CHAPTER II

REVIEW OF LITERATURE
2.1 For efficiency measurement there are two basic approaches, namely (i) the parametric approach (PA) and (ii) the non-parametric approach (NPA). In parametric approach either a Frontier Production Function* or Cost function is postulated and estimated, the later being done by (a) Corrected least squares approach,** (b) maximum likelihood method*** and (c) mathematical programming**** of which linear programming approach is a special case.

* A Frontier Production Function assigns maximum possible output to each input mix.


THE PARAMETRIC APPROACH:

In literature we come across studies of efficiency estimation based on (1) the Cobb-Douglas Production Function, (2) The Zellner–Revankar's Variable returns to scale (VRST) frontier production function, (3) the translog frontier production function, (4) the homothetic production function, (5) constant ratio of elasticity of substitution, homothetic (CRESH) frontier production function.

(1) THE COBB-DOUGLAS FRONTIER PRODUCTION FUNCTION

The mathematical form of Cobb-Douglas production function is,

\[ y^* = \Theta x_1^\alpha x_2^\beta \]  \hspace{1cm} (2.1.1)

where
\[ y^* \] = Frontier output
\[ x_1, x_2 \] = Inputs
\[ \Theta \] = Efficiency parameter
\[ \alpha, \beta \] = Frontier output elasticities with respect to first and second inputs respectively.
\[ \Theta > 0, \quad 0 < \alpha, \beta < 1 \]

The condition that \( \alpha \) and \( \beta \) are constrained to lie between 0 and 1 imply diminishing marginal products.

\( \alpha + \beta \) is returns to scale parameter.
Timmer (1971) armed with a Cobb-Douglas production frontier attempted to estimate technical efficiency of various production units in an industry. The Frontier Function is estimated with linear programming approach by minimizing,

\[ z = \beta \ln x_1 + \alpha \ln x_2 \]

Subject to

\[ \beta \ln x_{1i} + \alpha \ln x_{2i} \geq \ln y_i \]

\[ i = 1, 2, \ldots, n \]

\[ \alpha, \beta \geq 0. \]

\( \hat{x} \): unrestricted for sign.

Where \( \ln x_1 \) and \( \ln x_2 \) are means of logarithmic first and second inputs respectively of \( n \) production units.

Define technical efficiency of ith unit as follows:

\[ TF \left( i \right) = \frac{y_i^*}{y_i} \]

where \( \ln y_i^* = \ln x_{1i} + \ln x_{2i} \)

\[ 0 \leq TE \left( i \right) \leq 1. \]

Schmidt and Lovell** postulated a Cobb-Douglas production function with composite error term such as \( v - u \),


where \( u \) measures technical efficiency differences among different production units and \( u \) is a statistical disturbance term that represents PURE NOISE.

\[
y = Y \prod_{i=1}^{k} \hat{\alpha}_i x_i e^{(\mathcal{N} - u)}
\]  

(2.1.3)

The dual cost function of (2.1.3) is as follows:

\[
\ln C = \Theta + \frac{1}{r} \ln y + \sum_{i=1}^{k} \frac{\hat{\alpha}_i}{\hat{\sigma}_i} \ln p_i - \frac{1}{r} (\mathcal{N} - u)
\]  

(2.1.4)

Where \( r = \frac{k}{\sum_{i=1}^{k} \hat{\alpha}_i} \), \( p_i \) : price per unit of \( i \)th input. (2.1.4) is estimated by (i) the ordinary least squares procedure, (ii) the method of maximum likelihood and (iii) the method of moments, and the estimates thus obtained are used to estimate the technical efficiencies of individual production units.

Aigner, Lovell and Schmidt* (1977) proposed a stochastic frontier model with composite error term,

\[
\mathcal{E} = \mathcal{N} + u, \text{ where}
\]

\( \mathcal{N} \) is symmetric normal random variable and \( u \) is a truncated normal random variable that represents technical efficiency difference among individual production units.

The probability density function of $\xi$ is given by,

$$f(\xi) = \frac{2}{\sigma} f^*(\frac{\xi}{\sigma}) \left[ 1 - F^*(\xi; \lambda) \right], \quad -\infty \leq \xi \leq +\infty \quad (2.1.5)$$

where $\sigma^2 = \frac{2}{\sigma u} + \frac{2}{\sigma v}$

$$\lambda = \frac{\sigma u}{\sigma v}$$

$f^*(.)$ and $F^*(.)$ are standard normal density and distribution function respectively. $f(.)$ is symmetric around zero. It can be shown that

$$\mathbb{E} (\xi) = \frac{\sqrt{2}}{\sqrt{\pi}} \sigma u$$

$$\mathbb{V} (\xi) = \left( \frac{1}{\pi} \right) \sigma u^2 + \sigma v^2$$

Let either the frontier production function or its suitable transformation be given by,

$$y_i = x_i^T \beta + \xi_i \quad (2.1.6)$$

Let $(y_i, x_i^T)$ $(i=1,2,\ldots,n)$ be output-input data on the $i$th production unit.

By maximizing log-likelihood of the sample, with respect to $\sigma^2, \lambda$ and $\beta$ we obtain $(k+1)$ equations, where $k$ denotes the number of elements in $\beta$. 

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\[
\begin{align*}
\frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \hat{\beta}^T x_i) x_j + \frac{\lambda}{\sigma} \sum_{i=1}^{n} f_i^* x_i &= 0 \\
\frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \hat{\beta}^T x_i) \beta^T x_i + \frac{\lambda}{\sigma} \sum_{i=1}^{n} f_i^* y_i &= 0
\end{align*}
\]
(2.1.7)

where \( \frac{\sigma^2}{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{\beta}^T x_i)^2 \).

These equations have to be solved by iterative procedures.

If \( u (\bar{\xi}, 0) \) follows exponential distributions such as,

\[
f(u) = \frac{1}{\phi} \exp \left( \frac{u}{\phi} \right), \phi \geq 0 \text{ and } y \text{ follows } N(0, \sigma_y^2),
\]
the probability density function of the composite error term is given by,

\[
f(\xi) = \frac{1}{\phi} \left[ 1 - F^*( \frac{\xi}{\sigma_y} ) \right] \exp \left[ \frac{\xi}{\phi} + \frac{\phi}{2} \right] \frac{\sigma_y^2}{\phi}
\]
...(2.1.8)

For empirical analysis the frontier production function is assumed to be Cobb-Douglas.

Battese and Coelli* (1988) assume that the underlying frontier production function is Cobb-Douglas and the same is fitted to PANEL DATA in order to predict firm specific technical efficiencies.

The production function or its transformation which is linear in parameters is specified by,

$$ y_{it} = x_{it}' \beta + e_{it} \quad (2.1.9) $$

$$ i = 1, 2, \ldots, N $$

$$ t = 1, 2, \ldots, T $$

$$ e_{it} = v_{it} - u_i $$

There are $N$ units of production, whose input-output data are available for $T$ time points.

$$ v_{it} \sim N(0, \sigma^2) $$

$u_i$ are independently and identically distributed normal random variables, defined by the truncation of $N(\mu, \sigma^2)$. Further $v_{it}$ and $u_i$ are independently distributed.

