CHAPTER-I

INTRODUCTION

1.1 HISTORICAL BACKGROUND

In 1938, Fichtenholz [39] introduced the concept of two-norm spaces in some concrete Banach spaces, a kind of convergence weaker than that generated by the given norm. These results lead Alexiewicz [10] to introduce the notion of mixed convergence of sequences in a vector space $X$ equipped with two norms. Alexiewicz then developed the theory of such spaces in collaboration with Semadeni [9] which was later on found useful in the theory of sequences and the theory of Schauder Decomposition in Banach spaces.

It is natural to ask whether the concepts of mixed convergence of sequences in a linear space $X$ equipped with two norms can be extended to a linear space $X$ with two topologies, metrizable or non-metrizable? In 1954, A. Alexiewicz proved, if mixed convergence is metrizable then it is nothing, in this case the norm convergence induced by the first norm. Therefore, the problem waiting for solution was to find out a non-metrizable locally convex topology on $X$, for which the sequential convergence coincides with the mixed convergence. In 1957, Wiweger [152] first attempted in this direction. Wiweger extended the theory of two norm spaces in the most general setting of linear topological spaces. In his later publication Wiweger [153] dealt with analogue of Hahn-Banach theorem for $\gamma$-continuous linear functional under suitable restrictions, precise representation of semi-norms generating the mixed topology and the analogue of the Smulian and Eberlin theorem.

Wiweger’s mixed topology along with bitopological vector space was carried forward by some mathematician in various directions. Among them most significant are Cooper [29], Garling [47], Persson [115], Arima and Orihara [13] etc. Arima and Orihara [13] gave the details general method of forming neighbourhood systems of generalized mixed topology. Strict topology on spaces of continuous functions and
Mackey topology on vector spaces by Cooper [31]. Garling [47] introduced the mixed topology as the generalized types of inductive limit. Many applications of mixed topology were studied by Persson [115], Garling [47], Cooper [30, 31]. The summability theory, Schauder basis and Schauder decomposition are yet some area in which the study of two-normed spaces and mixed topologies has significant application. In this direction, significant contribution by Orlicz [110] on the theory of linear method can be referred. Chilana [28] on exhibiting some interesting properties of the space equipped with Wiweger’s mixed topology and Subramanian and Rothman [131] on the study of Schauder decompositions.

From above discussion, it is now clear that the trend of research as initiated by Polish mathematician A. Alexiewicz, Z. Semadeni, A. Wiweger etc. and pursued by their successors from different parts of the world were confines in the study of bitopological structures in the setting of vector space only. Therefore, in spite of many valuable contributions to functional analysis, a notable feature missing in this trend was embodiment of the bitopological structure in the most general form in relation to separation properties, connectedness, compactness and other vital concepts of general topological spaces.

Kelly [65] initiated an attempt to this effect. Unlike Wiweger's bitopology derived from two-norm on a vector space, Kelly derived his bitopological structure from two quasi-metrics on a set. His work also dealt with separation axioms for bitopological spaces in the same spirit. He introduced the terms pairwise regularity, pairwise Hausdroff and pairwise normality. In 1967, Patty [113] published one paper under the same title ‘Bitopological Spaces’ as Kelly [65]. In his paper, the topologies $P$ and $Q$ for $X$ are obtained from a quasi-pseudo metric and its symmetric associate. The difficulties of extending metrization theorem to these examined with various examples.

In 1967, Pervin [116] studied another important topological concept like connectedness of a bitopological spaces in the paper “Connectedness in Bitopological spaces”. In complete analogue with these kinds of things done by Kelly [65], Petty [113] and Pervin [116]. One would naturally be tempted to examine all the structures of the general topological spaces in the context of general bitopological spaces. However, as one can see the importance of $\gamma$-convergence in the two-norm spaces or mixed topology of Wiweger’s [153] in the bitopological linear spaces. Similarly, for an effective investigation of general bitopological spaces through the application of
general topological spaces, the creation of a mixed topology or something of that sort seems inevitable. In fact, this has partially fulfilled by the introduction of a mixed topology for a bitopological space jointly by Ganguli and Sinha [45] in 1984. They have added some more separation axioms for bitopological spaces. The construction of mixed topology for bitopological spaces is given in [45]. The results of this paper mainly characterize the topological properties like separation axioms, denseness, connectedness and quasi-metrizability of the bitopological space in relation to the mixed topology defined on it and vice-versa. In addition, the impact of mixed topology on the investigation of bitopological space has already been established.

Making use of topology $\tau_2$-constructed from a given topology $\tau_1$ by Robertson and Robertson [124] in study of Closed Graph theorem for a class of locally convex spaces. Hussain [56] proved another Closed Graph theorem in which a kind of mixing of the topologies $\tau_1$ and $\tau_2$ in the manner as derived in the Proposition 1.1.12. [56] to obtain the mixed topology $\tau_1(\tau_2)$ was used. Of course, such strategy of using mixed topology $\tau_1(\tau_2)$ became successful for the class of locally convex linear spaces, known as $B(C)$ linear space, a notion introduced by Hussain [56]. It is found that the Closed Graph theorem proved for $B(C)$ linear space does not need extra algebraic structures provided by the scalar multiplication. Therefore both the notion of $B(C)$ linear space and Closed Graph theorem proved for these spaces could be carried over to the context of topological group exhibiting the same role of the mixed topology $\tau_1(\tau_2)$ for $B(C)$ linear space. Using the concept of bitopological spaces, Ganguli and Sinha [45] introduced the concept of mixed topology that is determined in a natural way by two given topologies determined in a topological space. It is a natural neighbourhood topology corresponding to a bitopological space. Recently, some researchers working in this direction have established some results.

In early 1960’s Lotfi A. Zadeh, a professor at university of California at Berkly well respected for his contributions to the development of System Theories, began to feel that traditional system analysis techniques were too precise for many complex real life problems. The idea of grade of membership, which is the backbone of fuzzy set theory, struck to his mind in 1964 [114], which leads to the publication of his seminal paper [167] on fuzzy sets in 1965 and the birth of the fuzzy sets and fuzzy logic technology [170]. The concept of fuzzy sets and fuzzy logic encounter sharp criticism from the academic community. However, research scholars and scientist around the
world ranging from psychology, sociology, philosophy and economics to natural sciences and engineering were followers of L. A. Zadeh.

Fuzzy logic research in Japan started with two small university research groups established in late 1970’s, one was lead by T. Terano and H. Shibata in Tokyo and the other lead by K. Tanaka and K. Asai in Kanasia [159]. Like fuzzy logic researcher in U.S.A., these researchers encountered an “anti-fuzzy” atmosphere in Japan during early days. However, their persistence and hard work has proved to be worthwhile a decade later. These Japanese researchers and their followers would make many important contributions to the theories as well as the application of fuzzy logic. In 1974, S. Assilian and E. H. Mamdani in U.K. developed the notion of fuzzy logic controller for the first time, which was for controlling a steam generator [82]. Thereafter many researchers from different parts of the world developed in many directions and it is still in developing stage.

Fuzzy sets, linguistic variable and possibility distribution are three core concepts in fuzzy logic. A fuzzy set is a generalization to classical set to allow objects to take partial membership in vague concepts [167]. The degree of an object belongs to a fuzzy set, which is a real number between 0 and 1, is called the membership value in the set. The meaning of fuzzy set is thus characterising by a membership function that maps elements of a universe of discourse to their corresponding membership value. The membership function of a fuzzy set $A$ is denotes by $\mu_A$. In addition to membership function, a fuzzy set is also associated with a linguistically meaningful term. Associating a fuzzy set to a linguistic term offers two important benefits. First, the association make it easier for human expert to express their knowledge. Second, the knowledge expressed using linguistic term easily understandable. These benefits results in significant saving in the cost of designing, modifying and maintaining a fuzzy logic systems.

Zadeh [167] established the notion of fuzzy inclusion, union, intersection, complement and their various properties. This provides a natural framework for generalizing many algebraic and topological concepts in various directions. Accordingly, fuzzy group, fuzzy ideal, fuzzy rings, fuzzy vector spaces and many other branches has been developed during the last four decades. We have confined our attention mainly on mixed fuzzy topological structure and mixed fuzzy ideal topological structures only.
In 1968, Chang [25] introduced fuzzy topological space by using fuzzy sets introduced by Zadeh [167]. Since then many researchers like Goguen [50], Wong [155], Lowen [76], Warren [149, 150], Hutton and Reilly [58], Ming [87, 88], Azad [17] and many others, have carried out an extensive study of fuzzy topological space. Chang introduced the most basic concept like open set, closed sets, neighbourhood, interior of a set etc. and established many results similar to the crispy topological spaces as in [66]. Throughout this thesis, we have used the Chang’s definition of fuzzy topology and other concepts.

