \) with \( f''(0) = 0 \), and let \( \alpha \) and \( c \) be real numbers such that \( 0 \leq \alpha < 1 \) and \( c > 0 \). Then, the integral operator \( T_c[f] \) defined by (3.3.1) is bounded starlike of order \( \alpha \), whenever
\[ 0 < \lambda \leq \frac{3(c+2)(1-\alpha)}{c(3-\alpha)}. \quad (3.3.2) \]
Proof. Suppose that \( f \in \mathcal{M}(\lambda) \). One can see that the integral operator
\[
T_c[f](z) = cz \int_0^1 \int_0^1 r^{c-1} f'(rsz) \left( \frac{rsz}{f(rsz)} \right)^2 drds = F(z) \text{ (say)},
\]
satisfies the following differential equation
\[
F'(z) - zF''(z) - \frac{1}{c} z^2 F'''(z) = z^2 \left( \frac{z}{f(z)} \right)'' + \left( \frac{z}{f(z)} \right)^2 f'(z).
\] (3.3.3)

From the fact that \( f''(0) = 0 \) and (3.3.3), it easily follows that \( F''(0) = 0 \). Since \( f \in \mathcal{M}(\lambda) \), so
\[
z^2 \left( \frac{z}{f(z)} \right)'' + \left( \frac{z}{f(z)} \right)^2 f'(z) = 1 + \lambda w(z),
\]
for some \( w \in \mathcal{B}_1 \). Thus, we can rewrite (3.3.3) as
\[
F'(z) - zF''(z) - \frac{1}{c} z^2 F'''(z) = 1 + \lambda w(z),
\] (3.3.4)

Writing \( F(z) = z + \sum_{n=2}^{\infty} A_{n+1} z^n \) (\( F''(0) = 0 \)) and \( w(z) = \sum_{n=1}^{\infty} b_{n+1} z^{n+1} \), we obtain from (3.3.4)
\[
1 - \sum_{n=2}^{\infty} \frac{(n^2 - 1)(n+c)}{c} A_{n+1} z^n = 1 + \lambda \sum_{n=1}^{\infty} b_{n+1} z^{n+1}.
\]

This gives
\[
A_{n+1} = \frac{-c\lambda}{(n-1)(n+1)(n+c)} b_n, \text{ for } n \geq 2.
\] (3.3.5)

Therefore
\[
\frac{F(z)}{z} = 1 + \sum_{n=2}^{\infty} A_{n+1} z^n
\]
\[
= 1 + \sum_{n=2}^{\infty} \frac{-c\lambda}{(n-1)(n+1)(n+c)} b_n z^n
\]
\[
= 1 - c\lambda \left( \sum_{n=2}^{\infty} b_n z^n \right) * \left( \sum_{n=2}^{\infty} \frac{1}{(n^2 - 1)(n+c) z^n} \right).
\]

Case I. When \( c = 1 \), then
\[
\frac{F(z)}{z} = 1 - \lambda \left( \sum_{n=2}^{\infty} b_n z^n \right) * \left( \sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1) z^n} \right).
\]

Using a simple technique of partial fractions, we have
\[
\frac{F(z)}{z} = 1 - \lambda \left( \sum_{n=2}^{\infty} b_n z^n \right) * \left( \sum_{n=2}^{\infty} \frac{1}{4} \left[ \frac{1}{n-1} - \frac{1}{n+1} - \frac{2}{(n+1)^2} z^n \right] \right),
\]
or equivalently
\[
\frac{F(z)}{z} = 1 - \frac{\lambda}{4} \left[ \int_0^1 \frac{w(tz)}{t^2} (1 - t^2) \, dt - 2 \int_0^1 \int_0^1 w(stz) \, ds \, dt \right].
\]

Further, differentiating above equation w.r.t. \( z \), we get
\[
F'(z) = 1 - \frac{\lambda}{2} \int_0^1 \frac{w(tz)}{t^2} (1 - t^2) \, dt.
\]