Since $u$ are assumed to be non-negative, the truncated normal probability density function is given by,

$$ f(u) = \frac{1}{\sqrt{2\pi} \sigma} \cdot \frac{-(u-\mu)^2}{2 \sigma^2} \cdot \frac{1}{\Phi(-\frac{\mu}{\sigma})} $$

where $\Phi$ is the distribution function of standard normal distribution.
The time suffix is not augmented with the non-negative error term $U$. This amounts to assuming that the technical efficiency of each production unit is constant over time. In the above frame work the technical efficiency of $i$th unit is defined by,

$$E(y_{it}^* | U_i, x_{it}, t=1,2,...,T)$$

$$= \frac{1}{T} \sum_{t=1}^{T} x_{it}^T \beta - E(U_i)$$

$$= \frac{\bar{x}_i^T \beta - U_i}{x_i^T \beta} \quad (2.1.11)$$

The technical efficiency of the whole industry is given by,

$$TE = 1 - \frac{E(U|x_i^T \beta, t=1,2,...N)}{x_i^T \beta}$$

$$= 1 - (\bar{x}_i^T \beta)^{-1} \int_{-\infty}^{0} \frac{e^{-\frac{(u-\mu)^2}{2\sigma^2}}}{(2\pi)^{\frac{1}{2}}\sigma} \phi\left(\frac{u-\mu}{\sigma}\right) \, du$$

$$= 1 - \frac{\phi\left(\frac{\mu}{\sigma}\right)}{1 - \Phi\left(\frac{\mu}{\sigma}\right)} \quad (2.1.13)$$
where $\phi(\cdot)$ is the standard normal density function.

\[
\star \quad \int_0^\infty u f(u) \, du = a \int_0^\infty \frac{u}{\sqrt{2\pi} \sigma} \cdot e^{-\frac{(u-\mu)^2}{2 \sigma^2}} \cdot du,
\]

when \( a = \frac{1}{1 - \Phi(-\mu/\sigma)} \)

\[
= a \int_{\mu/\sigma}^{\infty} (\mu + \sigma z) \cdot e^{-z^2/2} \cdot dz - \frac{\sigma}{\sqrt{2\pi}} \int_{\mu/\sigma}^{\infty} z \cdot e^{-z^2/2} \cdot dz
\]

\[
= a \left[ \mu \left\{ 1 - \Phi\left( -\frac{\mu}{\sigma} \right) \right\} + \frac{\sigma}{\sqrt{2\pi}} \int_{\mu/\sigma}^{\infty} z \cdot e^{-z^2/2} \cdot dz \right]
\]

\[
= \frac{1}{1 - \Phi\left( \frac{-\mu}{\sigma} \right)} \left[ \mu \left\{ 1 - \Phi\left( -\frac{\mu}{\sigma} \right) \right\} + \sigma \Phi\left( -\frac{\mu}{\sigma} \right) \right]
\]

\[
\Phi\left( \frac{-\mu}{\sigma} \right) = \frac{\mu + \sigma}{1 - \Phi\left( \frac{-\mu}{\sigma} \right)}
\]

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If the underlying production function is Cobb-Douglas, the utility function for each unit becomes,

$$\text{TE} (i) = \exp (-u_i)$$  \hspace{1cm} (2.1.14)

The mean technical efficiency of the industry is given by

$$E \left[ \text{TE}(i) \right] = E \left[ \exp (-u_i) \right] = \int_{-\infty}^{\infty} \frac{e^{-u} \cdot \frac{-(u - \mu)^2}{\sigma^2}}{\sqrt{2\pi} \sigma} \cdot \frac{1 - \Phi \left[ \sigma - \left( \frac{\mu}{\sigma} \right) \right]}{1 - \Phi \left( -\frac{\mu}{\sigma} \right)} \cdot \exp \left\{ -\mu + \frac{\sigma^2}{2} \right\} \cdot \text{du}.$$  \hspace{1cm} (2.1.15)

Bettese and Coelli proceed as follows to suggest an alternative measure of technical efficiency in the above frame work.

Consider $e_i = (e_{i1}, e_{i2}, \ldots, e_{iT})^T$. The sample values of this random vector may be denoted by,

$$\mathbf{v}_{it} \sim N \left( 0, \sigma^2_{\theta} \right)$$

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follows truncated normal distribution as mentioned above. The conditional distribution of $e_i$ given $e_i = \varepsilon_i$ is also a truncated (at zero) normal distribution with mean $\mu_i^*$ and variance $\sigma_i^2$

where,  

$$
\mu_i^* = \frac{-\sigma_i^2 \varepsilon_i + T^{-1} \mu_i \sigma_i^2}{(\sigma_i^2 + T^{-1} \sigma_i^2)}
$$

$$
\sigma_i^2 = \frac{\sigma_i^2}{\sigma_i^2 + T \sigma_i^2}
$$

$$
\varepsilon_i = \sum_{i=1}^T \varepsilon_i / T
$$

Further, it can be shown that

$$
E \left( e^{-U_i} \mid e_i = \varepsilon_i \right) = \frac{1 - \Phi \left[ \sigma_i^2 - (\mu_i^*/\sigma_i^2) \right]}{1 - \Phi \left[ -\mu_i^*/\sigma_i^2 \right]} \exp \left[ -\mu_i^* / \sigma_i^2 \right]
$$

which is nothing but mean technical efficiency of $i$th production unit when the frontier is Cobb-Douglas production function.

Forsund and Hjalmarskَon* estimate the Zellner and Revankar's generalized production function for milk processing Swedish Dairy Plants using PNAEL DATA.

Production Frontier:
\[ g(y_*) = \frac{y(t)}{V_x} \exp\left[\beta(t) y_x\right] = A(t) \frac{a_1(t)}{V_1} \frac{a_2(t)}{V_2} - (2.1.17) \]

where  
- \( y \): Frontier output
- \( v_1 \): Labour input
- \( v_2 \): Capital input
- \( t \): Time point

\( A(t), \alpha(t), \beta(t), \alpha_1(t), \alpha_2(t) \) are parameters which vary across time.

It is assumed that the observed outputs of various production units either fall on the frontier if they are technically most efficient or fall below the frontier whenever they are inefficient.

Objective Function:
\[ \sum_{t=1}^{T} \sum_{i=1}^{\frac{W}{2}} \left[ \ln g(y_x(t)) - \ln g(y(t)) \right] \]

where  
- \( y_x \): Frontier output
- \( y \): Observed output
- \( n \): Number of production units.
- \( T \): Number of time points

\[ \sum_{t=1}^{T} \sum_{i=1}^{\frac{n}{2}} \left[ \ln A(t) + a_1(t) \ln x(t) + \alpha_2(t) \ln v_2(t) \right] \]
CONSTRAINTS:

\[ \ln A(t) + \alpha'_1(t) \ln \mathcal{M}_{1,i}(t) + \alpha'_2(t) \ln \mathcal{M}_{2,i}(t) - \alpha(t) \ln x_i(t) - \beta(t) x_i(t) \geq 0 \]

\[ \alpha'_1(t) + \alpha'_2(t) = 1. \]

\[ i = 1, 2, \ldots, n \]

\[ t = 1, 2, \ldots, T \]

\[ \ln A(t) = g + ht \]

\[ \alpha'_1(t) = a_1 + b_1 t \]

\[ \alpha'_2(t) = a_2 + b_2 t \]

\[ \alpha(t) = a + bt \]

\[ \beta(t) = c + dt. \]