Due to lack of proper fuzzification of a point, there were gaps in the study of local properties like convergence and continuity at a point in fuzzy topology. Wong [156] has filled up this gap by introducing the concepts of fuzzy point. This definition of a fuzzy point leads to the development of the study of convergence in a proper way. The results concerning local countability, separability and local compactness has been obtained. Many results of fuzzy topology differed from the general topology have nicely explained with examples by Wong [156]. C. K. Wong’s definition of fuzzy point is not generalization of crispy singleton set or an ordinary point. Moreover, the results of neighbourhood system of general topology does not reflect the results obtained in this neighbourhood system in fuzzy topological spaces. Pu and Liu [111] redefined the fuzzy points in such a way that it takes ordinary points as a special case of fuzzy points. They have also defined quasi relation between fuzzy points and fuzzy sets and quasi-neighbourhood ($Q$-nbd.) structure. The concept of neighbourhood systems in general topology and $Q$-neighbourhood systems in fuzzy topology have become similar.

In 1995, Das and Baishya [33] introduced the concept of mixed fuzzy topological spaces and studied various properties of that space. Mainly, fuzzy open maps, closed maps and fuzzy continuous maps in fuzzy bitopological spaces with reference to mixed fuzzy topology. A pairwise separation axiom has been introduced in fuzzy bitopological space and relations with ordinary separation axioms are established. Later in 2008, Rashid and Ali [118] have studied in details about the separation axioms in mixed fuzzy topological spaces.
1.2 DEFINITIONS AND NOTATIONS

In this section, we list some standard notations, definitions and concepts those will be use throughout the thesis.

Throughout the thesis $N$, $R$, $C$, $I$ and $\Delta$ denote the set of natural numbers, real numbers, complex numbers, the unit interval $[0, 1]$ and the index set respectively. $0^\prime$ and $1^\prime$ denotes the fuzzy empty set and whole fuzzy set in a fuzzy topological spaces and mixed fuzzy topological spaces respectively. fts is the abbreviation of fuzzy topological space; mfts is the abbreviation of mixed fuzzy topological space; nbd. is the abbreviation of neighbourhood; iff is the abbreviation of if and only if; prenbd. is the abbreviation of pre-neighbourhood; $Q$-nbd. is the abbreviation of quasi-neighbourhood. Closure of $A$ with respect to the fuzzy topology $\tau_i$ is denoted by $\tau_i-cl(A)$ or $\tau_i-\overline{(A)}$.

**DEFINITION 1.2.1.** Let $X$ be a non-empty set and $A$ be subset of $X$. The function $\chi_A: X \rightarrow [0, 1]$ defined by

$\chi_A(x) = \begin{cases} 
1, & x \in A \\
0, & x \in A 
\end{cases}$

is called the characteristic function of $A$.

**DEFINITION 1.2.2.** Let $X$ be a non-empty set and $I$, the unit interval $[0, 1]$. A fuzzy set $A$ in $X$ is characterized by a function $\mu_A: X \rightarrow I$, where $\mu_A$ is called the membership function of $A$ and $\mu_A(x)$ representing the membership grade of $x \in A$.

**DEFINITION 1.2.3.** A fuzzy subset is empty if its grade of membership is identically zero on $X$. It is denoted by $\phi$ or $0^\prime$.

**DEFINITION 1.2.4.** A fuzzy subset is whole if its grade of membership is identically one on $X$. It is denoted by $X$ or $1^\prime$. 
DEFINITION 1.2.5. A fuzzy set $A$ in $X$ is said to be finite if there exists a finite ordinary set $\{x_1, x_2, \ldots, x_n\}$ of $X$ such that $\mu_A(x) = 0$ for all $x \in X - \{x_1, x_2, \ldots, x_n\}$. Otherwise, $A$ is said to be infinite fuzzy set.

DEFINITION 1.2.6. Let $X$ be a non-empty set and two fuzzy sets $A$ and $B$ are said to be equal if $\mu_A(x) = \mu_B(x)$ for all $x \in X$. A fuzzy set $A$ is said to be contained in a fuzzy set $B$, written as $A \subseteq B$, if $\mu_A \leq \mu_B$. Complement of a fuzzy set $A$ in $X$ is a fuzzy set $A^c$ in $X$ defined as $\mu_A^c = 1 - \mu_A$. We write $A^c = co A$. Union and intersection of a collection $\{A_i : i \in \Delta\}$ of fuzzy sets in $X$, are written as $\bigcup_{i \in \Delta} A_i$ and $\bigcap_{i \in \Delta} A_i$ respectively. The membership functions are defined as follows:

$$\mu_{\bigcup_{i \in \Delta} A_i}(x) = \sup \{ \mu_{A_i}(x) : i \in \Delta \}, \quad \text{for all } x \in X,$$

and

$$\mu_{\bigcap_{i \in \Delta} A_i}(x) = \inf \{ \mu_{A_i}(x) : i \in \Delta \}, \quad \text{for all } x \in X.$$

DEFINITION 1.2.7. Let $X$ be a non-empty set and $A, B$ be two fuzzy subsets of $X$. Then the union of $A$ and $B$ is a fuzzy subset of $X$, denoted by $A \cup B$ which is defined by $(A \cup B)(x) = \max \{A(x), B(x)\}$, for every $x \in X$.

DEFINITION 1.2.8. Let $X$ be a non-empty set and $A, B$ be two fuzzy subsets of $X$. Then the intersection of $A$ and $B$ is a fuzzy subset of $X$, denoted by $A \cap B$ which is defined by $(A \cap B)(x) = \min \{A(x), B(x)\}$, for every $x \in X$.

DEFINITION 1.2.9. Let $X$ be a non-empty set and $A, B$ be two fuzzy subsets of $X$. Then the difference of $A$ and $B$ is defined by $A \setminus B = A \cap B^c$.

DEFINITION 1.2.10. A collection $\mathcal{B}$ of open fuzzy sets in a fuzzy topological space $X$ is said to be an open base for $X$ if every open fuzzy set in $X$ is a union of members of $\mathcal{B}$.
**DEFINITION 1.2.11.** If $A$ is a fuzzy set in $X$ and $B$ is a fuzzy set in $Y$, then $A \times B$ is a fuzzy set in $X \times Y$ defined by $\mu_{A \times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$ for all $x \in X$ and for all $y \in Y$.

**DEFINITION 1.2.12.** Let $f$ be a function from $X$ into $Y$. Then for each fuzzy set $B$ in $Y$, the inverse image of $B$ under $f$, written as $f^{-1}[B]$, is a fuzzy set in $X$ defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$.

Conversely, let $A$ be a fuzzy subset of $X$. The image of $A$, written as $f[A]$, is a fuzzy subset of $Y$ defined by $f[A](y) = \begin{cases} \sup \{A(x)\}, & \text{if } f^{-1}[y] \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$ for all $y$ in $Y$, where $f^{-1}[y] = \{x : f(x) = y\}$.

**DEFINITION 1.2.13.** A fuzzy set $A$ in a fuzzy topological space $(X, \tau)$ is called a neighbourhood of a point $x \in X$ if there exists $B \in \tau$ such that $B \subseteq A$ and $A(x) = B(x) > 0$.

**DEFINITION 1.2.14.** A fuzzy set in $X$ is called a fuzzy point if it takes the value 0 for all $y \in X$ except one say $x \in X$. If its value at $x$ is $\lambda$, we denote this fuzzy point by $x_\lambda$, where the point $x$ is called its support.

**DEFINITION 1.2.15.** The fuzzy point $x_\lambda$ is said to be contained in a fuzzy set $A$, or to belongs to $A$, denoted by $x_\lambda \in A$ if $\lambda \leq A(x)$. Evidently, every fuzzy set $A$ can be expressed as the union of the fuzzy points, which belongs to $A$.

**DEFINITION 1.2.16.** Two fuzzy sets $A$ and $B$ in $X$ are said to be intersecting if there exists a point $x \in X$ such that $(A \cap B)(x) \neq 0$. In this case we say that $A$ and $B$ intersect at $x$.

**DEFINITION 1.2.17.** A fuzzy point $x_\alpha$ is said to be quasi-coincident with $A$, denoted by $x_\alpha qA$, if and only if $\alpha + A(x) > 1$ or $\alpha > (A(x))^c$. 
DEFINITION 1.2.18. A fuzzy set $A$ is said to be quasi-coincident with $B$ and is denoted by $AqB$, if there exists an $x \in X$ such that $A(x) + B(x) > 1$.

It is clear that if $A, B$ are quasi-coincident at $x$, both $A(x)$ and $B(x)$ are not zero at $x$ and hence $A$ and $B$ intersect at $x$.