Combining above two equations, we get
\[
\frac{zF'(z)}{F(z)} = \frac{1 - \frac{\lambda}{2} \int_0^1 \frac{w(tz)}{t^2} (1 - t^2) \, dt}{1 - \frac{\lambda}{4} \left[ \int_0^1 \frac{w(tz)}{t^2} (1 - t^2) \, dt - 2 \int_0^1 \int_0^1 w(stz) \, ds \, dt \right]} = 1 - \frac{\lambda}{4} \left[ \int_0^1 \frac{w(tz)}{t^2} (1 - t^2) \, dt - 2 \int_0^1 \int_0^1 w(stz) \, ds \, dt \right].
\]

Taking absolute value, we have
\[
\left| \frac{zF'(z)}{F(z)} - 1 \right| = \left| \frac{\frac{\lambda}{4} \left[ \int_0^1 \frac{w(tz)}{t^2} (1 - t^2) \, dt + 2 \int_0^1 \int_0^1 w(stz) \, ds \, dt \right]}{1 - \frac{\lambda}{4} \left[ \int_0^1 \frac{w(tz)}{t^2} (1 - t^2) \, dt - 2 \int_0^1 \int_0^1 w(stz) \, ds \, dt \right]} - 1 \right| < \frac{\lambda}{4} \left[ \int_0^1 \frac{|w(tz)|}{t^2} (1 - t^2) \, dt + 2 \int_0^1 \int_0^1 |w(stz)| \, ds \, dt \right] - \frac{\lambda}{2} \int_0^1 \int_0^1 w(stz) \, ds \, dt \]
\[
< \frac{\lambda}{4} \left[ \int_0^1 (1 - t^2) \, dt + 2 \int_0^1 \int_0^1 s^2 t^2 \, ds \, dt \right] - \frac{\lambda}{2} \int_0^1 \int_0^1 s^2 t^2 \, ds \, dt = \frac{2\lambda}{9 - \lambda}.
\]

Thus, using (3.3.2) with \( c = 1 \), we get
\[
\left| \frac{zF'(z)}{F(z)} - 1 \right| < \frac{2\lambda}{9 - \lambda} \leq 1 - \alpha.
\]

**Case II.** When \( c \neq 1 \), then
\[
\frac{F(z)}{z} = 1 - c\lambda \left( \sum_{n=2}^{\infty} b_n z^n \right) * \left( \sum_{n=2}^{\infty} \frac{1}{(n^2 - 1)(n + c)^n} z^n \right).
\]

Using a simple technique of partial fractions, we have
\[
\frac{F(z)}{z} = 1 - c\lambda \left( \sum_{n=2}^{\infty} b_n z^n \right) * \left( \sum_{n=2}^{\infty} \frac{1}{2n^2 - 1} \left[ \frac{2}{n + c} - \frac{c + 1}{n + 1} + \frac{c - 1}{n - 1} \right] z^n \right)
\]
or equivalently
\[
\frac{F(z)}{z} = 1 - \frac{c\lambda}{2(c^2 - 1)} \int_0^1 \frac{w(tz)}{t^2} \left[ 2t^{c+1} - (c+1)t^2 + (c-1) \right] \, dt.
\]
Further, differentiating above equation w.r.t. $z$ and using integration by parts implies that

$$F'(z) = 1 - \frac{c\lambda}{(c+1)} \int_0^1 \frac{w(tz)}{t^2} \left(1-t^{c+1}\right) dt.$$

Combining above two equations, we get

$$\frac{zF'(z)}{F(z)} = \frac{1 - \frac{c\lambda}{(c+1)} \int_0^1 \frac{w(tz)}{t^2} \left(1-t^{c+1}\right) dt}{1 - \frac{c\lambda}{2(c^2-1)} \int_0^1 \frac{w(tz)}{t^2} \left[2t^{c+1} - (c+1)t^2 + (c-1)\right] dt}.$$

