CHAPTER-V

FUZZY $\delta^*$-CONTINUITY, FUZZY $\delta^*$- ALMOST CONTINUITY IN MIXED FUZZY TOPOLOGICAL SPACES

5.1 INTRODUCTION AND PRELIMINARIES

We procure some existing definitions and results those will be use for establishing the results of this chapter for the shake of completeness.

**DEFINITION 5.1.1.** A fuzzy set $A$ is said to be fuzzy preopen if $A \leq \text{int}(\overline{A})$.

**DEFINITION 5.1.2.** A fuzzy set $A$ in a fuzzy topological space $X$ is said to be fuzzy $\delta$-preopen if $A \leq \text{int}(\overline{\delta(A)})$. The complement of a fuzzy $\delta$-preopen set is called fuzzy $\delta$-preclosed. The set of all fuzzy $\delta$-preopen sets in $X$ will be denoted by $\delta$-$\text{PO}(X)$.

**DEFINITION 5.1.3.** A fuzzy set $A$ in a fuzzy topological space $X$ is called a fuzzy $\delta$-$q$-nbd. of a fuzzy point $x_a$ in $X$ if there exists a fuzzy $\delta$-preopen set $V$ in $X$ such that $x_a \leq V \leq A$.

A fuzzy point $x_a$ in a fuzzy topological space $X$ is called a fuzzy $\delta$-precluster point of a fuzzy set $A$ in $X$ if every fuzzy $\delta$-$q$-nbd. of $x_a$ is $q$-coincident with $A$. The union of all fuzzy $\delta$-precluster points of $A$ is called the fuzzy $\delta$-preclosure of $A$ and will be denoted by $\delta$-$\text{pcl}(A)$.

A fuzzy set $G$ in a fts $X$ is called a fuzzy $\delta$-pre-nbd. of a fuzzy point $x_a$ in $X$ if there exists a fuzzy $\delta$-preopen set $U$ in $X$ such that $x_a \leq U \leq G$. The union of all fuzzy $\delta$-preopen sets in a fts $X$, each contained in a fuzzy set $A$ in $X$, is called the fuzzy $\delta$-preinterior of $A$ and is denoted by $\delta$-$\text{pint}(A)$. 
**DEFINITION 5.1.4.** A function $f: X \rightarrow Y$ is said to be almost continuous if for each $x \in X$ and each open nbd. $V$ of $f(x)$, there exists an open nbd. $U$ of $x$ such that $f(U) \subseteq \text{int}(\text{cl}(V))$.

**DEFINITION 5.1.5.** A mapping $f: X \rightarrow Y$ from a fts $X$ to a fts $Y$ is called fuzzy completely continuous if $f^{-1}(A)$ is fuzzy regularly open in $X$ for any fuzzy open set $A$ in $Y$.

**DEFINITION 5.1.6.** A mapping $f: X \rightarrow Y$ from a fts $X$ to a fts $Y$ is said to be fuzzy strongly continuous if for every fuzzy set $A$ of $X$, $f(cA) \leq f(A)$.

**DEFINITION 5.1.7.** A mapping $f: X \rightarrow Y$ from a fts $X$ to a fts $Y$ is said to be fuzzy $\delta$-almost continuous if for each fuzzy point $x_a$ in $X$ and every nbd. $V$ of $f(x_a)$ in $Y$, $\delta-\text{cl}(f^{-1}(V))$ is a fuzzy nbd. of $x_a$ in $X$.

**DEFINITION 5.1.8.** A function $f: X \rightarrow Y$ from a fts $X$ to a fts $Y$ is said to be fuzzy $\delta^*$-almost continuous if the inverse image of a fuzzy $\delta$-preopen set in $Y$ is a fuzzy $\delta$-preopen set in $X$.

**DEFINITION 5.1.9.** A fts $X$ is said to be fuzzy $\delta$-preregular if for each fuzzy $\delta$-preclosed set $F$ in $X$ and each fuzzy point $x_a$ with $x_a q(1-F)$, there exists a fuzzy open set $U$ and a fuzzy $\delta$-preopen set $V$ such that $x_a qU$, $F \subseteq V$ and $U$ not quasi-coincident with $V$.