Substituting the above specifications in the objective functions and the constraints, we obtain,

\[ \sum_{t=1}^{T} \sum_{i=1}^{n} \left[ g + ht + a_1 \left\{ \ln \mathcal{M}_{1,i}(t) \right\} + b_1 \left\{ t \ln \mathcal{M}_{1,i}(t) \right\} \right] + a_2 \left\{ \ln \mathcal{M}_{2,i}(t) \right\} + \]

\[ + b_2 \left\{ t \ln \mathcal{M}_{2,i}(t) \right\} - a \left\{ \ln x_i(t) \right\} - b \left\{ t \ln x_i(t) \right\} \]

\[ - e x_i(t) - d \left\{ t \ x_i(t) \right\} \]

\[ g + ht + a_1 \left\{ \ln \mathcal{M}_{1,i}(t) \right\} + b_1 \left\{ t \ln \mathcal{M}_{1,i}(t) \right\} + a_2 \left\{ \ln \mathcal{M}_{2,i}(t) \right\} \]
\[ +b_2 \left\{ t \ln v_{2i}(t) \right\} - a \left\{ \ln x_i(t) \right\} - b \left\{ t \ln x_i(t) \right\} \]
\[ - c \left\{ x_i(t) \right\} - d \left\{ t \ln x_i(t) \right\} \geq 0 \]

\[ a_1 + a_2 + b_1 t + b_2 t = 1 \]

We now solve the following LP:

\[
\text{Min } \Pi = h t + a_1 \ln v_1 + b_1 t \ln v_1 + a_2 \ln v_2 + b_2 t \ln v_1 - a \ln x - b t \ln x - cx - d tx
\]

(2.1.18)

subject to

\[ g + h t + a_1 \left\{ \ln v_{1i}(t) \right\} + b_1 \left\{ t \ln v_{1i}(t) \right\} + a_2 \left\{ \ln v_{2i}(t) \right\} \]
\[ + b_2 \left\{ t \ln v_{2i}(t) \right\} - a \left\{ \ln x_i(t) \right\} - b \left\{ t \ln x_i(t) \right\} \]
\[ - c \left\{ x_i(t) \right\} - d \left\{ t x_i(t) \right\} \geq 0 \]

\[ a_1, a_2, b_1, b_2, c, d \geq 0 \]

\[ g \text{ and } h \text{ unrestricted for sign} \]
EFFICIENCY MEASURES:

![Graph showing efficiency measures](image)

Figure 2.1.1

Let \((v_o, y_o)\) be an observed point. It belongs to an inefficient production unit. If the unit be efficient it could have used \(\mu v_o\) units of inputs to produce \(y_o\) units of output, where \(0 \leq \mu < 1\). \(\mu\) is 'INPUT REDUCING TECHNICAL EFFICIENCY MEASURE', which we shall indicate by \(TE\). Thus,

\[ TE_1 = \mu \]
\[ y_o^\alpha e^{\beta y_o} = A \left( \mu_{10}^{d_1} \right)^{\alpha_1} \left( \mu v_{20} \right)^{\alpha_2} \]
\[ = \mu^{\alpha_1 + \alpha_2} A^{d_1} v_{10}^{d_1} v_{20}^{d_2} \]
\[ \Rightarrow \mu = \frac{y_o^{\alpha} e^{\beta y_o}}{A^{d_1} v_{10}^{d_1} v_{20}^{d_2}} \]

\[ \text{TE}_1 = \frac{y_o^{\alpha} e^{\beta y_o}}{y_o^{\alpha} e^{\beta y_o}} = \mu, \text{ since } \alpha_1 + \alpha_2 = 1 \]

\[ \text{TE}_2 = \frac{y_o^{\alpha} e^{\beta y_o}}{y_o^{\alpha} e^{\beta y_o}} = \frac{f(\mu v_o)}{f(v_o)} \]

The 'OUTPUT INCREASING TECHNICAL EFFICIENCY MEASURE' is defined by,

\[ \text{TE}_2 = \frac{y_o^{\alpha} e^{\beta y_o}}{y_o^{\alpha} e^{\beta y_o}} = \frac{f(\mu v_o)}{f(v_o)} \]

where \( y_o \) is the solution of

\[ y_o^{\alpha} e^{\beta y_o} = A \left( \mu_{10}^{d_1} \right)^{\alpha_1} \left( \mu v_{20} \right)^{\alpha_2} \]

The above two measures, in general, do not coincide with each other, unless the production function is homogeneous of degree one.

Let \( \varepsilon \) denote the scale elasticity. It can be shown that \( f(\mu v_o) = \mu^\varepsilon f(v_o) \).

\[ \Rightarrow \varepsilon = \ln \left( \frac{f(\mu v_o)}{f(v_o)} \right) / \ln \mu \]
\[ \zeta = \frac{\ln TE_2}{\ln TE_1} \]

or

\[ \ln TE_2 = \zeta \ln TE_1 \]

- (2.1.21)

Thus \( TE_1 \geq TE_2 \) whenever \( \zeta \geq 1 \).

In empirical studies the choice between the measures has to be determined by objective. If the amount of resources is fairly constant, then \( TE_2 \) is the relevant measure. On the other hand, if the framing of the problem is such that output is assumed to be constant then \( TE_1 \) is to be choosen for efficiency measurement.

**SCALE EFFICIENCY**

Whenever the returns to scale are constant then the production unit is said to achieve OPTIMAL SCALE. There exists three measures of SCALE EFFICIENCY.

**SCALE EFFICIENCY RELEVANT TO INPUT REDUCTION**:

Scale efficiency measures are often based on unit input requirements, there by the production function for a micro unit is transformed from the factor space into a space of input co-efficients.
Figure (2.1.2)

\[
\begin{align*}
\text{COD} &= \frac{y_o}{v_o} \\
\text{AOE} &= \sqrt[3]{\frac{y_o}{v_o}} \\
\end{align*}
\]

Scale efficiency measure relative to input reduction is given by:

\[
\text{SE}_1 = \frac{\text{COD}}{\text{AOE}} = \frac{y_0/v_0}{y_0/v_o} = \mu_3
\]

This measure is the relative reduction in input co-efficients made possible by producing at OPTIMAL SCALE on the frontier production function with the observed factor proportions. Thus, \( \text{SE}_1 \) is NOT A PURE MEASURE OF SCALE EFFICIENCY.

Derivations of constant returns to scale frontier:

We have the frontier,

\[
q_1 e^{\beta y} = \frac{q_1}{v_1} + \frac{q_2}{v_2}
\]
Returns to scale

$$\xi = \frac{1}{\alpha + \beta y}$$  \hspace{1cm} (2.1.22)

If returns to scale are constant, then $$\xi = 1$$

$$\Rightarrow \quad y = \frac{1 - \alpha'}{\beta}$$

We rewrite the frontier as,

$$y^{\alpha'-1} e^{\beta y} = A \left( \frac{v_1}{-\gamma} \right)^{\alpha_1} \left( \frac{v_2}{-\gamma} \right)^{\alpha_2} = A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

Substitute $$y$$ for $$y$$ in the above expression to obtain

$$\left( \frac{1 - \alpha'}{\beta} \right)^{\alpha'-1} e^{1 - \alpha'} = A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$\Rightarrow \left( \frac{\alpha - \xi}{1 - \alpha} \right)^{1 - \alpha} = A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$  \hspace{1cm} (2.1.23)

which is nothing but the CRTS frontier. For a production unit whose inputs and output are given by, $$(v_{10}, v_{20}, v_0)$$, scale efficient input is $$M_3 v_0$$, where $$v_0 = (v_{10}, v_{20})$$, Substituting this input vector in CRTS frontier, we obtain

$$\left( \frac{\beta \xi}{1 - \alpha} \right)^{1 - \alpha} = A \begin{pmatrix} v_{10} \\ v_{20} \end{pmatrix}^{\alpha_1} M_3^{\alpha_2}$$

