CHAPTER – II
CALCULUS BEFORE NEWTON AND LEIBNIZ

2.1: INTRODUCTION

The history of Greek mathematics begins in the 6th century B.C. Historians traditionally place the beginning of the Greek mathematics proper to the age of Thales of Miletus (624-548 B.C). Thales and Pythagoras (570BC-) were the leading mathematicians of this age. Thales who lived in the first half of the 6th century B.C. proved the following theorems:

1. A diameter of a circle divides it into two equal parts.
2. The base angles of an isosceles triangle are equal.
3. The vertical angles formed by two intersecting straight lines are equal.
4. The angle-side-angle congruence theorem for triangles.
5. Angle in a semi circle is a right angle [26].

Thales and Pythagoras gave an abstract character to Greek mathematics. As for example the Greek considered the ratio \( \frac{a}{b} \) between two numbers \( a \) and \( b \) as a relationship between these and not the fraction \( \frac{a}{b} \) as a single entity. The journey of factual calculus begins with Pythagoras (569-500 B.C). Though geometry was the central theme of the Greek mathematics, yet arithmetic was the key attraction of Pythagoras. Based on this, the early Pythagoreans developed an elementary theory of proportionality.

2.2: INCOMMENSURABILITY OF LENGTH

Pythagoreans thought that ‘everything is number’. To them, numbers were positive integrals. They were of the view that our cosmos was created and governed by understandable and definite numerical principles. The Pythagoreans were also of the view that the derivation of the multiplicity of creations in the world from a unique unity is identical to the derivation of numbers from the numerical unit. They opined that material world is imitating the mathematical world.
During the later part of 5th century B.C. it was discovered that there exists incommensurable* line segments. Pythagoras thought that every point on a straight line can be represented by a rational number. Trouble starts when a point P situated on a line as indicated below to be associated with a number equal to $\sqrt{2}$.

The ratio of the edge and the diagonal of a unit square is not equal to the ratio of two integers. The square on the diagonal of a unit square has area 2 whereas $\sqrt{2}$ is not commensurable.

![Fig. 2.1](image)

\[
PQ^2 = PR^2 + RQ^2 = 1^2 + 1^2 = 1 + 1 = 2
\]

The discovery had raised questions about the idea of measurement and calculation of geometry as a complete one.

The Pythagorean School around 500 B.C. proved with the help of ‘reductio ad absurdum’ method that there is no rational number whose square is 2. The ‘reductio ad absurdum’ method is that which shows an absurd conclusion from supposing the theorem is false.

For this reason, the Greeks did not consider geometric magnitudes to be demonstrated by numbers at all. They did not allow numbers to play the key role of their mathematics. To them, numbers were discrete integral quantities having

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* Incommensurable length- The length which cannot be measured as integral multiples of segments of its length, hence the ratio between their lengths is not equal to the ratio of two integers.
arithmetic properties. But the geometrical magnitudes were continuous spatial objects.

2.3: IRRATIONALITY OF $\sqrt{2}$ AND THE INFINITE

The phenomenon of incommensurable lengths showed that geometric magnitudes have some sort of inherently continuous character which cannot be avoided. So the Greeks thought that geometric magnitudes cannot be used freely in algebraic computations.

The notion of irrational numbers such as $\sqrt{2}, \sqrt{3}, \pi$ took a distinguishing form throughout the Sulva period (600-300 BC) of Hindu mathematics in India. Though there are nine Sulvasutras on evidence, but only four of them are known to us. These are (i) Boudhayana (ii) Apastamba (iii) Kaatyayana and (iv) Maanava.

George Buhler is of the opinion that the slokas 9,10,11 of Boudhayana Sulvasutra deal with surd numbers dvikarani ($\sqrt{2}$), trikarani ($\sqrt{3}$) and tritiyakarani ($\frac{1}{\sqrt{3}}$) through geometrical construction of squares.

Irrationals were introduced in computation around 1200 AD in Europe by Leonardo of Pisa. To make ground work for irrationals, familiarity of mathematics of the infinite was mandatory. Till then, no mathematics of the infinite was exposed as it was thought to be incomprehensible, indescribable and unthinkable. Modern mathematics says that irrationals can be approximated by rationales.