**LEMMA 5.1.1.** (Bhattacharyya and Mukherjee [19], Theorem 3.2.). For a function $f: X \rightarrow Y$ the following are equivalent:

(a) $f$ is fuzzy $\delta$-almost continuous.

(b) $f^{-1}(B) \subseteq \text{int}(\delta\text{cl} f^{-1}(B))$, for each fuzzy open set $B$ in $Y$.

(c) $f(cA) \subseteq \text{cl}f(A)$, for every fuzzy $\delta$-open set $A$ in $X$. 


LEMMA 5.1.2. (Das and Baishya [33], Lemma 3.2.). Let $\tau_1$ and $\tau_2$ be two fuzzy topologies on a set $X$. If every $\tau_1$-quasi neighbourhood of $x_j$ is $\tau_2$-quasi neighbourhood of $x_j$, for all fuzzy point $x_j$, then $\tau_1$ is coarser than $\tau_2$.

LEMMA 5.1.3. (Bhattacharyya and Mukherjee [19], Theorem 4.2.). A function $f: X \to Y$ is fuzzy $\delta^*$-almost continuous if and only if for each fuzzy point $x_a$ in $X$ and for each fuzzy $\delta$-preopen q-nbd $V$ of $f(x_a)$ in $Y$, there exists a fuzzy $\delta$-preopen q-nbd $W$ of $x_a$ in $X$ such that $f(W) \leq V$.

LEMMA 5.1.4. (Bhattacharyya and Mukherjee [19], Theorem 4.3.). For a function $f: X \to Y$ the following are equivalent:

(a) $f$ is fuzzy $\delta^*$-almost continuous.

(b) For each fuzzy point $x_a$ in $X$ and fuzzy $\delta$-prenbd. $V$ of $f(x_a)$, $f^{-1}(V)$ is a fuzzy $\delta$-prenbd. of $x_a$.

(c) For each fuzzy point $x_a$ in $X$ and fuzzy $\delta$-prenbd. $V$ of $f(x_a)$, there is a fuzzy $\delta$-prenbd. $U$ of $x_a$ such that $f(U) \leq V$.

(d) For each fuzzy set $B$ in $Y$, $f^{-1}(\delta\text{-pint}B) \leq \delta\text{-pint} f^{-1}(B)$.

(e) For each fuzzy $\delta$-preclosed set $F$ in $Y$, $f^{-1}(F)$ is fuzzy $\delta$-preclosed in $X$.

(f) For each fuzzy set $A$ in $X$, $f(\delta\text{-pcl} A) \leq \delta\text{-pcl} f(A)$.

(g) For each fuzzy set $B$ in $Y$, $\delta\text{-pcl}(f^{-1}(B)) \leq f^{-1}(\delta\text{-pcl} B)$.

We introduced the following definitions in this chapter.

DEFINITION 5.1.10. A fuzzy set $A$ in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$ is said to be fuzzy regular $\delta$-open set (in short FR-$\delta$-open) if $\tau_1\text{-int}(\tau_2\text{-cl} A) = A$. and its complement is said to be fuzzy regular $\delta$-closed set.

DEFINITION 5.1.11. Let $(X, \tau_1(\tau_2))$ be a mixed fuzzy topological space. A fuzzy set $A$ in $X$ is said to be fuzzy preopen set if $A \leq \tau_1\text{-int}(\tau_2\text{-cl}A)$. 

DEFINITION 5.1.12. A fuzzy set \( A \) in a mixed fuzzy topological space \((X, \tau_1(\tau_2))\) is said to be fuzzy regular \( \delta \)-open nbd. of a fuzzy point \( x \), if there exists a fuzzy regular \( \delta \)-open set \( U \) such that \( x \in U \) and \( U \subseteq A \).