DEFINITION 1.2.19. A fuzzy set $A$ in a fts $(X, \tau)$ is called a quasi-neighbourhood of $x_\lambda$ if $A_1 \in \tau$ such that $\overline{A_1} \subseteq A$ and $x_\lambda qA_1$. The family of all $Q$-neighbourhoods of $x_\lambda$ is called the system of $Q$-neighbourhoods of $x_\lambda$. Intersection of two quasi-neighbourhoods of $x_\lambda$ is a quasi-neighbourhood.

1.3 FUZZY TOPOLOGICAL SPACES

A fuzzy topology and fuzzy topological space are defined by C. L. Chang as follows:

DEFINITION 1.3.1. Let $X = \{x\}$ be a space of points. A fuzzy topology is a family $\tau$, of fuzzy subsets of $X$, which satisfies the following conditions:

(i) $0', 1' \in \tau$,

(ii) If $A, B \in \tau$ then $A \land B \in \tau$,

(iii) If $A_i \in \tau$ for each $i \in \Delta$, then $\bigvee A_i \in \tau$, $\Delta$ being an arbitrary index set.

Then $\tau$ is called a fuzzy topology for $X$ and the pair $(X, \tau)$ is a fuzzy topological space. Members of $\tau$ are called fuzzy open sets and the complement of a fuzzy open set is called a fuzzy closed set. If $(X, \tau)$ is a fuzzy topological space (in short, fts), then the closure and interior of a fuzzy set $A$ in $X$, denoted by $clA$ and $intA$ respectively, are defined by $clA = \cap \{ B : B$ is a fuzzy closed set in $X$ and $A \subseteq B \}$ and $intA = \cup \{ V : V$ is a fuzzy open set in $X$ and $V \subseteq A \}$. Clearly, $clA$ (respectively $intA$) is the smallest (respectively largest) closed (respectively open) fuzzy set in $X$ containing (respectively contained in) $A$. If there is more than one topology on $X$, then the closure and interior of $A$ with respect to a fuzzy topology $\tau$ on $X$ will be denoted by $\tau-cl A$ and $\tau-int A$. 
An alternative and more natural definition of a fuzzy topological space was suggested by R. Lowen as follows:

**DEFINITION 1.3.2.** A fuzzy topology on a non-empty set $X$ is a collection $\tau$ of fuzzy subsets of $X$ such that

(i) all constant fuzzy subsets of $X$ belong to $\tau$,

(ii) $\tau$ is closed under formation of fuzzy union of arbitrary collection of members of $\tau$,

(iii) $\tau$ is closed under formation of intersection of finite collection of members of $\tau$.

The pair $(X, \tau)$ is called fuzzy topological space.

**NOTE 1.3.1.** Lowen’s definition involves the changing of condition (i) namely $0^{'}, 1^{'} \in \tau$ to for every $\alpha$, constant, $\alpha \in \tau$. Then there are mathematically more sophisticated reasons available as well. Indeed

(a) It follows at once from the definition of fuzzy topology defined by Lowen, that every topologically generated fuzzy topology fulfils Lowen’s condition (i).

(b) It can easily be seen that with Chang’s definition constant functions between fuzzy topological spaces are not necessarily continuous. This can only be true in general if we use the Lowen’s definition. This is of course the most important arguments for considering the Lowen’s definition.

**DEFINITION 1.3.3.** A fuzzy point $x_\lambda$ is called an adherent point of a fuzzy set $A$ if every $Q$-neighbourhood of $x_\lambda$ is quasi-coincident with $A$.

**DEFINITION 1.3.4.** A fuzzy point $x_\lambda$ is called an accumulation point of a fuzzy set $A$ if

(i) $x_\lambda$ is an adherent point of $A$,

(ii) every $Q$-neighbourhood of $x_\lambda$ and $A$ are quasi-coincident at some point different from the support $x$, where $x_\lambda \in A$.

**DEFINITION 1.3.5.** The union of all the accumulation points of $A$ is called the derived set of $A$, denoted by $A^d$ and is such that

(i) $\text{cl}A = A \cup A^d$.

(ii) A fuzzy set $A$ is closed if $A$ contains all the accumulation points.
**Definition 1.3.6.** The function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is called fuzzy continuous if, for every \( B \in \sigma \), \( f^{-1}(B) \in \tau \). The function \( f \) is called fuzzy homeomorphic if \( f \) is bijective and both \( f \) and \( f^{-1} \) is fuzzy continuous.

**Definition 1.3.7.** The function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is called fuzzy open if for each fuzzy open set \( A \) in \( X \), \( f(A) \) is fuzzy open set in \( Y \).

**Definition 1.3.8.** The function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is called fuzzy closed if for each fuzzy closed set \( A \) in \( X \), \( f(A) \) is fuzzy closed set in \( Y \).

**Definition 1.3.9.** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a fuzzy continuous function. Then the following properties hold:

\((i)\) For every \( \sigma \)-closed \( A \), \( f^{-1}(A) \) is \( \tau \)-closed.

\((ii)\) For each fuzzy point \( e \) in \( X \) and each neighbourhood \( V \) of \( f(e) \), there exists a neighbourhood \( U \) of \( e \) such that \( f(U) \subset V \).

\((iii)\) For each fuzzy point \( e \) in \( X \) and each \( Q \)-neighbourhood \( V \) of \( f(e) \), there exists a \( Q \)-neighbourhood \( U \) of \( e \) such that \( f(U) \subset V \).

\((iv)\) For any fuzzy set \( A \) in \( X \), \( f(\overline{A}) \subset \overline{f(A)} \).

\((v)\) For any fuzzy set \( B \) in \( Y \), \( (f^{-1}(B)) \subset f^{-1}(\overline{B}) \).

**Definition 1.3.10.** If two fuzzy topologies \( \tau_1 \) and \( \tau_2 \) in \( X \) are such that \( \tau_1 \subset \tau_2 \), we say that \( \tau_2 \) is finer than \( \tau_1 \) and \( \tau_1 \) is coarser than \( \tau_2 \). \( \tau^c \) denotes the family of all \( \tau \)-closed fuzzy sets.

**Definition 1.3.11.** Let \( (X, \tau) \) be a fuzzy topological space and \( Y \) be an ordinary subset of \( X \). The class \( \tau_Y = \{ Y \cap A: A \in \tau \} \) determines a topology on \( Y \). This topology is called the relative topology on \( Y \).
**DEFINITION 1.3.12.** Let \((X, \tau)\) be a fuzzy topological space. A subfamily \(\mathcal{B}\) of \(\tau\) is a base for \(\tau\) if each member of \(\tau\) can be expressible as the union of some members of \(\mathcal{B}\).

**DEFINITION 1.3.13.** Let \((X, \tau)\) be a fuzzy topological space. A subfamily \(S\) of \(\tau\) is a subbase for \(\tau\) if the family of finite intersection of members of \(S\) forms a base for \(\tau\).

**DEFINITION 1.3.14.** Let \((X, \tau)\) be a fuzzy topological space and \(p\) be a fuzzy point. A subfamily \(B_p\) of \(\tau\) is called a local base at \(p\) if for every member \(B\) of \(B_p\), and for every member \(A\) of \(\tau\) such that \(p \in A\) there exists a member \(B\) of \(B_p\) such that \(p \in B \subset A\).

**DEFINITION 1.3.15.** Let \(U_{Q}\) be a family of \(Q\)-neighbourhoods of a fuzzy point \(x_\lambda\) in \(X\). A subfamily \(B_{Q}\) of \(U_{Q}\) is said to be \(Q\)-neighbourhood base of \(U_{Q}\) if for any \(A \in U_{Q}\) there exists \(B \in B_{Q}\) such that \(B < A\).

**DEFINITION 1.3.16.** A fuzzy topological space \((X, \tau)\) is said to be \(Q\)-first countable space if every fuzzy point in \(X\) has countable \(Q\)-neighbourhood base.

**DEFINITION 1.3.17.** Let \(U_c\) be a family of neighbourhoods of a fuzzy point \(x_\lambda\) in \(X\). A subfamily \(B_c\) of \(U_c\) is said to be neighbourhood base of \(U_c\), if for any \(A \in U_c\), there exists \(B \in U_c\) such that \(B < A\).

**DEFINITION 1.3.18.** Let \((X, \tau)\) be a fuzzy topological space. Then \(X\) is said to be a first countable space, if for each fuzzy point \(x_\lambda\) \((0 \leq \lambda \leq 1)\) there exists a countable class of fuzzy open set \(B_{x_\alpha}\) such that \(\alpha < U(x)\), for all \(U \in B_{x_\alpha}\) and if \(\alpha < V(x)\) for some fuzzy open set \(V\) then there exists \(W \in B_{x_\alpha}\) such that \(W \subseteq V\).