Taking absolute value, we have

$$\left|\frac{zF'(z)}{F(z)} - 1\right| = \left|\frac{-\frac{c\lambda}{2(c^2-1)} \int_0^1 \frac{w(tz)}{t^2} \left[-2ct^{c+1} + (c+1)t^2 + (c-1)\right] dt}{1 - \frac{c\lambda}{2(c^2-1)} \int_0^1 \frac{w(tz)}{t^2} \left[2t^{c+1} - (c+1)t^2 + (c-1)\right] dt}\right|. \quad (3.3.6)$$

In order to prove our result, we need to consider the following two subcases:

**Subcase I.** When $0 < c < 1$. Equation (3.3.6) gives

$$\left|\frac{zF'(z)}{F(z)} - 1\right| < \frac{\frac{c\lambda}{2(1-c^2)} \int_0^1 \frac{|w(tz)|}{t^2} \left[2ct^{c+1} - (c+1)t^2 - (c-1)\right] dt}{1 - \frac{c\lambda}{2(1-c^2)} \int_0^1 \frac{|w(tz)|}{t^2} \left[-2ct^{c+1} + (c+1)t^2 + (c-1)\right] dt} < 2c\lambda = \frac{2c\lambda}{3(c+2) - c\lambda}.$$

**Subcase II.** When $c > 1$. Equation (3.3.6) implies

$$\left|\frac{zF'(z)}{F(z)} - 1\right| < \frac{\frac{c\lambda}{2(c^2-1)} \int_0^1 \frac{|w(tz)|}{t^2} \left[-2ct^{c+1} + (c+1)t^2 + (c-1)\right] dt}{1 - \frac{c\lambda}{2(c^2-1)} \int_0^1 \frac{|w(tz)|}{t^2} \left[2t^{c+1} - (c+1)t^2 + (c-1)\right] dt} < 2c\lambda = \frac{2c\lambda}{3(c+2) - c\lambda}.$$

Thus, in both the cases, we arrive at the same conclusion. Finally using (3.3.2), we get

$$\left|\frac{zF'(z)}{F(z)} - 1\right| < \frac{2c\lambda}{3(c+2) - c\lambda} \leq 1 - \alpha.$$

This completes the proof of our theorem. □
Taking $\alpha = 0$ in Theorem 3.3.1, we get the following result:

**Corollary 3.3.2.** Let $f \in M(\lambda)$ with $f''(0) = 0$, and let $c > 0$ be any real number. Then, the integral operator $T_c[f]$ defined by (3.3.1) is starlike, whenever

$$0 < \lambda \leq \frac{(c+2)c}{c}.$$  

Next, we present an example in support of Theorem 3.3.1.

**Example 3.3.3.** Consider

$$\frac{z}{f(z)} = 1 + \frac{1}{4}z^3.$$  

One can easily check that $f \in M(\lambda)$ with $\lambda = 1$. Also, the integral operator defined by (3.3.1) becomes

$$T_c[f](z) = F(z) = z - \frac{c}{8(c+3)}z^4, \text{ for } c > 0.$$  

Thus, $f$ satisfies the conditions of Theorem 3.3.1 for $\alpha = 0$. Further

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| = \left| \frac{1 - \frac{c}{2(c+3)}z^3}{1 - \frac{c}{8(c+3)}z^3} - 1 \right| = \left| \frac{\frac{3c}{8(c+3)}z^3}{1 - \frac{c}{8(c+3)}z^3} \right| < \frac{\frac{3c}{8(c+3)}}{1 - \frac{c}{8(c+3)}} = \frac{3c}{7c+24} < 1.$$  

Thus, the integral operator $T_c[f]$ is starlike.

Below, we give an example of a function $f$ which does not satisfy the hypothesis of Theorem 3.3.1 in the sense, that if $f \in M(\lambda)$ where $\lambda$ does not belong to the stated range, then the integral operator $T_c[f]$ is not starlike.