DEFINITION 5.1.13. A fuzzy set \( A \) in a mixed fuzzy topological space \((X, \tau_1(\tau_2))\) is said to be fuzzy \( \delta \)-preopen set if \( A \subseteq \tau_1 \text{-int}(\tau_2 \text{-}\delta \text{-cl} A) \). The complement of fuzzy \( \delta \)-preopen set is said to be fuzzy \( \delta \)-preclosed. The set of all fuzzy \( \delta \)-preopen sets in \( X \) is denoted by \( \delta \text{-PO}(X) \). The union of all fuzzy \( \delta \)-preopen sets in a mixed fuzzy topological space \( X \), each contained in a fuzzy set \( A \) in \( X \) is called the fuzzy \( \delta \)-pre-interior of \( A \) and is denoted by \( \delta \text{-pint} A \).

A fuzzy set \( A \) is fuzzy \( \delta \)-preopen if and only if \( A = \delta \text{-pint} A \).

DEFINITION 5.1.14. A fuzzy set \( A \) in a mixed fuzzy topological space \((X, \tau_1(\tau_2))\) is called a fuzzy \( \delta \)-pre-\( q \)-neighbourhood of a fuzzy point \( x \) in \( X \) if there exists a fuzzy \( \delta \)-preopen set \( V \) in \( X \) such that \( x \in qV \) and \( V \subseteq A \).

DEFINITION 5.1.15. A fuzzy set \( A \) in a mixed fuzzy topological space \((X, \tau_1(\tau_2))\) is said to be a fuzzy \( \delta \)-pre-neighbourhood of a fuzzy point \( x \) in \( X \) if there exists a fuzzy \( \delta \)-preopen set \( V \) in \( X \) such that \( x \in V \) and \( V \subseteq A \).

DEFINITION 5.1.16. A fuzzy \( \delta \)-interior of a fuzzy set \( A \) in a mixed fuzzy topological space \((X, \tau_1(\tau_2))\) is denoted by \( \delta \text{-int} A \) and is defined by \( \delta \text{-int} A = 1 \text{-}\delta \text{-cl}(A) \).

DEFINITION 5.1.17. A fuzzy point \( x \) in a mixed fuzzy topological space \((X, \tau_1(\tau_2))\) is called a fuzzy \( \delta \)-precluster point of a fuzzy set \( A \) in \( X \) if every fuzzy \( \delta \)-pre-\( q \)-nbd. of the fuzzy point \( x \) is \( q \)-coincident with \( A \). The union of all fuzzy \( \delta \)-precluster points of \( A \) is called fuzzy preclosure of \( A \) and is denoted by \( \delta \text{-pcl} A \).

DEFINITION 5.1.18. A function \( f: (X, \tau_1(\tau_2)) \rightarrow (Y, \tau_3(\tau_4)) \) is said to be fuzzy \( \delta^* \)-continuous function if \( f^{-1}(V) \) is fuzzy regular \( \delta \)-open set in \( X \), for every fuzzy \( \delta \)-preopen set \( V \) in \( Y \).
**DEFINITION 5.1.19.** A function \( f: (X, \tau_1(\tau_2)) \to (Y, \tau_3(\tau_4)) \) is said to be fuzzy \( \delta^* \)-almost continuous function if \( f^{-1}(V) \) is fuzzy \( \delta \)-preopen set in \( X \), for every fuzzy \( \delta \)-preopen set \( V \) in \( Y \).

**5.2 MAIN RESULTS**

In this section, we state the results of this chapter.

**THEOREM 5.2.1.** If \( f \) is fuzzy \( \delta^* \)-almost continuous function from a mixed fuzzy topological space \( X \) into another mixed fuzzy topological space \( Y \) and \( F \) be any fuzzy set in \( Y \), then \( f^{-1}(1-F) = 1 - f^{-1}(F) \).

**THEOREM 5.2.2.** For a function \( f: X \to Y \) from a mixed fuzzy topological space \( (X, \tau_1(\tau_2)) \) into another mixed fuzzy topological space \( (Y, \tau_3(\tau_4)) \), the following conditions are equivalent:

(i) \( f \) is fuzzy \( \delta^* \)-almost continuous function.

(ii) For each fuzzy point \( x_\lambda \) in \( X \) and fuzzy \( \delta \)-prenbd. \( V \) of \( f(x_\lambda) \), \( f^{-1}(V) \) is fuzzy \( \delta \)-prenbd. of \( x_\lambda \).