**DEFINITION 1.3.19.** A fuzzy topological space \((X, \tau)\) is said to be \(C_\beta\) if there exists a countable base for \(\tau\).
**THEOREM 1.3.1. (Pu, P. and Liu, Y. [112], Lemma 1.1(3-4)).** Let \( f \) be a function from \( X \) into \( Y \). Then

(i) for a fuzzy point \( x_\lambda \) in \( X \), \( f(x_\lambda) \) is a fuzzy point in \( Y \) and \( f(x_\lambda) = (f(x))_\lambda \).  

(ii) Let \( \{ A_\alpha : \alpha \in \Delta \} \) be a family of fuzzy sets in \( X \), then \( f( \bigcup_\alpha A_\alpha ) = \bigcup_\alpha f(A_\alpha ) \).

**THEOREM 1.3.2. (Malghan and Benchalli [81], Theorem 3.1.).**

Let \( f: (X, \tau) \to (Y, \sigma) \) be a fuzzy open function, and then the following properties hold:

(i) \( f(A'^\circ) \subseteq (f(A))'^\circ \), for each fuzzy set \( A \) in \( X \).

(ii) \( f^{-1}(\overline{B}) \subseteq f^{-1}(\overline{B}) \), for each fuzzy set \( B \) in \( Y \).

(iii) \( (f^{-1}(B))'^\circ \subseteq f^{-1}(B'^\circ) \), for each fuzzy set \( B \) in \( Y \).

**THEOREM 1.3.3. (Malghan and Benchalli [80], Theorem 1.5.).**

Let \( f: (X, \tau) \to (Y, \sigma) \) be a function. Then \( f \) is closed if and only if \( \overline{f(A)} \subseteq f(\overline{A}) \) for each fuzzy set \( A \) in \( X \).

### 1.4 MIXED FUZZY TOPOLOGICAL SPACES

The concept of mixing two fuzzy topologies to get a third topology was first studied by Das and Baishya [33] and the new topology is called mixed fuzzy topology. However, this newly formed topology is not the generalization of mixed topology in the sense of crispy set. Das and Baishya defined and established the followings:

**DEFINITION 1.4.1.** Let \( (X, \tau_1) \) and \( (X, \tau_2) \) be two fuzzy topological spaces and let \( \tau_1(\tau_2) = \{ A \in \mathcal{F}^X : \text{for every fuzzy point } x_\lambda \text{ with } x_\lambda \trianglerighteq A, \text{there exists a } \tau_2\text{-}\mathcal{Q}\text{-neighbourhood } A_\alpha \text{ of } x_\lambda \text{ such that } x_\lambda \trianglerighteq A_\alpha \text{ and } \tau_1\text{-closure, } \overline{A_\alpha} \subseteq A \}, \) then \( \tau_1(\tau_2) \) is a fuzzy topology on \( X \) and this is called mixed fuzzy topology and the space \( (X, \tau_1(\tau_2)) \) is called mixed fuzzy topological space.

**THEOREM 1.4.1. (Das and Baishya [33], Theorem 3.3.).** Let \( \tau_1 \) and \( \tau_2 \) be two fuzzy topologies on \( X \). Then the mixed fuzzy topology \( \tau_1(\tau_2) \) is coarser than \( \tau_2 \). In symbol, \( \tau_1(\tau_2) \subseteq \tau_2 \).
THEOREM 1.4.2. (Das and Baishya [33], Theorem 3.5.). If \( \tau_1 \) is fuzzy regular and \( \tau_1 \subseteq \tau_2 \), then \( \tau_1 \subseteq \tau_1(\tau_2) \).

THEOREM 1.4.3. (Das and Baishya [33], Theorem 3.8.). If \((X, \tau_1, \tau_2)\) be a fuzzy bitopological space and \( x_a \) be a fuzzy point. Let \( U_{x_a} = \{ A: \text{there exists } A_2 \in \tau_2, \ x_a \in A_2 \text{ and } \overline{A_2} \subseteq A, \text{ the closure being with respect to } \tau_1 \} \), then there exists a fuzzy topology \( \tau_1(\tau_2) \) with respect to which \( U_{x_a} \) is a Q-neighbourhood system of \( x_a \).

THEOREM 1.4.4. (Baishya [18], Proposition 3.6.1.). Let \((X, \tau_1, \tau_2)\) and \((Y, \tau_1', \tau_2')\) be any two fuzzy bitopological spaces and let \( f: X \rightarrow Y \) be a mapping such that \( f \) is \( \tau_1 \)-\( \tau_1' \) and \( \tau_2 \)-\( \tau_2' \) continuous. Then \( f \) is \( \tau_1(\tau_2) \)-\( \tau_1'(\tau_2') \) continuous.

THEOREM 1.4.5. (Baishya [18], Proposition 3.6.2.). Let \((X, \tau_1, \tau_2)\) and \((Y, \tau_1', \tau_2')\) be any two fuzzy bitopological spaces and \( \tau_1' \subseteq \tau_2' \) and \( \tau_1' \) is fuzzy regular space. If \( f \) is \( \tau_1(\tau_2) \)-\( \tau_1'(\tau_2') \) continuous, then \( f \) is \( \tau_2 \)-\( \tau_1' \) continuous.

THEOREM 1.4.6. (Baishya [18], Proposition 3.6.3.). Let \((X, \tau_1, \tau_2)\) and \((Y, \tau_1', \tau_2')\) be any two fuzzy bitopological spaces and let \( f: X \rightarrow Y \) is \( \tau_1(\tau_2) \)-\( \tau_1'(\tau_2') \) continuous, then \( f \) is \( \tau_2 \)-\( \tau_1' \) continuous.

THEOREM 1.4.7. (Baishya [18], Proposition 3.6.4.). Let \((X, \tau_1, \tau_2)\) and \((Y, \tau_1', \tau_2')\) be any two fuzzy bitopological spaces. If \( f: X \rightarrow Y \) is \( \tau_1 \)-\( \tau_1' \) continuous and \( \tau_1 \) is fuzzy regular and \( \tau_1 \subseteq \tau_2 \), then \( f \) is \( \tau_1(\tau_2) \)-\( \tau_1' \) continuous.

THEOREM 1.4.8. (Baishya [18], Proposition 3.6.5.). Let \((X, \tau_1, \tau_2)\) and \((Y, \tau_1', \tau_2')\) be any two fuzzy bitopological spaces. If \( f: X \rightarrow Y \) is \( \tau_1(\tau_2) \)-\( \tau_1' \) continuous function, then \( f \) is \( \tau_2 \)-\( \tau_1' \) continuous.
THEOREM 1.4.9. (Baishya [18], Proposition 3.6.6.) Let \((X, \tau_1, \tau_2)\) and \((Y, \tau_1^*, \tau_2^*)\) be any two fuzzy bitopological spaces. If \(f: X \to Y\) is \(\tau_1(\tau_2) - \tau_1^*\)-continuous function, then \(f\) is \(\tau_2 - \tau_2^*\)-continuous.

In Chapter-II of the thesis, we have introduced a new type of mixed fuzzy topological space and introduced the concepts of countability.

1.5 SOME CONTINUOUS LIKE FUNCTIONS IN TOPOLOGICAL SPACES AND FUZZY TOPOLOGICAL SPACES

Historically, topology began its development with the attempts of classification of spaces by Riemann [122] followed by Frechet [40] on metric spaces. The work of Riesz [123], where the notion of limit point were used for the first time to describe abstract spaces. In 1913, Weyl [151] introduced the notion of neighbourhood system and culminated in 1914 with F. Hausdorff where he introduced the right axiom system for neighbourhood and made them a suitable abstraction and thus gave birth of modern topology. Topology has followed two principle lines of development. In one direction, it gives us homotopy theory, dimension theory and manifolds. In other direction, it tries to make analysis independent of geometrical intuitions and leads us to some basic branches such as integration, abstract harmonic analysis, Banach spaces and Hilbert spaces.

The continuous function has a number of deep rooted and widespread applications in developing the subject topology. The main purpose for studying the general topology is to study the invariance of topological properties under homeomorphism. That is an open continuous bijections. Having enormous use and potentiality of continuous functions and the pivotal property like compactness, there has been a plenty of attempts by mathematicians to generalize these important concepts. Consequently, many weaker and stronger forms of continuity and compactness came into existence day to day. For these, many new types of sets that are near to open set introduced by some mathematician. Velicko [146] introduced the concept of \(0\)-open sets and \(\delta\)-open...
sets that are weaker forms of open sets. With the help of these nearly open sets, many generalized version of continuity have come into existence. Some of them are \( \delta \)-continuity, \( \theta \)-continuity, almost continuity, super continuity, \( \delta \)-almost continuity etc. In 1968, Singal and Singal [127] introduced the notion of almost continuous function, which is a generalization of the concept of continuity. After that, Noiri [108] defined the notion of \( \delta \)-continuous functions. They have established that the concept of continuity and \( \delta \)-continuity are independent and both the classes of continuity are contained in the class of almost continuous functions. In 1982, Munshi and Bassan [98] introduced the concept of super continuous mapping, which is a stronger form of both continuity and \( \delta \)-continuity. Later on, many researchers have characterized these classes of functions, made certain comparative studies and finally established some relationship among them. Reilly and Vamanamurthy [121] have investigated the super continuous mapping. Ganguli and Dutta [42, 43] rename super-continuous function as strongly \( \delta \)-continuous function.