Upon simplifications,

$$3 = \left( \frac{\beta \xi}{1 - \alpha} \right)^{1 - \alpha} / A \begin{pmatrix} v_{10} \\ v_{20} \end{pmatrix}^{\alpha_1} M_3^{\alpha_2} = SE_1$$  \hspace{1cm} (2.1.24)
Pareto Scale Efficiency Relevant to Input Reduction:

![Diagram with labels](image)

Figure (2.1.3)

\[
\frac{SE_2}{\mu_3} \frac{v_o}{\nu_o} = \frac{\mu_3}{\mu}
\]

\[
SE_2 = \frac{\mu_3}{\mu_1}
\]

For the production unit whose inputs and output are given by \((v_{10}, v_{20}, y_o)\), CRS frontier, and VRTS frontiers imply respectively.

\[
\left(\frac{\beta \epsilon}{1-\alpha}\right) = A \left(\mu_3 v_{10}\right)^{\alpha_1} \left(\mu_3 v_{20}\right)^{\alpha_2}
\]

\[
y_o = A \left(\mu_1 v_{10}\right)^{\alpha_1} \left(\mu_1 v_{20}\right)^{\alpha_2}
\]

Dividing the former equation by \(\alpha - 1\) we obtain,

\[
SE_2 = \frac{\mu_3}{\mu_1} = \left(\frac{v_o - \epsilon}{1-\alpha}\right)^{1-\alpha} \exp(-\beta y_o)
\]  

(2.1.25)

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SCALE EFFICIENCY RELEVANT TO INPUT REDUCTION:

\[ y \]

Figure (2.1.4)

Had the unit operated along CRTS frontier, to produce \( y_0 \) units of output it needed to use \( \mu_3 v_0 \) units of input. This unit becomes scale efficient if it used inputs at \( \mu_2 v_0 \) level.

\[
SE_3 = \frac{\mu_3 v_0}{\mu_2 v_0} = \frac{\mu_3}{\mu_2} - (2.1.26)
\]

PURE SCALE EFFICIENCY RELEVANT TO OUTPUT AUGMENTATION:

\[ y \]

Figure (2.1.5)

\[
SE = \frac{\frac{\nu_1 y_o}{\nu_2 y_o}}{\frac{\nu_1}{\nu_2}} = \frac{\nu_1}{\nu_2} - (2.1.26)
\]
The CRTS and VRTS frontiers respectively give,

\[
\begin{align*}
\begin{pmatrix}
\beta e^\alpha \\
1-\alpha
\end{pmatrix} & = A \begin{pmatrix}
\frac{V_{10}}{V_2} \\
\frac{V_{20}}{V_2}
\end{pmatrix} \\
\gamma_0 & = A \begin{pmatrix}
\frac{V_{10}}{V_1} \\
\frac{V_{20}}{V_2}
\end{pmatrix}^{\alpha_1}
\end{align*}
\]

Dividing the former equation with the latter,

\[
\frac{\beta}{1-\alpha} \frac{e^\alpha}{\gamma_0} = \frac{V_1}{V_2}
\]

\[
\Rightarrow SE_2 = SE_5 = e^{-\beta \gamma_0} - \frac{V_1}{V_2}^{1-\alpha}
\]

Thus, \( SE_2 = SE_5 \) \hspace{1cm} (2.1.28)

It can be easily shown that,

\[
\varepsilon = \frac{\ln SE_3 - \ln SE_5}{\ln SE_3 - \ln SE_4}
\]

\[
\text{STRUCTURAL EFFICIENCY - ITS MEASUREMENT}
\]

According to Farrell a measure of technical efficiency of the whole industry is a measure of STRUCTURAL EFFICIENCY, which an average of technical efficiencies of the micro production units.
According to Farrell the purpose of structural efficiency is to measure the extent to which an industry keeps up with the performance of its own best firms. The averaging process suggested by Farrell is to weigh the technical efficiencies of micro units with their respective observed outputs.

A second approach to find structural efficiency is to compute average output and inputs and regard these as output and inputs produced by some other unit and to compute efficiencies as described above.

**THE TRANSLOG PRODUCTION FUNCTION:**

It is a second order approximation of an arbitrary production function, which we may write as,

\[ \ln Q = \beta_o + \sum_i \beta_i \ln x_i + 1/2 \sum_i \sum_j \beta_{ij} \ln x_i \ln x_j \]  

(2.1.30)

Under certain conditions its dual cost function is given by,

\[ \ln C = \alpha'_o + \alpha'_Q \ln Q + 1/2 \alpha'_{QQ} (\ln Q)^2 + \sum_i \gamma_i \ln P_i \ln Q \]

\[ = \sum_i \alpha'_i \ln P_i + 1/2 \sum_i \sum_j \alpha'_{ij} \ln P_i \ln P_j \]  

(2.1.31)

Atkinson and Halvorsen* propose some parametric tests for the presence (or absence) of allocative efficiency. A

generalised cost function is postulated and minimized subject to (i) constant output and (ii) constraints imposed by its regulatory environment. The Lagrangian function which is minimized with respect to inputs is,

\[ L = \sum_{h} p_h x_h - \Phi [ f(x) - Q ] + \sum_{i} \lambda_i R_i (P, X) \quad (2.1.32) \]

\[ h = 1, 2, \ldots, n \]

\[ i = 1, 2, \ldots, m. \]

First order conditions for cost minimization problem for inputs \( j \) and \( k \) may be written as,

\[ \frac{f_i}{f_k} = \frac{p_j}{p_k} + \frac{\sum_{i} \lambda_i \frac{\partial R_i}{\partial x_i}}{\frac{\partial R_i}{\partial x_k}} \]

(2.1.33)

\[ j, k = 1, 2, \ldots, n \quad (j \neq k). \]

We write the above conditions as,

\[ f_j = \frac{p_j^*}{p_k^*} \quad, \quad j, k = 1, 2, \ldots, n \quad (j \neq k) \quad (2.1.34) \]

We call \( p_j^* \), \( p_k^* \) as SHADOW PRICES.
We shall assume, $P_j^* = K_j P_j$, where $K_j > 0$ is a constant and $P_j$ actual price. Thus, shadow price of $j$th input is assumed to be directly proportional to actual price of $j$th input.*

* Let $P_j^* = g_j(P_j)$

By Taylor series expansion, around $P_j = 0$, we have

$$P_j^* = g_j(0) + P_j g_j'(0) + \ldots.$$  

Assuming $g_j(0) = 0$, $g_j'(0) > 0$ we have,

$$P_j^* = K_j P_j,$$  

where $K_j = g_j'(0)$.
SHADOW COST FUNCTION:

\[ \ln c^S = \alpha_0 + \frac{1}{\gamma_{QQ}} \ln Q + \frac{1}{2} \gamma_{QQ} (\ln Q)^2 + \sum_i \gamma_{iQ} \ln Q \ln P_i^* + \sum_i \gamma_{ij} \ln P_i^* \ln P_j^* \]

\[ \ln c^S = \alpha_0 + \frac{1}{\gamma_{QQ}} \ln Q + \frac{1}{2} \gamma_{QQ} (\ln Q)^2 + \sum_i \gamma_{iQ} \ln Q \ln (k_i p_i) + \sum_i \gamma_{ij} \ln (k_i p_i) \ln (k_j p_j) \]

(2.1.35)

ACTUAL COST FUNCTION

\[ c^A = \sum_i p_i x_i \]
\[ c^S = c^S(k, p, q) \]

By Shephard's Lemma, we have,

\[ \frac{\partial c^S}{\partial k_i p_i} = x_i \]