(iii) For each fuzzy point \( x_\lambda \) in \( X \) and fuzzy \( \delta \)-prenbd. \( V \) of \( f(x_\lambda) \), there is a fuzzy \( \delta \)-prenbd. \( U \) of \( x_\lambda \) such that \( f(U) \leq V \).

(iv) For each fuzzy set \( B \) in \( Y \), \( f^{-1}(\delta \text{-pint}(B)) \leq \delta \text{-pint}(f^{-1}(B)) \).

(v) For each fuzzy \( \delta \)-preclosed set \( F \) in \( Y \), \( f^{-1}(F) \) is fuzzy \( \delta \)-preclosed in \( X \).

**THEOREM 5.2.3.** For any fuzzy set \( U \) in a mixed fuzzy topological space \( (X, \tau_1(\tau_2)) \), \( \delta \text{-pcl} U \) is the intersection of all fuzzy \( \delta \)-preclosed sets containing \( U \).

**THEOREM 5.2.4.** For any fuzzy subset \( A \) and \( B \) in a mixed fuzzy topological space \( (X, \tau_1(\tau_2)) \), the following results holds:

(a) \( A \leq B \Rightarrow \delta \text{-pcl} A \leq \delta \text{-pcl} B \).

(b) \( A \) is fuzzy \( \delta \)-preclosed if and only if \( A = \delta \text{-pcl} A \).

(c) \( \delta \text{-pcl} A \) is fuzzy \( \delta \)-preclosed in \( X \).

(d) \( \delta \text{-pcl}(\delta \text{-pcl} A) = \delta \text{-pcl} A \).
**THEOREM 5.2.5.** The union of any collection of fuzzy δ-preopen sets in a mixed fuzzy topological space is fuzzy δ-preopen.

**THEOREM 5.2.6.** Finite intersection of fuzzy δ-preopen sets is fuzzy δ-preopen set in mixed fuzzy topological space.

**THEOREM 5.2.7.** For a function $f: X \rightarrow Y$ from a mixed fuzzy topological space $(X, \tau_1(\tau_2))$ into another mixed fuzzy topological space $(Y, \tau_3(\tau_4))$, the following conditions are equivalent.

(i) $f$ is fuzzy δ*-continuous function.

(ii) For each fuzzy point $x_\lambda$ in $X$ and fuzzy δ-pre-nbd. $V$ of $f(x_\lambda)$, $f^{-1}(V)$ is fuzzy regular δ-open nbd. of $x_\lambda$.

(iii) For each fuzzy point $x_\lambda$ in $X$ and fuzzy δ-prenbd. $V$ of $f(x_\lambda)$, there is a fuzzy regular δ-open nbd. $U$ of $x_\lambda$ such that $f(U) \subseteq V$.

(iv) For each fuzzy δ-preclosed set $F$ in $Y$, $f^{-1}(F)$ is fuzzy regular δ-closed set in $X$.

### 5.3 PROOF OF THE RESULTS OF SECTION 5.2

In this section, we prove the results of this chapter.

**PROOF OF THEOREM 5.2.1.** Let $x_\lambda$ be a fuzzy point in $f^{-1}(1-F)$.

\[\text{i.e. } x_\lambda \in f^{-1}(1-F) \Rightarrow f(x_\lambda) \in I-F.\]

\[\Rightarrow f(x_\lambda) \not\subseteq F.\]

\[\Rightarrow x_\lambda \not\subseteq f^{-1}(F).\]

\[\Rightarrow x_\lambda \in I-f^{-1}(F).\]

Hence, $f^{-1}(1-F) \subseteq 1-f^{-1}(F)$.

Similarly, we can show that $f^{-1}(1-F) \supseteq 1-f^{-1}(F)$.

Thus we conclude that $f^{-1}(1-F) = 1-f^{-1}(F)$. 

**Proof of Theorem 5.2.2.** (i)⇒(ii) Consider any fuzzy point \(x_\lambda\) in \(X\) and fuzzy \(\delta\)-prebd. \(V\) of \(f(x_\lambda)\).

Then there exists a fuzzy \(\delta\)-preopen set \(B\) such that \(f(x_\lambda)\in B\) and \(B \leq V\).