Since the perception of different types of continuous functions in topological spaces, many researchers all over the world have gave contributed for developing different class of functions and studied various properties of such mappings and established relationship among themselves. It is very difficult task to cover all the literature. Here we give some of the important references [42, 43, 44, 45, 53, 74, 79, 81, 91, 92, 93, 95, 96, 98, 104, 106, 108, 110, 119, 121, 127, 129, 132, 140, 148, 149] etc.

Since the introduction of fuzzy topological spaces by Chang [25], many researchers all over the world have tried to develop these concepts and most of them successfully developed fuzzy topological spaces in various directions. C. L. Chang introduced the notion of continuity in fuzzy setting. He also proved one very important property that in fuzzy setting constant function many not be continuous. Later on, Lowen [76] modified the Chang’s definition of the fuzzy topology and proved constant function in fuzzy topological spaces is continuous.

In 1980, Pu and Liu [111] introduced the concept of quasi-coincidence and quasi-neighbourhood that opened a new avenue by providing with a new fuzzy methodology by which such extensions of functions can be very interesting and effectively carried out. After then, Azad [17] introduced some weaker form of fuzzy continuity. In 1988, Ganguli and Saha [44] introduced a new type of continuity namely fuzzy \( \delta \)-continuity using the concept of quasi-coincidence and
quasi-neighbourhood as introduced by Pu and Liu [111, 112]. Later on, Mukherjee and Ghosh [86] introduced and studied some stronger form of fuzzy continuity in fuzzy setting in the light of quasi-coincidence and quasi-neighbourhood. Mukherjee and some others [91, 93, 94, 95, 96] introduced and investigated various continuous like functions with the help of quasi-coincidence and quasi-neighbourhood. In 1991, Ajmal and Tyagi [5] redefined the notions $\delta$-open fuzzy set, $\delta$-closed fuzzy set, $\delta$-adherent point of a fuzzy set and $\delta$-closure of fuzzy sets. Several related characterization of fuzzy almost continuous functions in normal fuzzy topological spaces were obtained. The notion of fuzzy almost quasi-compact function has been introduced and its relation with fuzzy almost continuous functions is discussed. Recently, Bhattacharyya and Mukherjee [19] introduced and investigated two new classes of functions between fuzzy topological spaces namely $\delta$-almost continuous and $\delta^*$-almost continuous functions and proved that fuzzy $\delta^*$-almost continuous function is weaker than fuzzy $\delta$-almost continuous and both of them are independent of fuzzy almost continuous functions. In the same year, Geogiou et al. [48] introduced and studied fuzzy strongly and fuzzy super-continuous functions chiefly with the help of quasi-coincidence in fuzzy setting. Arya and Singal [16] introduced and studied fuzzy sub-weakly continuous functions, a weaker form of fuzzy continuity, fuzzy locally weak*­continuity and fuzzy sub-weakly $\alpha$-continuity in fuzzy topological spaces.

Fascinated by the various applications of mixed fuzzy topological spaces and various stronger and weaker forms of continuity in fuzzy topological spaces established by many researchers all over the world, the idea of studying different types of $\delta$-continuous functions between mixed fuzzy topological spaces has occurred in our mind. In this thesis we have introduced fuzzy $\delta$-continuity, fuzzy $\delta^*$-continuity, fuzzy $\delta^*$-almost continuity and weakly continuity and investigate some properties in mixed fuzzy topological spaces and we have established relationship among them.

Now, we procure some definitions and results collected from different research papers, journals, books, thesis of various mathematicians all over the world devoted to the development of topological spaces and fuzzy topological spaces.
**DEFINITION 1.5.1.** Let $X$ be a topological space. A set $S$ in $X$ is said to be regular open (respectively regular closed) if $\text{int(cl } S) = S$ (respectively $\text{cl(int } S) = S$). A point $x \in S$ is said to be a $\delta$-cluster point of $S$ if $S \cap U \neq \emptyset$, for every regular open set $U$ containing $x$. The set of all $\delta$-cluster points of $S$ is called the $\delta$-closure of $S$ and is denoted by $[S]_\delta$. If $[S]_\delta = S$, the $S$ is said to be $\delta$-closed. The complement of a $\delta$-closed set is called $\delta$-open set.

For every topological space $(X, \tau)$, the collection of all $\delta$-open sets form a topology on $X$, this is weaker than $\tau$. This topology is denoted by $\tau^*$ which has a base consisting of all regular open sets in $(X, \tau)$. In case of regular open sets, finite intersection of regular open sets is regular open whereas finite union of regular closed sets is regular closed.

**DEFINITION 1.5.2.** A function $f: X \to Y$ is said to be $\delta$-continuous if for each $x \in X$ and each open nbd. $V$ of $f(x)$, there exists an open nbd. $U$ of $x$ such that $f(\text{int(cl } U)) \subseteq \text{int(cl } V))$.

**DEFINITION 1.5.3.** A function $f: X \to Y$ is said to be almost continuous if for each $x \in X$ and each open nbd. $V$ of $f(x)$, there exists an open nbd. $U$ of $x$ such that $f(U) \subseteq \text{int(cl } V))$.

**DEFINITION 1.5.4.** A function $f: X \to Y$ is called strongly $\delta$-continuous at a point $x \in X$ if for any open nbd. $V$ of $f(x)$ in $Y$, there exist $\delta$-open nbd. $U$ of $x$ in $X$ such that $f(U) \subseteq V$.

**DEFINITION 1.5.5.** A fuzzy point $x_a \in cl(A)$ if each $q$-neighbourhood of $x_a$ is $q$-coincident with $A$.

**DEFINITION 1.5.6.** A fuzzy set $A$ in a fts $X$ is called fuzzy regularly open (closed) if $\text{int(cl } (A)) = A$ (respectively $\text{cl(int } (A)) = A$).

**DEFINITION 1.5.7.** A fuzzy point $x_a$ is called a fuzzy $\delta$-cluster point of a fuzzy set $A$ in a fuzzy topological space $X$ if every fuzzy regularly open $q$-nbd. of $x_a$ is $q$-coincident with $A$. The set of all fuzzy $\delta$-cluster points of $A$ is called the fuzzy...
\( \delta \)-closure of \( A \), to be denoted by \([A]_{\delta} \) or \( \delta-cl(A) \). The fuzzy set \( A \) is fuzzy \( \delta \)-closed if \( A = [A]_{\delta} \) and complement of a fuzzy \( \delta \)-closed set is called fuzzy \( \delta \)-open.

**DEFINITION 1.5.8.** Fuzzy \( \delta \)-interior of a fuzzy set \( A \) in a fuzzy topological space \( X \), denoted by \( \delta-int(A) \), is defined by \( \delta-int(A) = 1-(\delta-cl(A)) \). A fuzzy set \( A \) in a fuzzy topological space \( X \) is fuzzy \( \delta \)-open if \( A = \delta-int(A) \).

**DEFINITION 1.5.9.** A fuzzy set \( A \) is said to be fuzzy preopen if \( A \leq int(cl(A)) \).

**DEFINITION 1.5.10.** A fuzzy set \( A \) in a fuzzy topological space \( X \) is said to be fuzzy \( \delta \)-preopen if \( A \leq int(\delta-cl(A)) \). The complement of a fuzzy \( \delta \)-preopen set is called fuzzy \( \delta \)-preclosed. The set of all fuzzy \( \delta \)-preopen sets in \( X \) will be denoted by \( \delta-PO(X) \).

**DEFINITION 1.5.11.** A fuzzy set \( A \) in a fuzzy topological space \( X \) is called a fuzzy \( \delta \)-pre-\( q \)-nbd. of a fuzzy point \( x_a \) in \( X \) if there exists a fuzzy \( \delta \)-preopen set \( V \) in \( X \) such that \( x_aqV \leq A \).

A fuzzy point \( x_a \) in a fuzzy topological space \( X \) is called a fuzzy \( \delta \)-precluster point of a fuzzy set \( A \) in \( X \) if every fuzzy \( \delta \)-pre-\( q \)-nbd. of \( x_a \) is \( q \)-coincident with \( A \). The union of all fuzzy \( \delta \)-precluster points of \( A \) is called the fuzzy \( \delta \)-preclosure of \( A \) and will be denoted by \( \delta-pcl(A) \).