**Example 3.3.4.** Consider

$$\frac{z}{f(z)} = 1 + \frac{3}{2}z^3.$$  

One can easily check that $f \in M(\lambda)$ with $\lambda = 6$. For $c = 2$, the integral operator defined by (3.3.1) becomes

$$T_c[f](z) = F(z) = z - \frac{3}{10}z^4.$$  

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Thus, $f$ satisfies the criterion of Theorem 3.3.1, but $\lambda$ does not belong to the required range (by (3.3.2) the range of $\lambda$ is $(0, 2]$ for $\alpha = 0$ and $c = 2$).

Further
\[
\left| \frac{zF'(z)}{F(z)} - 1 \right| = \left| \frac{1 - \frac{6}{5}z^3}{1 - \frac{3}{10}z} - 1 \right|
= \left| \frac{\frac{9}{10}z^3}{1 - \frac{3}{10}z} \right|
< \frac{\frac{9}{10}}{1 - \frac{3}{10}} = \frac{9}{7} \leq 1.
\]

Thus, $T_c[f]$ is not starlike.

**Theorem 3.3.5.** Let $f \in M(\lambda)$ with $f''(0) = 0$, and let $\alpha$ and $c$ be real numbers such that $0 \leq \alpha < 1$ and $c > 0$. Then, the integral operator $T_c[f]$ satisfies
\[
\left| \frac{zT''_c[f](z)}{T'_c[f](z)} \right| < 1 - \alpha
\]
and so, is convex of order $\alpha$, whenever
\[
0 < \lambda \leq \frac{(c + 2)(1 - \alpha)}{c(3 - \alpha)}.
\quad (3.3.7)
\]

**Proof.** For $f \in M(\lambda)$, let $T_c[f] = F$ (say) be the integral operator defined by (3.3.1).

Now proceeding as in Theorem 3.3.1, we have
\[
A_{n+1} = \frac{-c\lambda}{(n-1)(n+1)(n+c)} b_n, \quad \text{for} \quad n \geq 2,
\quad (3.3.8)
\]
where
\[
F(z) = z + \sum_{n=2}^{\infty} A_{n+1} z^{n+1}
\]
and
\[
w(z) = \sum_{n=1}^{\infty} b_{n+1} z^{n+1}
\]
is some function $w \in B_1$.

Therefore
\[
F'(z) = 1 + \sum_{n=2}^{\infty} (n+1) A_{n+1} z^n
= 1 + \sum_{n=2}^{\infty} \frac{-c\lambda}{(n-1)(n+c)} b_n z^n
= 1 - c\lambda \left( \sum_{n=2}^{\infty} b_n z^n \right) \ast \left( \sum_{n=2}^{\infty} \frac{1}{(n-1)(n+c)} z^n \right).
\]
Using partial fractions, we have
\[
F'(z) = 1 - \frac{c\lambda}{(c+1)} \int_0^1 \frac{w(tz)}{t^2} (1 - tc^{+1}) \, dt.
\] (3.3.9)

Since \( F(z) = z + \sum_{n=2}^{\infty} A_n z^{n+1} \), therefore
\[
z F''(z) = \sum_{n=2}^{\infty} A_n n(n + 1) z^n.
\]

In the view of (3.3.8)
\[
z F''(z) = \sum_{n=2}^{\infty} \frac{-nc\lambda}{(n-1)(n+c)} b_n z^n
\]
\[
= \sum_{n=2}^{\infty} \frac{-nc\lambda}{(n-1)(n+c)} z^n * w(z).
\]

Resolving into partial fractions, we get
\[
z F''(z) = \frac{-c\lambda}{c+1} \int_0^1 \frac{w(tz)}{t} (1 - tc^{+1}) dt.
\] (3.3.10)