Since \(f\) is fuzzy \(\delta^*\)-almost continuous, therefore \(f^{-1}(B)\) is fuzzy \(\delta\)-preopen set in \(X\) containing \(x_\lambda\).

Also we have \(f(x_\lambda)\in B \Rightarrow x_\lambda \in f^{-1}(B)\) and \(B \leq V \Rightarrow f^{-1}(B) \leq f^{-1}(V)\).

Hence, \(f^{-1}(V)\) is fuzzy \(\delta\)-prebd. of \(x_\lambda\).

(ii)⇒(iii) Easy, so omitted.

(iii)⇒(iv) Since \(\delta\)-pint \(B\) is fuzzy \(\delta\)-preopen set in \(Y\), so \(\delta\)-pint \(B\) is \(\delta\)-prebd. of each of its fuzzy points. Therefore, for every fuzzy point \(x_\lambda\) in \(X\) and each fuzzy \(\delta\)-prebd. \((\delta\)-pint \(B\)) of \(f(x_\lambda)\), there exists a fuzzy \(\delta\)-prebd. \((\delta\)-pint \(f^{-1}(B)\)) of \(x_\lambda\) such that

\[
f^{-1}(\delta\text{-pint}(B)) \leq \delta\text{-pint}(f^{-1}(B)).
\]

(iv)⇒(v) Let \(F\) be a fuzzy \(\delta\)-preclosed set in \(Y\).

\(\Rightarrow 1-F\) is fuzzy \(\delta\)-preopen set in \(Y\).

\(\Rightarrow 1-F = \delta\text{-pint}(1-F)\).

\(\Rightarrow f^{-1}(1-F) = f^{-1}(\delta\text{-pint}(1-F))\).

By condition (iv), we get

\[
f^{-1}(1-F) \leq \delta\text{-pint}(f^{-1}(1-F)).
\]

\(\Rightarrow 1-f^{-1}(F) \leq \delta\text{-pint}(f^{-1}(1-F))\) \hspace{1cm} (since \(f^{-1}(1-F) = 1-f^{-1}(F)\)).

\(\Rightarrow 1-f^{-1}(F) \leq \delta\text{-pint}(1-f^{-1}(F))\).

\(\Rightarrow 1-f^{-1}(F)\) is fuzzy \(\delta\)-preopen in \(X\).

\(\Rightarrow f^{-1}(F)\) is fuzzy \(\delta\)-preclosed in \(X\).

(v)⇒(i) Since for every fuzzy \(\delta\)-preclosed set \(F\) in \(Y\), \(f^{-1}(F)\) is fuzzy \(\delta\)-preclosed set in \(X\).

\(\Rightarrow 1-f^{-1}(F)\) is fuzzy \(\delta\)-preopen set in \(X\).

\(\Rightarrow f^{-1}(1-F)\) is fuzzy \(\delta\)-preopen set in \(X\).

Hence for each fuzzy \(\delta\)-preopen set \((1-F)\) in \(Y\), \(f^{-1}(1-F)\) is fuzzy \(\delta\)-preopen set in \(X\).

Therefore \(f\) is fuzzy \(\delta^*\)-almost continuous function.
PROOF OF THEOREM 5.2.3. Let $V= \bigwedge \{ A \in I^X : A \text{ is } \delta\text{-preclosed and } U \leq A \}$.

We need to show that $V= \delta\text{-pcl}U$.

Let $x_{\lambda} \in V$ be any fuzzy point in $X$.

We assume that $x_{\lambda} \notin \delta\text{-pcl}U$. Then $x_{\lambda}$ is not a fuzzy $\delta$-precluster point of $U$ and so there exist fuzzy $\delta\text{-pre-q-nbd.} B$ of $x_{\lambda}$ such that $UqB$ ($U$ is not quasi-coincident with $B$).

Therefore there exists a fuzzy $\delta$-preopen set $C$ in $X$ such that $x_{\lambda} \in C$ and $C \leq B$.

But $UqB \Rightarrow UqC$.

$\Rightarrow U \leq 1-C$.

Since $C$ is fuzzy $\delta$-preopen set, so $1-C$ is fuzzy $\delta$-preclosed set and $V \leq (1-C)$ as $V$ is the smallest fuzzy $\delta$-preclosed set containing $U$.