A fuzzy set \( G \) in a fts \( X \) is called a fuzzy \( \delta \)-pre-nbd. of a fuzzy point \( x_a \) in \( X \) if there exists a fuzzy \( \delta \)-preopen set \( U \) in \( X \) such that \( x_a \leq U \leq G \). The union of all fuzzy \( \delta \)-preopen sets in a fts \( X \), each contained in a fuzzy set \( A \) in \( X \), is called the fuzzy \( \delta \)-preinterior of \( A \) and is denoted by \( \delta-pint(A) \).

**DEFINITION 1.5.12.** A mapping \( f: X \rightarrow Y \) is said to be fuzzy \( \delta \)-almost continuous if for each fuzzy point \( x_a \) in \( X \) and every nbd. \( V \) of \( f(x_a) \) in \( Y \), \( \delta-cl( f^{-1}(V) ) \) is a fuzzy nbd. of \( x_a \) in \( X \).
**DEFINITION 1.5.13.** A function \( f: X \to Y \) is said to be fuzzy \( \delta^* \)-almost continuous if the inverse image of a fuzzy \( \delta \)-preopen set in \( Y \) is a fuzzy \( \delta \)-preopen set in \( X \).

**DEFINITION 1.5.14.** A fts \( X \) is said to be fuzzy \( \delta \)-preregular if for each fuzzy \( \delta \)-preclosed set \( F \) in \( X \) and each fuzzy point \( x_a \) with \( x_a q(1-F) \), there exists a fuzzy open set \( U \) and a fuzzy \( \delta \)-preopen set \( V \) such that \( x_a qU, F \leq V \) and \( U \) is not quasi-coincident with \( V \).

**DEFINITION 1.5.15.** A mapping \( f: X \to Y \) from a fts \( X \) to a fts \( Y \) is said to be fuzzy super-continuous at a fuzzy point \( x_a \) of \( X \) if for every \( q \)-nbd. \( U \) of \( f(x_a) \), there is a \( q \)-nbd. \( V \) of \( x_a \) such that \( f(int(cl(V))) \leq U \).

**DEFINITION 1.5.16.** A mapping \( f: X \to Y \) from a fts \( X \) to a fts \( Y \) is said to be fuzzy continuous at a fuzzy point \( x_a \) of \( X \) if for every \( q \)-nbd. \( U \) of \( f(x_a) \), there is a \( q \)-nbd. \( V \) of \( x_a \) such that \( f(V) \leq U \). The function \( f \) is called fuzzy continuous on \( X \) if and only if \( f \) is fuzzy continuous at each fuzzy point of \( X \).

**DEFINITION 1.5.17.** A mapping \( f: X \to Y \) from a fts \( X \) to a fts \( Y \) is said to be fuzzy almost open (closed) if the image of every fuzzy regularly open (regularly closed) set in \( X \) is fuzzy open (closed) in \( Y \).

**DEFINITION 1.5.18.** Let \( X \) and \( Y \) be two fts’s and \( f: X \to Y \) be a mapping. Then \( f \) is said to be fuzzy almost continuous if corresponding to any fuzzy point \( x_a \) of \( X \) and any fuzzy regular open \( q \)-nbd. \( V \) of \( f(x_a) \) there is a fuzzy open \( q \)-nbd. \( U \) of \( x_a \) such that \( f(V) \leq U \).

**DEFINITION 1.5.19.** A mapping \( f: X \to Y \) is said to be fuzzy \( \delta \)-continuous if for each fuzzy point \( x_a \) of \( X \) and for any fuzzy regularly open \( q \)-nbd. \( U \) of \( f(x_a) = y_a \) of \( Y \) (where \( y = f(x) \) ), there exists a fuzzy regularly open \( q \)-nbd. \( V \) of \( x_a \) such that \( f(V) \leq U \).
**DEFINITION 1.5.20.** A mapping $f: X \rightarrow Y$ from a fts $X$ to a fts $Y$ is called fuzzy completely continuous if $f^{-1}(A)$ is fuzzy regularly open in $X$ for any fuzzy open set $A$ in $Y$.

**DEFINITION 1.5.21.** A mapping $f: X \rightarrow Y$ from a fts $X$ to a fts $Y$ is said to be fuzzy strongly continuous if for every fuzzy set $A$ in $X$, $f(clA) \subseteq f(A)$.

**DEFINITION 1.5.22.** Let $X$ and $Y$ be two fuzzy topological spaces. A mapping $f: X \rightarrow Y$ is said to be fuzzy weakly continuous if for each open fuzzy set $B$ in $Y$, $1_{f^{-1}[B]} \subseteq 1_{f^{-1}[clB]}$.

**DEFINITION 1.5.23.** Let $X$ and $Y$ be two fuzzy topological spaces. A mapping $f: X \rightarrow Y$ is said to be fuzzy sub-weakly continuous if there exists an open base $\mathcal{B}$ for $Y$ such that $cl f^{-1}[V] \subseteq f^{-1}[clV]$ for each $V \in \mathcal{B}$.

**THEOREM 1.5.1.** (Noiri [108], Theorem 2.2.) For a function $f: X \rightarrow Y$, the following are equivalent:

(i) $f$ is $\delta$-continuous.

(ii) For each $x \in X$ and each regular open set $V$ containing $f(x)$, there exists a regular open set $U$ containing $x$ such that $f(U) \subseteq V$.

(iii) For every regularly closed set $F$ of $Y$, $f^{-1}(F)$ is $\delta$-closed in $X$.

(iv) For every $\delta$-closed set $F$ of $Y$, $f^{-1}(F)$ is $\delta$-closed in $X$.

(v) For every $\delta$-open set $V$ of $Y$, $f^{-1}(V)$ is $\delta$-open in $X$.

(vi) For every regularly open set $V$ of $Y$, $f^{-1}(V)$ is $\delta$-open in $X$.

(vii) For each $x \in X$ and each net $x_\lambda \xrightarrow{\delta} x$, the net $f(x_\lambda) \xrightarrow{\delta} f(x)$.

**THEOREM 1.5.2.** (Ganguli and Saha [44], Theorem 2.15.) For a function $f: X \rightarrow Y$ from a fts $(X, \tau)$ into a fts $(Y, \sigma)$.

(i) $f$ is almost continuous.

(ii) For any fuzzy subset $A \subseteq X$, $\overline{f(A)} \subseteq [f(A)]_\delta$.

(iii) For any fuzzy subset $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}([B]_\delta)$. 
(iv) For every $\delta$-closed set $B \subseteq Y$, $f^{-1}(B)$ is closed in $X$.

(v) For every $\delta$-open set $B \subseteq Y$, $f^{-1}(B)$ is open in $X$.

(vi) For every regular open set $B \subseteq Y$, $f^{-1}(B)$ is open in $X$.

**THEOREM 1.5.3. (Ganguli and Saha [44], Theorem 2.10).** For a function $f: X \to Y$ the following are equivalent:

(i) $f$ is fuzzy $\delta$-continuous.

(ii) $f([A]_\delta) \subseteq [f(A)]_\delta$, for every fuzzy set $A$ in $X$.

(iii) $[f^{-1}(A)]_\delta \subseteq f^{-1}([A]_\delta)$, for every fuzzy set $A$ in $Y$.

(iv) For every fuzzy $\delta$-closed set $A$ in $Y$, $f^{-1}(A)$ is fuzzy $\delta$-closed in $X$.

(v) For every fuzzy $\delta$-open set $A$ in $Y$, $f^{-1}(A)$ is fuzzy $\delta$-open in $X$.

(vi) For every fuzzy regular open set $A$ in $Y$, $f^{-1}(A)$ is fuzzy $\delta$-open in $X$.

**THEOREM 1.5.4. (Ganguli and Saha [44], Proposition 2.5).** Every $\delta$-closed set is closed in a fuzzy topological space $(X, \tau)$.

**THEOREM 1.5.5. (Ganguli and Saha [44], Theorem 2.12).** The following statements are true for any fuzzy sets $A$ and $B$ in a fuzzy topological space $X$:

(i) $A \leq B \Rightarrow \delta$-pcl$A \leq \delta$-pcl$B$.

(ii) $A$ is fuzzy $\delta$-preclosed if and only if $A = \delta$-pcl$A$.

(iii) $\delta$-pcl$A$ is fuzzy $\delta$-preclosed in $X$.

(iv) $\delta$-pcl$(\delta$-pcl$A)$ = $\delta$-pcl$A$.

**THEOREM 1.5.6. (Bhattacharyya and Mukherjee [19], Theorem 2.16).** The following statements are true for any fuzzy sets $A$ and $B$ in a fuzzy topological space $X$:

(i) $A \leq B \Rightarrow \delta$-pint$A \leq \delta$-pint$B$.

(ii) $\delta$-pint$A$ is fuzzy $\delta$-preopen in $X$.

(iii) $A$ is fuzzy $\delta$-preopen if and only if $A = \delta$-pint$A$.