Substituting this result in actual cost function.

\[ C^A = \sum_{i} p_i \frac{\partial c^S}{\partial k_i p_i} \]

Thus, by specifying an appropriate functional form for shadow costs, one can derive a parametric expression for the firm's total actual costs. Shadow cost share of input i is,

\[ M_i^S = \frac{p_i^* x_i}{c^S} = \frac{k_i p_i x_i}{c^S} \]

\[ \Rightarrow x_i = c^S m_i^S (k_i p_i)^{-1} \quad \ldots \quad (2.1.37) \]

Substituting this in actual cost functions,

\[ c^A = \sum_i p_i m_i^S c^S (k_i p_i)^{-1} \]

\[ = c^S \sum_i k_i^{-1} m_i^S \]

Taking logs on both sides, we write

\[ \ln c^A = \ln c^S + \ln \sum_k k_i^{-1} m_i^S \]

Thus, we write actual cost function as follows:

\[ \ln c^A = \alpha_0 + \alpha_Q \ln Q + 1/2 \gamma_{QQ} (\ln Q)^2 + \sum_{i}^{\psi_{ij}} \ln (k_i p_i) \]

\[ + \sum_{i}^{\psi_{ij}} \ln (k_i p_i) \ln (k_i p_i) + \gamma_{ij} \ln (k_i p_i) \ln (k_j p_j) \]

\[ + \ln \left( \sum_k k_i^{-1} \right) \ln k_i^{-1} p_i + \gamma_{ij} \ln (k_i p_i) + \ln Q \]

\[ \ldots \quad (2.1.38) \]
RELATIVE PRICE EFFICIENCY (ALLOCATIVE EFFICIENCY):

If the marginal rate of technical substitution can be equated with appropriate price ratios, then relative prices efficiency exists.

\[
\frac{f_i}{p_i} = \frac{p^*_i}{p^*_j} = \frac{K_i}{K_j} \frac{p_i}{p_j}
\]

\[
K_i = K_j \Rightarrow \frac{f_i}{p_i} = \frac{f_j}{p_j} \quad \ldots \quad (2.1.39)
\]

⇒ relative price efficiency.

Relative price efficiency with respect to all pairs of inputs implies that output is produced at minimum cost.

ABSOLUTE PRICE EFFICIENCY:

Absolute price efficiency exists if the value of marginal product for each input is equated to the input's market price.

\[
f_i = K_i p_i \quad i = 1, 2, \ldots n \quad \ldots \quad (2.1.40)
\]

\[
K_i = 1, \quad i = 1, 2, \ldots n \quad \text{implies absolute price efficiency.}
\]
Thus, a test of the null hypothesis,

\[ H_0: K_i = K_j, \quad j = 1, 2, \ldots, n \quad (i \neq j) \]

reveals the presence or absence of ALLOCATIVE EFFICIENCY. The null hypothesis of cost minimization is tested statistically by estimating the systems of equations with and without the appropriate equality restrictions imposed. The results are compared. The test statistic is, \(-2 \log_\lambda\), where \(\lambda\) is the ratio of the maximum value of the likelihood function for the restricted equations to the maximum value of the likelihood function for the unrestricted equations. The test statistic follows \(\chi^2\) distribution with appropriate degrees of freedom.

2.2 NON-PARAMETRIC APPROACH–DATA ENVELOPMENT ANALYSIS (DEA)

DATA ENVELOPMENT ANALYSIS* is an approach to compute and compare productive efficiency of units of production such as firms in an industry, departmental stores, hospitals, schools and so on. In a production process where one output is produced employing only one input, efficiency is defined as the ratio of output to input. If multiple outputs are produced combining multiple inputs TECHNICAL EFFICIENCY may be defined as the ratio of weighted

------------------

sum of outputs to weighted sum of inputs. These weights are determined by a TYPICAL PRODUCTION UNIT by comparing its performance with other production units. To measure technical efficiency of a target unit \( k_o \) we solve the following FRACTIONAL LINEAR PROGRAMMING PROBLEM:

\[
\begin{align*}
\text{Max } & \quad h_o = \sum_{r \in R} u_r \cdot \frac{y_{rj}}{x_{ij}} \cdot \frac{1}{r k_o} \\
\text{Subject to } & \quad \sum_{i \in I} \phi_i \cdot x_{ij} \geq 1, \quad j = 1, 2, \ldots, n \\
& \quad u_r, \phi_i > 0
\end{align*}
\]

where

- \( y_{rj} \) = rth output from jth unit
- \( x_{ij} \) = ith input of jth unit
- \( u_r \) = weight assigned to rth output
- \( \phi_j \) = weight assigned to jth input
- \( n \) = number of units of production

In this fractional programming problem the efficiency of unit \( k_o \) is maximized subject to the efficiencies of all units including unit \( k_o \), which are bound above by UNITY.
Problem (2.2.1) can be converted into a linear programming problem as follows:

Max $h_o = \sum_r u_r y_{rk_o}$

Subject to

$$\sum_i u_i x_{ik_o} = 1 \quad (2.2.2)$$

$$\sum_r u_r y_{rj} - \sum_i \lambda_i^r x_{ij} \leq 0$$

$$-u_r \leq -\epsilon, \quad r = 1, 2, \ldots, s$$

$$-\lambda_i^r \leq -\epsilon, \quad i = 1, 2, \ldots, m$$

where $\epsilon$ is a small positive quantity. Problems (2.2.1) and (2.2.2) are precisely one and the same.

Problem (2.2.2) has a DUAL which we may write as,

$$\min_{(0, \lambda)} 0$$

subject to

$$\sum \lambda_j y_{rj} \geq y_{rk}, \quad r = 1, 2, \ldots, s \quad (2.2.3)$$

$$\sum \lambda_j x_{ij} \leq \theta x_{ik}, \quad i = 1, 2, \ldots, m$$

$$\lambda_j \geq 0, \quad j = 1, 2, \ldots, n$$

$\theta$ is a free parameter
Min $\theta$ in the Problem (2.2.3) is the INPUT ORIENTED TECHNICAL EFFICIENCY MEASURE that satisfies STRONG DISPOSABILITY OF INPUTS* constructed by SHEPHARD**.

In a production process the existing inputs and outputs combination may not reflect constant returns to scale.*** In fact, returns to scale could be increasing, constant or decreasing. Variable returns to scale can be modelled as follows:

$$\max \sum_{r} U_{r} y_{rk0} + u_{*}$$

Subject to

$$\sum_{r} U_{r} y_{rj} - \sum_{c} \nu_{i} x_{ij} \leq 0, j = 1, 2, \ldots, n$$ (2.2.4)

$$\sum_{c} \nu_{i} x_{ik0} = 1$$

$$u_{r} \geq \epsilon > 0$$

$$\nu_{i} \geq \epsilon > 0$$

$U_{*}$ unrestricted for sign.

$U_{*} < 0 \iff$ returns to scale are increasing.

*A production technology where it is not possible to increase output by reducing the inputs is said to satisfy STRONG DISPOSABILITY OF INPUTS.