Hence, $x_{\lambda} \notin V$. This leads to a contradiction.

Thus, we must have $x_{\lambda} \in \delta\text{-pcl}U$ and so $V \leq \delta\text{-pcl}U$.

Conversely, we shall show that $V \geq \delta\text{-pcl}U$.

Suppose $x_{\lambda} \in \delta\text{-pcl}U$ but $x_{\lambda} \notin V$.

If $x_{\lambda} \notin V$, then there exist a fuzzy $\delta$-preclosed set $F$ in $X$ containing $U$ such that $x_{\lambda} \notin F$.

$\Rightarrow x_{\lambda} \in 1-F$ and $Uq(1-F)$. This leads to the contradiction that $x_{\lambda}$ is fuzzy $\delta$-precluster point of $U$.

Therefore, $x_{\lambda} \in V$ and so we get $V \geq \delta\text{-pcl}U$.

Thus $V = \delta\text{-pcl}U$, i.e. for any fuzzy set $U$ in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$, $\delta\text{-pcl}U$ is the intersection of all fuzzy $\delta$-preclosed sets containing $U$.

PROOF OF THEOREM 5.2.4. (a) Let $A$ and $B$ be two fuzzy subsets of a mixed fuzzy topological space $X$ and $A \leq B$.

Let $x_{\lambda} \in \delta\text{-pcl}A$. Then there exist a fuzzy $\delta\text{-pre-q-nbd.} U$ of $x_{\lambda}$ such that $AqU$ and $U \leq A$.

Since $A \leq B$. Therefore $UqA$ and $A \leq B \Rightarrow UqB$.

Hence, we get a fuzzy $\delta\text{-pre-q-nbd.} U$ of $x_{\lambda}$ such that $BqU$ and $U \leq B$.

i.e. $x_{\lambda}$ is a fuzzy $\delta$-precluster point of $B$.

Therefore, $x_{\lambda} \in \delta\text{-pcl}B$ and consequently we get $\delta\text{-pcl}A \leq \delta\text{-pcl}B$.

(b) Let $A$ is fuzzy $\delta$-preclosed set in $X$.

Clearly, $\delta\text{-pcl}A \leq A$.

Since $A$ is fuzzy $\delta$-preclosed set.
\( \Rightarrow A \) contains all its \( \delta \)-precluster points.
\( \Rightarrow A \leq \delta \text{-} pclA. \)
Hence \( A = \delta \text{-} pclA. \)
Converse part is straightforward from definition of fuzzy \( \delta \)-preclosed set.

\((c)\) This result follows directly from Theorem 5.2.3.

\((d)\) From \((b)\) and \((c)\) it is obvious.

**PROOF OF THEOREM 5.2.5.** Let \((X, \tau_1(t_2))\) be a mixed fuzzy topological space.
Consider the collection \(\{A_a: a \in \Delta\}\) of fuzzy \(\delta\)-preopen set in \(X\).
We need to show that \(\bigcup_{a \in \Delta} A_a\) is fuzzy \(\delta\)-preopen set.
Since each \(A_a\) is fuzzy \(\delta\)-preopen set, so by definition of fuzzy \(\delta\)-preopen set
\(A_a \leq \tau_1\text{-}int(\tau_2\text{-}clA_a)\) for each \(a \in \Delta\).
Now, \(\bigcup_{a \in \Delta} A_a \leq \bigcup_{a \in \Delta} (\tau_1\text{-}int(\tau_2\text{-}clA_a))\)
\(\leq \tau_1\text{-}int(\bigcup_{a \in \Delta} (\tau_2\text{-}clA_a))\)
\(\leq \tau_1\text{-}int(\tau_2\text{-}cl(\bigcup_{a \in \Delta} A_a)).\)
Hence, \(\bigcup_{a \in \Delta} A_a\) is a fuzzy \(\delta\)-preopen set in \(X\). Thus, arbitrary union of fuzzy \(\delta\)-preopen sets is fuzzy \(\delta\)-preopen.