Combining (3.3.9) and (3.3.10)
\[
z F''(z) = \frac{-c\lambda}{c+1} \int_0^1 \frac{w(tz)}{t^2} (1 + ct^{c+1}) \, dt
\]
\[
1 - \frac{c\lambda}{(c+1)} \int_0^1 \frac{w(tz)}{t^2} (1 - tc^{+1}) \, dt,
\] (3.3.11)

which further implies that
\[
\left| \frac{z F''(z)}{F'(z)} \right| = \left| \frac{-c\lambda}{c+1} \int_0^1 \frac{w(tz)}{t^2} (1 + ct^{c+1}) \, dt}{1 - \frac{c\lambda}{(c+1)} \int_0^1 \frac{w(tz)}{t^2} (1 - tc^{+1}) \, dt} \right| < \frac{c\lambda}{c+1} \int_0^1 \frac{w(tz)}{t^2} (1 + ct^{c+1}) \, dt
\]
\[
1 - \frac{c\lambda}{(c+1)} \int_0^1 \frac{w(tz)}{t^2} (1 - tc^{+1}) \, dt
\]
\[
< \frac{c\lambda}{c+1} \int_0^1 (1 + ct^{c+1}) \, dt
\]
\[
1 - \frac{c\lambda}{(c+1)} \int_0^1 (1 - tc^{+1}) \, dt
\]
\[
= \frac{2c\lambda}{(c+2) - c\lambda}
\]
\[
\leq 1 - \alpha, \quad \text{by (3.3.7)}.
\]

This completes the proof of our theorem. \(\square\)

Taking \(\alpha = 0\) in Theorem 3.3.5, we obtain the following result:

**Corollary 3.3.6.** Let \( f \in \mathcal{M}(\lambda) \) with \( f''(0) = 0 \), and \( c > 0 \) be any real number. Then, the integral operator \( T_c[f] \) is convex, whenever
\[
0 < \lambda \leq \frac{(c+2)}{3c}.
\]

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3.4 Radius Property

According to a result due to Aksentiev [2] (also see Ozaki and Nunokawa [90] for a reformulated version), the functions in $\mathcal{U}(\lambda)$ are known to be univalent for $0 < \lambda \leq 1$. As $\mathcal{M} \subset \mathcal{U}$, so, functions in $\mathcal{M}$ are univalent in $E$. But functions in $S$ need not necessarily belong to the class $\mathcal{M}$. For example, consider

$$f(z) = \frac{z}{1 + z^2 + z^3/2}.$$ 

One can see that,

$$\frac{z}{f(z)} = 1 + z^2 + z^3/2,$$

$$(\frac{z}{f(z)})' = 1/2 + 3z^2/2,$$

$$(\frac{z}{f(z)})'' = 3z.$$

Therefore,

$$\left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| = \left| -z \left( \frac{z}{f(z)} \right)' + \left( \frac{z}{f(z)} \right)^2 - 1 \right| = | -z^3 | < 1.$$

Also,

$$\left| z^2 \left( \frac{z}{f(z)} \right)'' + f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| = | 2z^3 | < 1.$$ 

for all $z \in E$. Thus, $f \in \mathcal{U}$ and hence is univalent, but $f \notin \mathcal{M}$.

In this last section, we determine the value of $r_0$ which is the $\mathcal{M}$-radius in $S$. In the proof of our result, we shall need the following well-known result concerning the class $S$ which reveals the importance of the area theorem in the theory of univalent functions.

**Lemma 3.4.1.** [37, p.193] Let $\mu > 0$ and $f \in S$ be in the form

$$\left( \frac{z}{f(z)} \right)^\mu = 1 + \sum_{n=1}^\infty b_n z^n.$$

Then, we have

$$\sum_{n=1}^\infty (n-\mu) |b_n|^2 \leq \mu.$$ 

We shall also need the following result of Obradović and Ponnusamy [85] for the proof of our result.
Lemma 3.4.2. [85] Let $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a non-vanishing analytic function in $E$ that satisfies the coefficient condition
\[
\sum_{n=2}^{\infty} (n-1)^2 |b_n| \leq \lambda,
\tag{3.4.1}
\]
for some $\lambda > 0$. Then the function $f$ defined by $f(z) = z/\phi(z)$ is in $\mathcal{M}(\lambda)$.