$U_x > 0 \iff$ returns to scale are constant

The DUAL of the problem (2.2.4) is given as,

$$\text{Min } \begin{array}{c} h_k \\ _o \end{array}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} \lambda_j \leq h_k \lambda_{ik}$$

$$\sum_{j=1}^{n} y_{rj} \lambda_j \geq y_{rk}$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

OUTPUT ORIENTED TECHNICAL EFFICIENCY can be calculated by solving,

$$\text{Max } \phi_k \begin{array}{c} \_o \end{array}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \lambda_j \leq x_{ik}$$

(2.2.6)

$$\sum_{j=1}^{n} y_{rj} \lambda_j \geq \phi_k y_{rk}$$

The technology implied by (2.2.6) satisfies STRONG DISPOSABILITY OF OUTPUTS. More over, $\text{Max } \phi_k$ is output oriented technical efficiency measure. $\text{Max } \phi_k \gg 1$.

$$\text{Min } h_k = [\text{Max } \phi_k]^{-1} \text{ if and only if returns to scale are CONSTANT.}$$

Max $\phi_k$ is the WEAK OUTPUT MEASURE OF TECHNICAL EFFICIENCY derived by ROLF FARE, which is consistent with strong disposability of OUTPUTS.
Variable returns to scale are modelled into output orientation problems by specifying and solving,

\[
\begin{align*}
\text{Max } & \phi_{k_0} \\
\text{subject to } & \\
\sum_j x_{ij} \lambda_j & \leq x_{ik_0} \\
\sum_j y_{rj} \lambda_j & \geq \phi_{k_0} y_{rk_0} \\
\sum_j \lambda_j & = 1 \\
\lambda_j & \geq 0
\end{align*}
\] (2.2.7)

NON-INCREASING RETURNS TO SCALE are introduced into both INPUT and OUTPUT orientation linear programming problems by replacing the constraint $\sum_j \lambda_j = 1$ by $\sum_j \lambda_j \leq 1$. Thus, one solves,

\[
\begin{align*}
\text{Min } & h_{k_0} \\
\text{subject to } & \\
\sum_j x_{ij} \lambda_j & \leq h_{k_0} x_{ik_0} \\
\sum_j y_{rj} \lambda_j & \geq y_{rk_0} \\
\sum_j \lambda_j & \leq 1 \\
\lambda_j & \geq 0
\end{align*}
\] (2.2.8)
Max $\phi_{k_0}$
subject to
\[
\begin{align*}
\sum_{j} x_{ij} \lambda_j & \leq x_{i k_0} \\
\sum_{j} y_{rj} \lambda_j & \geq y_{r k_0} \\
\lambda_j & \leq 1 \\
\lambda_j & \geq 0
\end{align*}
\] (2.2.9)

Let the optimal values of the objective functions of the problems (2.2.6) and (2.2.7) be,

$\phi_{k_0}$ (\*) and $\phi_{k_0}$ (**) 

SCALE EFFICIENCY is defined as,

\[
SE (k_0) = \frac{\phi_{k_0} (**)}{\phi_{k_0} (*)} \quad \cdots (2.2.10)
\]

2.3 The basic tool for efficiency measurement is
PRODUCTION FUNCTION. A production function is a relationship
that explains how inputs are combined to obtain MAXIMUM
POSSIBLE OUTPUT. This definition assumes that the firm(s)
under consideration is (are) TECHNICALLY EFFICIENT. However,
in an industry, for example, all the firms
may not be technically efficient because the firms may vary
in the adoption of latest technology. These firms may vary
with respect to managerial efficiency also. Thus, there is
need to model inefficiency into production and hence into
the production function. A production process in which two
inputs are combined to produce a single output is often explained through ISOQUANTS.

Figure (2.3.1) represents isoquant map. These curves satisfy the following:

(i) They are smoothly continuous, hence differentiable at every point of the definition.

(ii) They are convex to be origin,

(iii) no two isoquants intersect each other,

(iv) farther an isoquant from the origin greater the output it represents.

The production units whose inputs fall on an isoquant are technically efficient. But, there is need to introduce inefficiency in an isoquant.

The locus of all input combinations which yield the same level of output is called an ISOQUANT.
A production process, for example, where two outputs are produced using a single input, is often explained by means of PRODUCT TRANSFORMATION CURVE.

![Product Transformation Curve](image)

Figure (2.3.2)

A product transformation curve is defined as the locus of all output combinations which can be produced by the same input level. (i) The product transformation curves are smoothly continuous so that they are differentiable at every point of definition, (ii) Farther a product transformation curve from the origin greater the input it represents, (iii) no two product transformation curves intersect each other. The production units whose outputs fall on a product transformation curve are technically efficient. There is need to model inefficiency into a product transformation curve too.

A production function, its isoquants and product transformation curves are related with efficient production. In general, not all production units are technically the most efficient nor they are equally efficient. This warrants
the introduction of inefficiency into production function and its isoquants, product transformation curves where the production process combines multiple inputs to produce multiple outputs. An approach for this purpose is due to SHEPHARD and Färe*

**INPUT CORRESPONDENCE:**

Let \( x \in \mathbb{R}_+^n \) be a vector of inputs and \( U \in \mathbb{R}_+^m \) be a vector of outputs such that the vectors of inputs which can produce at least \( u \) be denoted by,

\[
L(u) = \left\{ x : x \text{ produces at least } u \right\},
\]

where \( L(u) \) is called an **INPUT CORRESPONDENCE**.

\( L(u) \) is assumed to satisfy the following AXIOMS:

1. \( u \geq 0 \Rightarrow 0 \notin L(u) \).  

Positive output cannot be produced by null vector.

2. Let \( \| u^t \| \rightarrow \infty \) as \( t \rightarrow \infty \), then, \( \bigcap_{t=1}^{\infty} L(u^t) = \emptyset \)

Infinite output is produced by infinite inputs.

3. \( x \in L(u) \Rightarrow \lambda x \in L(u) \), \( \lambda \geq 1 \)

Proportional increase in inputs do not decrease output.

4. \( L \) is a closed correspondence

5. \( L (\emptyset u) \subseteq L(u), \emptyset u \geq \lambda \)


** \( u \geq 0 \) means, \( u_i \geq 0 \) for at least one \( i \), \( u \geq 0 \) represents null vector also.
Proportional increase in outputs is not possible by reduced inputs.

AXIOM-3 is weak disposability of inputs while AXIOM-5 represents weak disposability of outputs.

6. \( y \succ x, \; x \in L(u) \Rightarrow y \in L(u) \)

This axiom consists axiom-3 as a particular case, therefore, it is called STRONG DISPOSABILITY OF INPUTS.

7. \( v \succ u \Rightarrow L(v) \subseteq L(u) \).

This axiom contains axiom-5 as a particular case and is referred to STRONG DISPOSABILITY OF OUTPUTS.

Figure (2.3.3)

\[ L(u) \]

Figure (2.3.4)

L (u) in figure (2.3.3) is consistent with strong disposability of inputs.
$L(u)$ in figure (2.3.4) is consistent with weak disposability of inputs.

The production technology as described by $L(u)$ in figure (2.3.3) satisfies axioms 1-5 while that in figure (2.3.4) satisfies axioms 1-6.

To measure efficiency it is necessary to isolate certain sub-sets of $L(u)$ and $P(x)$.

**INPUT ISOQUANT:**

$$\text{Isoq } L(u) = \{ x : x \in L(u), \lambda x \notin L(u), \lambda \in [0,1) \}$$

**WEAK EFFICIENT SUBSET* OF $L(u)$:**

$$\text{WEff } L(u) = \{ x : x \in L(u), \ y \prec x \Rightarrow y \notin L(u) \}$$

$$\text{WEff } L(o) = \{ o \}$$

$$(*) \ y \prec x \iff y_i \prec x_i \ for \ i = 1, 2, \ldots, n \$$

$$y^* \prec x \iff y_i = x_i \ for \ i = 1, 2, \ldots, n$$
EFFICIENT SUBSET OF $L(u)$:

$$\text{Eff } L(u) = \left\{ x : x \notin L(u), \ y \leq x \Rightarrow y \notin L(u) \right\}$$

$$\text{Eff } L(o) = \{ o \}$$

Figure (2.3.5)

The bold line segments alone form Eff $L(u)$, while dotted as well as bold lines define WEff $L(u)$. In this case WEff $L(u) = \text{Isoq } L(u)$.