**PROOF OF THEOREM 5.2.6.** Suppose \(A_n\) is a fuzzy \(\delta\)-preopen set in a mixed fuzzy topological space \((X, \tau_1(t_2))\).
Then by definition of fuzzy \(\delta\)-preopen set we have \(A_n \leq \tau_1\text{-}int(\tau_2\text{-}clA_n)\).

We need to show that
\[ \bigwedge_{n=1}^{m} A_n \leq \tau_1\text{-}int(\tau_2\text{-}cl(\bigwedge_{n=1}^{m} A_n)). \]
Since
\[ \bigwedge_{n=1}^{m} (\tau_1\text{-}int(\tau_2\text{-}clA_n)) = \tau_1\text{-}int(\bigwedge_{n=1}^{m} (\tau_2\text{-}clA_n)). \]
(Since \(int A \land int B = int(A \land B)\), for any fuzzy sets \(A\) and \(B\) in a fuzzy topological space)

\[ \Rightarrow \bigwedge_{n=1}^{m} (\tau_1\text{-}int(\tau_2\text{-}clA_n)) \leq \tau_1\text{-}int(\tau_2\text{-}cl(\bigwedge_{n=1}^{m} A_n)) \quad \text{-------------------(5.3.1.)} \]

(since \(\overline{A \land B} \leq \overline{A} \land \overline{B}\) and \(\text{int } A \leq \text{int } B\)).

Also
\[ \bigwedge_{n=1}^{m} A_n \leq \bigwedge_{n=1}^{m} (\tau_1\text{-}int(\tau_2\text{-}clA_n)) \quad \text{-------------------(5.3.2.)} \]
(since \(A \leq B \Rightarrow \text{int } A \leq \text{int } B\)).
From (5.3.1) and (5.3.2) we can concluded that
\[ m \bigwedge_{n=1}^{m} A_n \leq (\tau_1\text{-int}(\tau_2\text{-}\delta\text{-cl}(m \bigwedge_{n=1}^{m} A_n))). \] Hence, finite intersection of fuzzy \( \delta \)-preopen sets is fuzzy \( \delta \)-preopen set.

**NOTE 5.3.1.** From the definitions of fuzzy regular \( \delta \)-open set and fuzzy \( \delta \)-preopen set, it is clear that a fuzzy regular \( \delta \)-open set is always fuzzy \( \delta \)-preopen set. However, the converse may not be necessarily true in general.

The following is an example that a fuzzy \( \delta \)-preopen set need not be fuzzy regular \( \delta \)-open set.

**EXAMPLE 5.3.1.** Let us consider a non-empty set \( X=\{x, y\} \) and consider the following fuzzy sets in \( X \).
\[ A = \{(x, .7), (y, .3)\} \] and \( B = \{(x, .3), (y, .7)\} \). Then the collection of fuzzy sets \( \tau_1 = \{0', 1', B\} \) and \( \tau_2 = \{0', 1', A\} \) will form two fuzzy topologies on \( X \).

Now we construct a mixed fuzzy topology on \( X \) from these fuzzy topologies \( \tau_1 \) and \( \tau_2 \) and we get \( \tau_1(\tau_2) = \{0', 1', A\} \).

Consider the fuzzy set \( C \) in \( X \) defined by \( C(x) = 0.3 \) and \( C(y) = 0.7 \), then \( \tau_2\text{-}\delta\text{-cl}(C) = \bigwedge\{F: F \text{ is } \delta\text{-closed and } C \leq F\} = 1' \).

Also \( \tau_1\text{-int}(1') = 1' \). Hence \( 1' \text{-int}(\tau_2\text{-}\delta\text{-cl}(C)) \Rightarrow C \) is fuzzy \( \delta \)-preopen set.

But \( C \) is not fuzzy regular \( \delta \)-open set.

**PROOF OF THEOREM 5.2.7. (i) \Rightarrow (ii) Let \( x_\lambda \) be a fuzzy point in \( X \) and \( V \) be the fuzzy \( \delta \)-preenbd. of \( f(x_\lambda) \).

Since \( f \) is fuzzy \( \delta^* \)-continuous function and \( V \) is fuzzy \( \delta \)-preenbd. of \( f(x_\lambda) \).

Therefore there exists a fuzzy \( \delta \)-preopen set \( A \) in \( Y \) such that \( f(x_\lambda) \in A \) and \( A \leq V \).