2.4: THE INFINITE AND THE GREEK MATHEMATICS

The Infinite and unintelligible to aperion (not limited) that was propounded by Anaximander of Miletus (610-546 BC) was thought to be present in the continuous or unbroken magnitudes of geometry. To him, discrete numbers are unable to represent the infinite. The mathematical “horror of infinity” arose from the time of Zeno. In those days, popular Pythagorean idea was that a line consists of points and time is composed of a series of detached moments. Zeno pointed out the absurdity of “infinite divisibility” of space and time by his famous paradoxes that involved application of infinite processes to geometry.
The Greeks considered arithmetic and geometry to be essentially distinct. Their reflection was that the irrational quantities might disprove their philosophy.

This division of mathematics manifests the fundamental division between finite (arithmetic of whole numbers) and infinite (geometrical continuum).

In respect of $\sqrt{2}$ Michael Stifel wrote in 1544, “this cannot be a true number which is of such a nature that it lacks precision. ………, so an irrational number is not a true number, but lies hidden in a kind of infinity” [42].

2.5: LIMITATION OF PYTHAGOREANS

Numbers to the Pythagoreans are termed today as positive integers. They could not imagine numbers beyond the positive integers. Even Pythagoreans had little idea that the theorem of Pythagoras could be relevant in different contexts. They have no idea that the algebraic equation $x^2 + y^2 = a^2$ may remind one of a geometric curve known as circle of radius $a$, as they developed little abbreviated symbolism. The interconnection between geometry and algebra was first observed only by two French mathematicians Pierre de Fermat (1601-1665) and Rene Descartes (1596-1650).

If the Greek would have emphasized measurement by numbers and not preferred geometric approach of mathematics by excluding algebraic aspects, the development of calculus would have been vigorous!

2.6: NICHOLAS OF CUSA AND THE INFINITE

Nicholas of Cusa (1401-1464) was a German cardinal, philosopher and theologian. The logic of Cusa opened the access for the discussion and debate for the new field of mathematics of the Infinite. To him, our finite mind is not competent enough to know the infinite. Nicholas called infinity “absolute”, it must be perceived in a full and unrestrained sense. The actual infinite would be known in a transcendental domain through mystical insight only. Mystics called it Brahman, God, Allah, Tao etc. He believed that there is no comparative relation between the infinite and the finite. He argued that there cannot be an opposite to the indescribable infinite. The Absolute Infinite includes all and encompasses all. It can be understood by considering an arc of a sequence of circles of larger and larger diameter which coincides with a line at infinite.
His doctrine of opposites of the finite and the infinite is known as *coincidence of opposites*. It is the idea that all kinds of multiplicity in the finite world become one in the infinite. It was used not only for studying theology or philosophy, but for mathematics also. For example,

(i) A curve and a straight line are opposites, but if the radius of a circle is made infinitely long, then its curved circumference coincides with a straight line (Fig-2.2).

(ii) If the number of sides of a polygon inscribed in a circle are increased from a square to pentagon, to hexagon and so on, the area of the polygon will become closer to that of the circle. If the number of sides are increased to infinity, the polygon coincides with the circle [60].

Having intensively studied on infinite, he is considered to be an important primogenitor of infinitesimal calculus. Infinitesimal calculus involves absurd manipulations of *infinite sum of infinitesimal quantities* that gives finite result strangely. Calculus was deep and sudden, an important mathematical development with paradoxes of infinite at its heart.

### 2.7: AREA PROBLEM

This problem dates back to ancient Greeks. The area of figures with boundaries as straight sides were determined by fitting squares of a standard size into the figures. And this posed the question as to what would be the area of a region bounded by curved sides?

#### 2.7.1: Method of Exhaustion

The first problem of finding area of a curvilinear figure confronted by the Greek mathematicians was the computation of the area of a circle. Hippocrates
of Chios (470-410 B.C) proved that the ratio of the areas of two circles is equal to the ratio of the squares of their diameters (or radius). He proved it by drawing similar regular polygons inside two concentric circles [Fig.2.3]. In each case, the ratio of the areas of the inscribed polygons is equal to the ratio of the squares of the radii of the two circles. Exhausting the areas of the circles by increasing the number of sides of the polygons indefinitely the same result will be true. But Hippocrates had no idea of limit sufficient to settle the infinitesimal argument. Also, Antiphon, a contemporary of Plato exhausted a circle with inscribed polygons by drawing a square, a octagon, a 16-gon, a 32-gon etc. He was aware that this process could be continued indefinitely in order to find the approximate value of the area of a circle, and not the precise value.