Figure (2.3.5a)

Bold line segment form Eff $L(u)$, while dotted line and bold lines form Isoq $L(u)$.
Eff L(u) $\subseteq$ WEff L(u) $\subseteq$ Isoq L(u)

An example of a technology for which
Eff L(u) = WEff L(u) = Isoq L(u) is given by,

$$L(u) = \left\{ x : \prod_{i=1}^{n} x_i \alpha_i \succcurlyeq u, \ \alpha_i > 0 \right\}.$$  

**OUTPUT CORRESPONDENCE:**

For a given input vector x the collection of output vectors producible by x is defined as output correspondence. The output correspondence implied by the five axioms of input correspondence is,

$$P(x) = \{ u : \exists x \in L(u) \}$$

There is duality between P(x) and L(u).

$$x \in L(u) \Leftrightarrow u \in P(x)$$

**AXIOMS of output correspondence:**

1. **P(o) = \{ o \}**
   Null input vector yields null output vector.

2. For $x \in \mathbb{R}_{+}^n$, P(x) is bounded.
   Finite input cannot produce infinite output.

3. **P(x) $\subseteq$ P(\lambda x), \lambda \geq 1**
   Proportionate increase in inputs does not decrease output. Weak disposability of inputs.
4. \( P \) is a closed correspondence.

5. \( u \in P(x) \Rightarrow \Theta u \in P(x), \Theta \in [0, 1] \)
   Proportional decrease in output requires no additional input. Weak disposability of outputs.

6. \( y \geq x \Rightarrow P(x) \subseteq P(y) \), strong disposability of inputs.

7. \( u \in P(x) \Rightarrow v \in p(x), y \leq u \), strong disposability of outputs.

![Figure (2.3.6)](image)

The shaded region in the above figure represents an output correspondence for a given input vector \( x \).

---

**SUBSETS OF OUTPUT CORRESPONDENCE** :

Output Isoquant:

\[
\text{Isoq } P(x) = \{ u : u \in p(x), \Theta x \notin P(x), \Theta > 1 \}
\]

\[
P(x) = \{ o \} \Rightarrow \text{Isoq } P(x) = \{ o \}
\]

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Weak Efficient Subset of $p(x)$:

$\text{Weff } P(x) = \left\{ u : u \in P(x), \quad \forall u \Rightarrow \forall \tilde{u} \in P(x) \right\}$

$P(x) = \left\{ o \right\} \Rightarrow \text{Weff } P(x) = \left\{ o \right\}$

Efficient sub-set of $P(x)$:

$\text{Eff } P(x) = \left\{ u : u \in P(x), \quad \forall u \Rightarrow \forall \tilde{u} \in P(x) \right\}$

Eff $P(x) \subseteq \text{Weff } P(x) \subseteq \text{isoq } P(x)$

An example of a technology for which

$\text{Eff } P(x) = \text{Weff } P(x) = \text{Isoq } P(x)$ is

$P(x) = \left\{ u : \left\{ \sum_{i=1}^{m} s_i u_i - p_i \right\} \leq x \right\}$

GRAPH:

The GRAPH of a production technology is defined as,

$GR = \left\{ (x,u) : x \in L(u), \quad u \in R^m_+ \right\}$

$= \left\{ (x,u) : u \in P(x), \quad x \in R^n_+ \right\}$

AXIOMS:

1. $0 \in GR, \ (0,u) \in GR \Rightarrow u = 0$

   Zero - Input produces zero output.

2. $GR \cap \left\{ (x,u) : x \leq \bar{x} \right\}$ is bounded for each $\bar{x} \in R_+^n$

3. $(x,u) \in GR \Rightarrow (\lambda x, \lambda u) \in GR, \ \lambda \geq 1$
4. GR is a closed set.

5. 

\[(x,u) \in \text{GR} \Rightarrow (x, \theta u) \in \text{GR}, \quad 0 \leq \theta \leq 1\]

\[\text{Figure (2.3.7)}\]

The shaded region bound by x-axis and the line segments represent the GRAPH of the production technology.

Axioms - 3 and 5 refer to weak disposability of inputs and outputs respectively.

6. 

\[(x,u) \in \text{GR} \Rightarrow (y,v) \in \text{GR} \text{ whenever } x \leq y, \ u \geq v.\]

This is the axiom of Strong Disposability of inputs and outputs.

**SUB-SETS OF GRAPH:**

**Graph Isoquant:**

\[\text{Isoq } \text{GR} = \left\{ (x,u) : (x,u) \in \text{GR}, (\lambda x, \omega^{-1} u) \in \text{GR} \text{ for } 0 \leq \lambda < 1 \right\}\]
Weak Efficient Subset of GR:

\[ \text{WEff } \text{GR} = \{ (x,u) : (x,u) \in \text{GR}, \quad y \preceq x, \quad u \preceq v, \quad (y,v) \notin \text{GR} \} \]

Efficient Subset of GR:

\[ \text{Eff } \text{GR} = \{ (x,u) : (x,u) \in \text{GR}, \quad y \preceq x, \quad v \succeq u \Rightarrow (y,v) \notin \text{GR} \}. \]

\[ \text{Eff } \text{GR} \subset \text{WEff } \text{GR} \subset \text{Isoq } \text{GR} \]

In terms of these concepts, Färe* defines a number of efficiency measures.

Input oriented Efficiency Measures:

In his path breaking article, Farrell* (1957) introduces various measures of efficiency such as technical and allocative efficiencies. For this purpose he uses a reference technology that is unit isoquant of a production function that is homogeneous of degree one, which admits only constant returns to scale. His unit isoquant reflects storage disposability of inputs.

According to Fare,** et al., an isoquant need not reflect strong disposability of inputs, it may admit only weak disposability of inputs as shown in Figure (2.3.4).


** Färe, Rolf, op.cit.,
Farell Input Technical Efficiency

Farell input oriented efficiency measure, $F_i(u,x)$ is obtained by minimizing $\lambda$ such that $\lambda x \in L(u)$. Min $\lambda = F_i(u,x)$ measures the possible input reduction in producing a given level of output, when the reference technology is provided by production isoquant. Therefore, $F_i(u,x) \times \in Isoq L(u)$.

Technical Weak Efficiency Measures:

The technical weak efficiency measure is found by minimizing $\lambda$ such that $\lambda x \in L^5(u)$. $w_i(u,x) = \min \lambda$ measures the possible input reduction to achieve the output level $u$, where the reference technology is assumed to satisfy STRONG DISPOSABILITY OF INPUTS. Therefore, $w_i(u,x) \times \in Weff L(u)$.

**OVERALL INPUT EFFICIENCY MEASURE**:

The overall input efficiency measure compares the input vector $X$ that produces $u$ with an input vector $X \in CM(u,p)$ where CM $(u,p)$ is the collection of input vectors which produce $u$ which are not only technically efficient but also cost efficient. The reference set for computing overall input efficiency measure is CM $(u,p)$. The overall input efficiency measure is thus defined as,

$$Q_i(u,p) = \frac{Q(u,p)}{p^T X}$$
where $Q(u,p) = \min \lambda$

such that

$Q(u,p) p^T x = p^T x^*$

where $x^* \in CM(u,p)$.