Since \( f \) is fuzzy \( \delta^* \)-continuous function, therefore \( f^{-1}(A) \) is fuzzy regular \( \delta \)-open set in \( X \).

Now \( f(x_\lambda) \in A \Rightarrow x_\lambda \in f^{-1}(A) \) and \( A \leq V \Rightarrow f^{-1}(A) \leq f^{-1}(V) \).

Hence \( f^{-1}(V) \) is fuzzy \( \delta \)-regular open set in \( X \).

(ii) \Rightarrow (iii) Straightforward.

(iii) \Rightarrow (iv) Let \( F \) be a fuzzy \( \delta \)-preclosed set in \( Y \). Then \( 1-F \) is fuzzy \( \delta \)-preopen set in \( Y \). Let \( x_\lambda \) be a fuzzy point in \( X \) such that \( f(x_\lambda) \in 1-F \).
Then $1-F$ itself fuzzy $\delta$-prenbd. of $f(x_i)$. Hence, by the given condition ($iii$), we have a fuzzy $\delta$-regular-nbd. $V$ of $x_i$ such that $f(V) \leq (1-F)$. This implies that

$$V \leq f^{-1}(1-F) \Rightarrow V \leq 1- f^{-1}(A) \Rightarrow f^{-1}(A) \leq V.$$ 

Hence $f^{-1}(A)$ is fuzzy $\delta$-regular open set in $X$.

$(iv) \Rightarrow (i)$ Easy, so omitted.
This completes the proof.

**NOTE 5.3.2.** From the above definitions of fuzzy $\delta^*$-almost continuity and fuzzy $\delta^*$-continuity, it is clear that fuzzy $\delta^*$-continuous functions between mixed fuzzy topological spaces is always fuzzy $\delta^*$-almost continuous function. But, every fuzzy $\delta^*$-almost continuous function may not be fuzzy $\delta^*$-continuous. This will follows from the following example:

**EXAMPLE 5.3.2.** Consider a non-empty set $X=\{x, y\}$ and consider the following fuzzy sets in $X$.

$A=\{(x, .7), (y, .3)\}$ and $B=\{(x, .3), (y, .7)\}$. Then the collection of fuzzy sets $\tau_1 = \{0', 1', B\}$ and $\tau_2 = \{0', 1', A\}$ are two fuzzy topologies on $X$ and from these two topologies we get the mixed fuzzy topology $\tau_1(\tau_2) = \{0', 1', A\}$ on $X$.

Consider another mixed fuzzy topology on $X$ defined as follows:

Consider the following fuzzy sets in $X$

$A_1=\{(x, .2), (y, .8)\}, A_2=\{(x, .2), (y, .2)\}, A_3=\{(x, .8), (y, .2)\}$ and $A_4=\{(x, .8), (y, .8)\}$. Then the collection $\tau_3 = \{0', 1', A_1, A_2, A_3, A_4\}$ will form a fuzzy topology in $X$.

Also, consider the fuzzy sets $B_1 = \{(x, .3), (y, .7)\}, B_2 = \{(x, .7), (y, .3)\}$, $B_3 = \{(x, .3), (y, .3)\}$ and $B_4 = \{(x, .7), (y, .7)\}$ in $X$.

Then the collection of fuzzy sets $\tau_4 = \{0', 1', B_1, B_2, B_3, B_4\}$ will form a fuzzy topology on $X$ and from these two topologies we can form the mixed fuzzy topology $\tau_3(\tau_4) = \{0', 1'\}$.

Now, consider the identity function $E: (X, \tau_1(\tau_2)) \rightarrow (Y, \tau_3(\tau_4))$, then $E$ is fuzzy $\delta^*$-almost continuous function but not fuzzy $\delta^*$-continuous.

Consider the fuzzy set $C = \{(x, .7), (y, .5)\}$, then $C$ is fuzzy $\delta$-preopen set in $X$ and $E^{-1}(C) = \{(x, .7), (y, .5)\}$, the inverse image under $E$, is not fuzzy regular $\delta$-open. Hence $E$ is not fuzzy $\delta^*$-continuous function.