Theorem 3.4.3. If $f \in S$ is given by
\[
f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n},
\]
then $\frac{1}{r} f(rz) \in \mathcal{M}$ for $0 < r \leq r_0$, where $r_0 = 0.55738$, correctly rounded off to five decimal places, is the unique real root of the equation $8r^6 - 5r^4 + 4r^2 - 1 = 0$.

Proof. As $f \in S$ be given by
\[
f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n}.
\]
Then $z/f(z)$ is non-vanishing in $E$. Thus
\[
\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n.
\tag{3.4.2}
\]
Using Lemma 3.4.1, with $\mu = 1$, we obtain that
\[
\sum_{n=2}^{\infty} (n-1)|b_n|^2 \leq 1.
\tag{3.4.3}
\]
Also, $|b_1| = |f''(0)/2| \leq 2$ (by the Bieberbach inequality for the second coefficient of $f \in S$). For $0 < r \leq 1$, the series representation in (3.4.2) implies that
\[
\frac{rz}{f(rz)} = 1 + \sum_{n=1}^{\infty} b_n r^n z^n.
\]
Therefore,
\[
\frac{1}{r} f(rz) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n r^n z^n}.
\]
According to Lemma 3.4.2 (with $\lambda = 1$), to prove that $\frac{1}{r} f(rz) \in \mathcal{M}$, it suffices to show that
\[
\sum_{n=2}^{\infty} (n-1)^2 |b_n r^n| \leq 1.
\]
Now, by application of Cauchy-Schwarz inequality, we have
\[
\sum_{n=2}^{\infty} (n-1)^2 |b_n r^n| = \sum_{n=2}^{\infty} \left( \sqrt{n-1} |b_n| \right) \left( (n-1)^{3/2} |r^n| \right).
\]

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\[
\begin{align*}
&\leq \left( \sum_{n=2}^{\infty} (n-1)|b_n|^2 \right)^{1/2} \left( \sum_{n=2}^{\infty} (n-1)^3 r^{2n} \right)^{1/2} \\
&\leq \left( \sum_{n=2}^{\infty} (n-1)^3 r^{2n} \right)^{1/2} \quad \text{(using (3.4.3))} \\
&= \left( r^2 \sum_{n=1}^{\infty} n^3 r^{2n} \right)^{1/2} \\
&= \left( 6r^2 \left( \sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} r^{2n} - 2 \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} r^{2n} \right. \\
&\quad \left. + \frac{7}{6} \sum_{n=0}^{\infty} (n+1)r^{2n} - \frac{1}{6} \sum_{n=0}^{\infty} r^{2n} \right) \right)^{1/2} \\
&= \left( 6r^2 \left( \frac{1}{(1-r^2)^3} - 2 \frac{1}{(1-r^2)^3} + \frac{7}{6} \frac{1}{(1-r^2)^2} - \frac{1}{6} \frac{1}{(1-r^2)} \right) \right)^{1/2} \\
&= \frac{\sqrt{r^4(r^4 + 4r^2 + 1)}}{(1-r^2)^2}.
\end{align*}
\]

Consequently, \( \frac{1}{r} f(rz) \in M \) for \( 0 < r \leq r_0 \), where \( r_0 \) is a positive solution of the equation

\[
g(r) := \frac{\sqrt{r^4(r^4 + 4r^2 + 1)}}{(1-r^2)^2} = 1.
\]

We see that \( g(r) \) is an increasing function of \( r^2 \) from zero to infinity in the interval \( (0,1) \) and so the equation \( g(r) = 1 \), which is equivalent to the equation

\[
8r^6 - 5r^4 + 4r^2 - 1 = 0,
\]

has unique real root \( r_0 \) in the interval \( (0,1) \). Upon solving the equation, we have \( r_0 = 0.55738 \), which completes the proof. \( \square \)