2.7.2: Eudoxus’ Principle

Later the theory was made mathematically precise by Eudoxus. To find the area \( a(S) \) of a curvilinear figure \( S \), Eudoxus attempted by means of a sequence \( P_1, P_2, P_3 \ldots \ldots \) of polygons that exhaust the area. His principle says-two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continuously, then some magnitude would be left-less than the lesser magnitude set out.

The term ‘exhaustion’ was first used in 1647 by Gregoire de Saint-Vincent in *Opus geometricum quadratura circuli et sectionum* to describe the process devised by Eudoxus. Later Archimedes exploited the technique with his
praiseworthy skill and mastery in different ways for determination of area, volume, surface etc. of different curvilinear figures.

Antiphon, Plato, Leucippus, Democritus all made contributions to the method of exhaustion. But the scientific basis was founded by Eudoxus (408-355 BC) in about 370 B.C.

2.7.3: Area of a circle

Eudoxus’ principle provided the basis for a rigorous proof of the theorem “Circles are to one another as the squares on the diameter” [Elements- XII ]

\[ \frac{a(C_1)}{a(C_2)} = \frac{d_1^2}{d_2^2} \]

Where \( a(C_1), a(C_2) \) are the areas and \( d_1, d_2 \) are the diameters of the circles.

\[ \frac{a(C_1)}{r_1^2} = \frac{a(C_2)}{r_2^2} \]

Where \( r_1 \) and \( r_2 \) are radii of the circles.

However, the Greek could not proceed further. The ancient Greeks calculated not with quantities and equations, instead, they used proportionalities to express relationship between quantities. The Greeks were of the view that this relation was in fact a proportion between ratios of areas, rather than a numerical equality. So they could not introduce \( \pi \) at this moment.

2.7.4: Archimedes’ use of method of exhaustion

Archimedes (287-212 B.C) was very much involved to conical shapes like parabola, ellipse and hyperbola. He started with parabola since it had a lot of applications. A parabolic mirror can reflect light coming from a very great distance and concentrate it at a point namely the focus (which means fireplace).

Archimedes used the Method of Exhaustion to prove the followings:

1. The area bounded by the intersection of a line and a parabola is \( \frac{4}{3} \) times the area of the triangle having the same base and height.
2. The area of an ellipse is proportional to a rectangle having sides equal to its major and minor axes.
3. The volume of a sphere is 4 times that of a cone having a base and height of the same radius.
(4) The volume of a cylinder with a height equal to its diameter is $\frac{3}{2}$ that of a sphere having the same diameter.

(5) The area bounded by one spiral rotation and a line is $\frac{1}{3}$ that of a circle having a radius equal to the length of line segment. Archimedes’ works on area and volume can be arranged in the following order:

(i) Measurement of a circle
(ii) Quadrature of a parabola
(iii) On the sphere and cylinder
(iv) On spirals
(v) On conoids (Paraboloid and Hyperboloid)
(vi) The Method

Method of exhaustion was proved to be a technique of amazing power in the first five of these above. But Archimedes failed to find areas of elliptical and hyperbolic sectors. These computations become possible after the development of integral calculus only.

Archimedes developed the “method of compression” also. By this method, the area of a circle was compressed between the areas of inscribed and circumscribed polygons [Fig.2.4], the volume of a paraboloid was compressed between the volumes of inscribed and circumscribed cylinders [Fig.2.5(a) & (b)]. The method was extensively used by Archimedes (287-212 BC) to derive arithmetic formulae for areas of other geometric figures also.

Fig.2.4: Exhausting the area of a circle
Archimedes’ efforts were dependent on the shape of the curves and solids. It required a lot of algebraic skills; because of non-availability of any kind of algorithms at that time. The absence of sufficient knowledge of algebra made Archimedes to refrain himself from finding areas and volumes of many other curves and solids. He used cumbersome ‘double reductio ad absurdum’ method to find the areas and volumes.
2.7.6: Euclid’s use of Method of exhaustion

To the Greeks, the age old concept of finding area of planer regions and volumes of solids were important. Appearing in Book XII of Euclid’s *Elements*, the method was proved to be fundamentally critical in the following centuries. Euclid used the Method of Exhaustion for finding the relative volumes of cones, pyramids, cylinders and spheres in the following six propositions of Book XII of *Elements*.