(iv) $\delta$-pint$(1-A) = 1- \delta$-pcl $A$. 

THEOREM 1.5.7. (Bhattacharyya and Mukherjee [19], Theorem 3.2.). For a function
\( f: X \rightarrow Y \) the following are equivalent:
(i) \( f \) is fuzzy \( \delta \)-almost continuous.
(ii) \( f^{-1}(B) \subseteq \text{int}(\delta \text{-cl } f^{-1}(B)) \), for each fuzzy open set \( B \) in \( Y \).
(iii) \( f(\text{cl}A) \subseteq \text{cl}(f(A)) \), for every fuzzy \( \delta \)-open set \( A \) in \( X \).

THEOREM 1.5.8. (Bhattacharyya and Mukherjee [19], Theorem 4.3.). For a function
\( f: X \rightarrow Y \) the following are equivalent:
(a) \( f \) is fuzzy \( \delta^* \)-almost continuous.
(b) For each fuzzy point \( x_a \) in \( X \) and fuzzy \( \delta \)-prebd. \( V \) of \( f(x_a) \), \( f^{-1}(V) \) is a fuzzy \( \delta \)-prebd. of \( x_a \).
(c) For each fuzzy point \( x_a \) in \( X \) and fuzzy \( \delta \)-prebd. \( V \) of \( f(x_a) \), there is a fuzzy \( \delta \)-prebd. \( U \) of \( x_a \) such that \( f(U) \subseteq V \).
(d) For each fuzzy set \( B \) in \( Y \), \( f^{-1}(\delta \text{-pint}B) \subseteq \delta \text{-pint } f^{-1}(B) \).
(e) For each fuzzy \( \delta \)-preclosed set \( F \) in \( Y \), \( f^{-1}(F) \) is fuzzy \( \delta \)-preclosed in \( X \).
(f) For each fuzzy set \( A \) in \( X \), \( f(\delta \text{-pcl } A) \subseteq \delta \text{-pcl } f(A) \).
(g) For each fuzzy set \( B \) in \( Y \), \( \delta \text{-pcl}( f^{-1}(B)) \subseteq f^{-1}(\delta \text{-pcl } B) \).

In Chapter-III of the thesis, we have introduced fuzzy weakly continuous mappings between mixed fuzzy topological spaces and investigated some of their properties.
In Chapter-IV of the thesis, we have introduced fuzzy \( \delta \)-continuous mapping between mixed fuzzy topological spaces and have examined some properties and established relationship between fuzzy continuous mapping and fuzzy \( \delta \)-continuous mapping.
In Chapter-V of the thesis, we introduced the notions of the fuzzy \( \delta^* \)-continuous and fuzzy \( \delta^* \)-almost continuous mappings in case of mixed fuzzy topological spaces and established relationship between fuzzy \( \delta^* \)-continuous mapping and fuzzy \( \delta^* \)-almost continuous mapping.
1.6 IDEALS IN TOPOLOGICAL SPACES AND FUZZY TOPOLOGICAL SPACES

The concept of ideal in topological spaces has been introduced by Vaidyanathaswamy [144] in 1945. His work was published in an epic paper titled “The localisation theory in set topology”. Later in 1966, Kuratowski [71] studied in detail about the ideal topological space. Applications of various fields were further investigated by Jankovic and Hamlett [62], Dontchev, Ganster and Rose [35], Arenas et al. [12], Mukherjee et al. [97], Navaneethakrishnan and Paulraj [102] etc. Recently, Yuksel et al. [164] introduced the concepts of $\delta$-$I$-continuous functions between topological spaces. This concept has lots of scope for further investigation and applications.

The use of the notion of ideal in general topology historically developed along two main directions. The first one is concerned with the study of the local properties of topological spaces and extended global properties. The central concept in this investigation is the compatibility of an ideal with a topology. The second one is the use of ideals to generalize certain properties of topological spaces such as compactness and the separation axioms. In 2006, Zvina [177] generalized the topological space via ideals. We find various developments of ideal topological spaces in recent times in [55, 60, 61, 67, 105, 125].

In 1997, Sarkar [126] introduced the concept of fuzzy ideal topological space. The concept of fuzzy local function has also introduced on utilizing the quasi-nbd. structures for a fuzzy topological space. In the same year, Mahmoud [78] independently presented some of the ideal concepts in the fuzzy topological space and established many of their properties. They have extended some characterization theorems for ideal concepts existing in general topology. These findings generated new fuzzy topologies for any fuzzy topological space.

In 2002, Nasef and Mahmoud [101] introduced $\tau$-co-density in fuzzy ideal, the fuzzy ideal open set, fuzzy ideally restriction etc. Thereafter, Malakar [79] introduced the concepts of fuzzy semi-irresolute and fuzzy strongly irresolute functions. Hatir and Jafari [51] introduced and studied fuzzy semi-$I$-open sets and fuzzy $I$-continuity, fuzzy idealization and studied about a decomposition of fuzzy $I$-continuity. Recently, Yuksel et al. [166] introduced $b$-$I$-continuous functions via fuzzy ideal. Many
researchers are still trying to generalize some concepts of fuzzy topological spaces via fuzzy ideal that are already established.

In the present reviewing of literature on fuzzy ideal topological spaces and different types of continuity, it is observe that there is a gap of mixed fuzzy ideal topological spaces and different types of continuity between mixed fuzzy ideal topological spaces. Our aim is to generalize the concept of different $\delta$-continuous functions via fuzzy ideal in mixed fuzzy ideal topological spaces.

Now, we procure some definitions and results collected from different research papers, journals, books, thesis of various mathematicians all over the world devoted to the development of ideal concepts in topological spaces and fuzzy topological spaces.

**Definitions**

**Definition 1.6.1.** An ideal $I$ on a topological space $(X, \tau)$ such that $I$ is a non-empty collection of subsets of $X$ satisfying the following two conditions:

(i) $A \in I$ and $B \subseteq A$ implies $B \in I$,

(ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

If $I$ is an ideal on $X$, then $(X, \tau, I)$ is called an ideal topological spaces or simply an ideal space.

**Definition 1.6.2.** A local function of $A$ with respect to $\tau$ and $I$ is defined by

$A^* (I, \tau) = \{ x \in X : U \cap A \not\subseteq I, \text{ for every } x \in U \text{ and } U \in \tau \}$, for $A \subseteq X$.

**Definition 1.6.3.** A subset $A$ of an ideal space $(X, \tau, I)$ is said to be $R$-I-open if $A = int (cl^*(A))$, where $cl^*(A) = A \cup A^*(I, \tau)$ describes Kuratowski closure operators $Cl^*(.)$ for a topology $\tau^*(I, \tau)$. A point $x$ in ideal space $(X, \tau, I)$ is called a $\delta$-I-cluster point of $A$ if $A \cap int(cl^*(V)) \neq \emptyset$, for each neighbourhood $V$ of $x$. The set of all $\delta$-I-cluster points of $A$ is called $\delta$-I-closure of $A$ and is denoted by $Cl_{\delta} (A)$. $A$ is said to be $\delta$-I-closed if $Cl_{\delta} (A) = A$.

**Definition 1.6.4.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be pre-open if $A \subseteq int (cl(A))$.

**Definition 1.6.5.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be $\delta$-pre-open if $A \subseteq int (Cl_{\delta} (A))$. 
**Definition 1.6.6.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be pre*-I-open if $A \subseteq \text{int}(\text{cl}_I(A))$.

**Definition 1.6.7.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be $\beta*$-I-open if $A \subseteq \text{Cl}^*(\text{int}(\text{cl}_I(A)))$ and a subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be $\beta*$-t-I-set if $\text{int}(A) = \text{Cl}^*(\text{int}(\text{cl}_I(A)))$.

**Definition 1.6.8.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be $\delta*$-$\beta$-I-open if $A \subseteq \text{Cl}^*(\text{Int}(\text{cl}_I(A)))$. The complement of a $\delta*$-$\beta$-I-open set is said to be $\delta*$-$\beta$-I-closed set.

**Definition 1.6.9.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be $\delta^*$-$\beta$-I-open if $A \subseteq \text{Cl}^*(\text{Int} \text{cl}_I(A))$. The complement of a $\delta^*$-$\beta$-I-open set is said to be $\delta^*$-$\beta$-I-closed set.

**Definition 1.6.10.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be $\delta$-$\beta$-open if $A \subseteq \text{Cl}(\text{Int}(\text{cl}_I(A)))$.

**Definition 1.6.11.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be strongly $t$-I-set if $\text{int}(A) = \text{int}(\text{cl}_I(A))$.

**Definition 1.6.12.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be $B^*$-I-set if there exists a $U \in \tau$ and a $\beta^*$-t-I-set $V$ in $X$ such that $A = U \cap V$.