**THE INPUT CONGESTION MEASURE:**

In a production process, we say, input congestion takes place, if it is possible to increase output by decreasing one or more inputs. Whenever input congestion takes place at a point on production frontier, then at that point marginal products of some inputs are negative. Input congestion occurs when production technology is not consistent with strong disposability of inputs.

From the definitions of $F_i(u,x)$ and $W_i(u,x)$ it follows that,

$$F_i(u,x) \geq W_i(u,x).$$

Input congestion measure is defined as,

$$C_i(u,x) = \frac{W_i(u,x)}{F_i(u,x)}$$

To compute the denominator the reference set is Isoq $L(u)$, while to compute the numerator the reference set is Weff $L(u)$. Thus, $C_i(u,x) = 1$ if and only if Isoq $L(u) =$ Weff $L(u)$. 

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ALLOCATIVE INPUT EFFICIENCY MEASURE

A production unit whose input combination falls on Weff L(u) is technically efficient but may be allocatively not efficient, because there may be points on Weff L(u) at which total input cost is minimized. Input oriented allocative efficiency is defined as,

\[ A_i (x, u, p) = \frac{O_i (x, u, p)}{W_i (u, x)} \]

Decomposition of Overall Input Efficiency Measure:

It can be shown that,

(i) \[ O_i (u, x, p) = F_i (u, x) \cdot C_i (u, x) \cdot A_i (u, x, p) \]

(ii) \[ O'_i (u, x, p) = W_i (u, x) \cdot A_i (u, x, p) \]

OUTPUT ORIENTED EFFICIENCY MEASURES

FARRELL OUTPUT ORIENTED TECHNICAL EFFICIENCY:

For a given input vector the collection of all input vectors producible by x defines an output correspondence, p(x). Farrell output technical efficiency measures the possible output augmentation if the reference set is Isoq p(x).
Farrell output measure of technical efficiency is obtained by maximizing $\theta$ such that $\theta u \in \mathcal{P}(x)$.

$$\text{Max } \theta = F_\circ (x,u), \quad F_\circ (x,u) \geq 1.$$  

$F_\circ (u,x) \in \text{Isoq } \mathcal{P}(x)$. 

**WEAK OUTPUT MEASURE OF TECHNICAL EFFICIENCY:**

The weak output measure of technical efficiency can be obtained by maximizing $\theta$ such that $\theta u \in \mathcal{S}(x)$.

$$\text{Max } \theta = W_\circ (x,u) \quad \text{measures the possible output augmentation when input vector is given as } x. \quad \text{The reference set used for the purpose of computing } W_\circ (x,u) \quad \text{is } \text{Weff }\mathcal{P}(x) \quad \text{which is consistent with strong disposability of outputs.}$$

$$W_\circ (x,u) \geq 1.$$  

Thus, $W_\circ (x,u), \quad u \in \text{WEff }\mathcal{P}(x)$. 

Whenever, $\text{Isoq } \mathcal{P}(x)$ satisfies strong disposability of outputs, then

$$F_\circ (u,x) = W_\circ (u,x),$$

otherwise,

$$F_\circ (u,x) < W_\circ (u,x)$$

**OVERALL OUTPUT EFFICIENCY MEASURE:**

Let $\text{RM } (x,r)$ be the collection of output vectors
which are not only technically efficient but also revenue efficient, where \( r \in (R^m_+ - \{o\}) \). The overall output efficiency measure is obtained by maximizing \( \Theta \) such that,

\[
(\text{Max } \Theta ) \quad r^T u = r^T u_*, \text{ where }
\]

\[ u_* \not\in \text{RM } (x,r). \]

The overall output efficiency measure is defined as,

\[
C_o (x,u,r) = \frac{R(x,r)}{ru}, \quad ru > 0.
\]

where \( R(x,r) = r^T u_*, \) \( u_* \not\in \text{RM } (x,r) \)

**MEASURE OF OUTPUT CONGESTION:**

It is a derived measure defined as,

\[
C_o (x,u) = \frac{W_o (x,u)}{P_o (x,u)}
\]

The numerator is computed with \( \text{WEff } p(x) \) as reference set, while the denominator is realised when \( \text{Isoq } p(x) \) is considered to be the reference set. Thus, output loss due to lack of disposability is absent if and only if,

\[
\text{WEff } p(x) = \text{Isoq } p(x).
\]

**THE ALLOCATIVE OUTPUT EFFICIENCY MEASURE:**

An output vector that lies on \( \text{WEff } p(x) \) although
is technically the most efficient, may not be revenue efficient. Suppose $\text{RM}(x,r)$ is the collection of output vectors which belong to $\text{WEff } p(x)$ which are revenue efficient.

The allocative output efficiency measure is defined as,

$$A_o(x,u,r) = \frac{O_o(x,u,r)}{W_o(x,u)}$$

**DECOMPOSITION OF OVERALL OUTPUT EFFICIENCY MEASURE**:

It can be proved that,

(i) $O_o(x,u,r) = F_o(x,u) \cdot C_o(x,u) \cdot A_o(x,u,r)$

(ii) $O_o(x,u,r) = W_o(x,u) \cdot A_o(x,u,r)$.

**Relationship between input and output oriented efficiency measures**:

While the input oriented efficiency measures are bound between 0 and 1, the output oriented efficiency measures are greater than or equal to unity. However, the inverse of an output oriented efficiency measure equals the corresponding input oriented efficiency measure if and only if the input correspondence, equivalently the output correspondence is LINEAR HOMОGENEOUS.

$L(\lambda u) = \lambda L(u) \Leftrightarrow P(\theta x) = \theta P(x)$. 

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2.4 The parametric approach for efficiency measurement a priori assumes a mathematical form for the production frontier. Any deviation of the true frontier from the assumed frontier leads to SPECIFICATION ERROR in the technical efficiency estimates. Once the frontier production function is specified a priori, its estimation may be performed either by statistical methods or by mathematical programming techniques. In order to estimate productive efficiency by statistical methods either a one sided disturbance, for example $-\mu \leq \sigma$ following truncated normal distribution, or a composite disturbance term $v-u$, for example $v$ following normal distribution with zero mean and variance $\sigma^2$, is to be augmented with the frontier production function. Many of the statistical methods failed to specify unit-wise technical and/or allocated efficiencies. If the parameters of a frontier production function are estimated by the methods of mathematical programming, the estimates fail to possess any statistical properties.

In the non-parametric approach to measure productive efficiency the technology is assumed to be piecewise linear and such a technology is constructed by the input sets $L(u)$, the output sets $p(x)$, the graph $GR$ and the production function $\delta(x)$. There is duality among $L(u)$, $p(x)$ $GR$ and $\delta(x)$. Therefore, one determines the other uniquely.
For example, the classical production function $\bar{\Phi}(x)$ is defined as a maximization problem as follows:

$$\bar{\Phi}(x) = \text{Max } \left\{ u \mid x \in L(u) \right\}.$$  

$L^*_\Phi(u) = \{ x \mid \bar{\Phi}(x) \geq u \}$ are the input sets induced by the production function $\bar{\Phi}(x)$. It can be shown that $L^*_\Phi(u) = L(u)$.

Since the production technology is assumed to be piecewise linear, the various productive efficiencies can be expressed as mathematical programming problems as such the estimates obtained as solutions of these problems possess no statistical properties. This is a limitation of the present study.

However, the proposed procedure to compute productive efficiencies is free from specification error and it gives unit-wise productive efficiencies of various types.

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