**Definition 1.6.13.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be $\delta^*$-t-set if $\text{int}(A) = \text{cl}(\text{Int}(\text{cl}_I(A)))$.

**Definition 1.6.14.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be $\delta^*$-$B$-set if there exists a $U \in \tau$ and a $\delta$-$t$-set $V$ in $X$ such that $A = U \cap V$. 


DEFINITION 1.6.15. A function \( f: (X, \tau, I) \to (Y, \sigma) \) is said to be pre-continuity if for every \( V \in \sigma \), \( f^{-1}(V) \) is a pre-open of \( (X, \tau, I) \).

DEFINITION 1.6.16. A function \( f: (X, \tau, I) \to (Y, \sigma) \) is said to be \( \delta \)-almost continuity if for every \( V \in \sigma \), \( f^{-1}(V) \) is a \( \delta \)-pre-open of \( (X, \tau, I) \).

DEFINITION 1.6.17. A function \( f: (X, \tau, I) \to (Y, \sigma) \) is said to be \( \beta^*\)-I-continuity (respectively \( \pre^*\)-I-continuity) if for every \( V \in \sigma \), \( f^{-1}(V) \) is a \( \beta^*\)-I-open (respectively \( \pre^*\)-I-open) of \( (X, \tau, I) \).

DEFINITION 1.6.18. A function \( f: (X, \tau, I) \to (Y, \sigma) \) is said to be \( \delta\beta\)-B-continuity if for every \( V \in \sigma \), \( f^{-1}(V) \) is a \( \delta\beta\)-B-open of \( (X, \tau, I) \).

DEFINITION 1.6.19. A function \( f: (X, \tau, I) \to (Y, \sigma) \) is said to be \( \delta^*\beta\)-I-continuity (respectively \( \delta^*\beta\)-B-continuity) if for every \( V \in \sigma \), \( f^{-1}(V) \) is a \( \delta^*\beta\)-open (respectively \( \delta^*\beta\)-set) of \( (X, \tau, I) \).

DEFINITION 1.6.20. A function \( f: (X, \tau, I) \to (Y, \sigma) \) is said to be strongly \( B\)-I-continuous if \( f^{-1}(V) \) is a strongly \( B\)-I-set in \( (X, \tau, I) \), for every \( V \in \sigma \).

DEFINITION 1.6.21. A function \( f: (X, \tau, I) \to (Y, \sigma) \) is said to be \( B^*\)-I-continuous if \( f^{-1}(V) \) is a strongly \( B^*\)-I-set in \( (X, \tau, I) \), for every \( V \in \sigma \).

DEFINITION 1.6.22. A non-empty collection of fuzzy sets \( \mathcal{J} \) of a set \( X \) is called a fuzzy ideal on \( X \) if (i) \( \mu \in \mathcal{J} \) and \( \nu \subseteq \mu \) implies \( \nu \in \mathcal{J} \) [heredity], (ii) \( \mu \in \mathcal{J} \) and \( \nu \in \mathcal{J} \) implies \( \mu \cup \nu \in \mathcal{J} \) [finite additivity].
**DEFINITION 1.6.23.** Let \((X, \tau)\) be a fuzzy topological space and \(I\) be a fuzzy ideal on \(X\). Let \(A\) be any fuzzy set of \(X\). Then the fuzzy local function \(A^*(I, \tau)\) of \(A\) is the union of all fuzzy points \(x \_i\) such that if \(\mu \in N(x \_i)\), where \(N(x \_i)\) is a nbd. of \(x \_i\) and \(I \in \mathcal{I}\) then there is at least one \(y \in X\) for which \(\mu(y) + A(y) -1 > I(y)\). We shall occasionally write \(A^*\) or \(A^*(I)\) for \(A^*(I, \tau)\) and it will cause no ambiguity.

**DEFINITION 1.6.24.** A fuzzy closure operator \(\psi: \mathcal{F}(X) \rightarrow \mathcal{F}(X)\) is defined by

(i) \(\psi(0_X) = 0_X\),

(ii) \(A \in \mathcal{F}(X) \Rightarrow A \subseteq \psi(A)\),

(iii) \(A, B \in \mathcal{F}(X) \Rightarrow \psi(A \cup B) = \psi(A) \cup \psi(B)\),

(iv) \(A \in \mathcal{F}(X) \Rightarrow \psi(\psi(A)) = \psi(A)\).

Obviously, \(\{A: \psi(A) = A\}\) constitutes a collection of fuzzy closed sets for a fuzzy topology on \(X\).

**DEFINITION 1.6.25.** A fuzzy set \(\mu\) of \(X\) is called fuzzy closed and discrete if \(d^d = 0_X\).

**THEOREM 1.6.1.** *(Yazlik and Hatir [158], Lemma 1.)* Let \((X, \tau, I)\) be an ideal topological space and \(A, B\) be subsets of \(X\).

(i) If \(A \subseteq B\), then \(Cl^*(A) \subseteq Cl^*(B)\).

(ii) \((A \cap B)^* \subseteq A^* \cap B^*\).

(iii) \(A^* = Cl^*(A) \subseteq Cl(A)\).

(iv) \((A \cup B)^* \subseteq A^* \cup B^*\).

(v) If \(U \in \tau\), then \(U \cap A^* \subseteq (U \cap A)^*\).

**THEOREM 1.6.2.** *(Yazlik and Hatir [158], Lemma 2.)* Let \((X, \tau, I)\) be an ideal topological space and \(A, B\) be subsets of \(X\). Then

(i) \(A \subseteq Cl^g(A)\).
(ii) If \( A \subseteq B \), then \( \text{Cl}_A(A) \subseteq \text{Cl}_B(B) \).

**THEOREM 1.6.3. (Yazlik and Hatir [158], Lemma 3.).** Let \((X, \tau, I)\) be an ideal topological space and \( A^* \subseteq A \), then \( A^* = \text{cl}(A^*) = \text{cl}^*(A) = \text{cl}(A) \).

**THEOREM 1.6.4. (Sarkar [126], Theorem 3.4.).** Let \((X, \tau)\) be a fuzzy topological space and \( I_1, I_2 \) be two fuzzy ideals on \( X \). Then for any fuzzy sets \( A, B \) in \( X \),

(i) \( A \subseteq B \Rightarrow A^*(I_1, \tau) \subseteq B^*(I_1, \tau) \).

(ii) \( I_1 \subseteq I_2 \Rightarrow A^*(I_2, \tau) \subseteq A^*(I_1, \tau) \).

(iii) \( A^* = \text{cl}(A^*) \subseteq \text{cl}(A) \).

(iv) \( (A^*)^* \subseteq A^* \).

(v) \( (A \cup B)^* = A^* \cup B^* \).

(vi) \( A \in I_1 \Rightarrow (B \cup A)^* = B^* \).

**THEOREM 1.6.5. (Sarkar [126], Theorem 3.6.).** Let \( f: \mathcal{T}(X) \rightarrow \mathcal{T}(X) \) be a function such that

(i) \( f(0_X) = 0_X \),

(ii) \( f(A \cup B) = f(A) \cup f(B) \),

(iii) \( f(f(A)) \subseteq f(A) \).

Where \( A, B \) are any fuzzy sets of \( X \). Then \( \psi: \mathcal{T}(X) \rightarrow \mathcal{T}(X) \) defined by \( \psi(A) = A \cup f(A) \) is a fuzzy closure operator. Clearly, \( f \) does not necessarily coincide with fuzzy derived set operator in the generated fuzzy topology.

**THEOREM 1.6.6. (Sarkar [126], Theorem 3.8.).** Let \( \tau_1, \tau_2 \) be two fuzzy topologies on \( X \). Then for any fuzzy ideal \( \mathcal{I} \) on \( X \), \( \tau_1 \subseteq \tau_2 \) implies

(i) \( A^*(\tau_2, \mathcal{I}) \subseteq A^*(\tau_1, \mathcal{I}) \), for every \( A \in \mathcal{T}(X) \).

(ii) \( \tau_1^*(\mathcal{I}) \subseteq \tau_2^*(\mathcal{I}) \).

**THEOREM 1.6.7. (Sarkar [126], Theorem 3.9.).** \( A^{d*} \subseteq A^d \) and \( A^{d*} \subseteq A^* \) for all fuzzy set \( A \) of \( X \), where \( A^{d*} \) denotes the fuzzy derived set of \( A \) in \( \tau^* \) fuzzy topology.
In Chapter-VI of the thesis, we introduced the notions of mixed fuzzy ideal topological spaces and fuzzy \( \delta-I \)-continuous function between mixed fuzzy ideal topological spaces and investigated some of their properties.

As the wheel of time roles on, it is hoped that more and more results about mixed fuzzy topologies and mixed fuzzy ideal topologies would explore with its application to justify fruitfulness of such theories.