(1) The area of a circle is proportional to the square of its diameter (*Prop – 2*)
(2) The volumes of two tetrahedrons of the same height are proportional to the areas of their triangular bases. (*Prop – 5*)
(3) The volume of a cone is a third of the volume of the corresponding cylinder having the same base and height. (*Prop – 10*)
(4) The volume of a cone (or cylinder) of the same height is proportional to the area of the base. (*Prop – 11*)
(5) The volume of a cone (or cylinder) similar to another is proportional to the cube of the ratio of the diameters of the bases. (*Prop – 12*)
(6) The volume of a sphere is proportional to the cube of its diameter. (*Prop – 18*)

The concepts of infinity and limit were absorbed in Eudoxus’ principle. The Greek mathematicians used magnitudes that can be made as large or as small in place of infinitely large or small.

It only provides a technique for calculating certain continuous magnitudes and gives rise to approximate calculations.

2.8: QUADRATURE

The problem of quadrature was a problem for the ancient mathematicians also. The problem exhibits a square of the same area of a given figure. To find a quadrature of a rectangle of length \( a \) and breadth \( b \), one must have a square of side equal to \( \sqrt{ab} \). To find a quadrature of a triangle, we must have a rectangle of same area equal to that of a triangle and then a square equal to the area of the rectangle.
Calculus before Newton and Leibniz

The Greek mathematicians took the challenge of ‘squaring of a circle’ and ‘rectification of a circle’. Archimedes was able to perform a quadrature by dividing it into small rectangular strips of two types. The

(i) first gives the area of the rectangles inscribed in the circle
(ii) the other gives the area of the rectangles circumscribing the Circle.

![Fig. 2.8](image)

Difference between the two areas becomes smaller and smaller when the number of rectangles is increased or the width of the rectangles is decreased. However, this of course is an early example of integration.

2.8.1: The quadrature of segment of a parabola

The quadrature of a circle or segment of a circle was successfully found out by the earlier mathematicians. But nobody attempted to find the area of segments of a parabola. Archimedes was the first to attempt to find the area of a segment of a parabola using the method of exhaustion with high standard of originality. It was clearly mentioned in the preface of the Quadrature of the parabola.

To find the area of a parabolic segment, Archimedes considered a certain inscribed triangle, the base of which is a given chord of the parabola and the third vertex is the point of tangency such that the tangent at the third vertex is parallel to the chord.

![Fig. 2.9](image)
The following facts related to parabolic segment APB were known during Archimedes’ time

(i) Tangent line at P is parallel to AB.

(ii) The straight line through P parallel to the axis intersects the base AB at its midpoint M.

(iii) Every chord parallel to the base AB is bisected by the diameter PM.

(iv) With the notation in Fig.2.9, \( \frac{PV}{PM} = \frac{PS^2}{MB^2} \) [26]

Applying these, Archimedes found the area of a parabolic segment

\[
= a + \frac{a}{4} + \frac{a}{4^2} + \cdots \cdots , + \frac{a}{4^n} \quad \ldots \ldots \quad (1)
\]

where \( a \) is the area of the triangle APB.

To determine the sum of the series, Archimedes used the following method-- given a series of areas A,B,C,D, ……, Y, Z such that A is the greatest and each is equal to 4 times the next i.e

\[
A=4B, \quad B=4C, \quad C=4D, \quad \ldots \ldots , Y=4Z.
\]

Now,

\[
B+C+D+ \ldots \ldots +Y+Z+B/3+C/3+ \ldots \ldots +Y/3+Z/3 \ldots \ldots (1)
\]

\[
= 4B/3+4C/3+4D/3+ \ldots \ldots +4Y/3+4Z/3
\]

\[
= (4B+4C+4D+ \ldots \ldots +4Y+4Z)/3
\]

\[
= A/3+ (B+C+ \ldots \ldots +Y)/3
\]

On the other hand,

\[
B/3+C/3+ \ldots \ldots +Y/3 = (B+C+ \ldots \ldots +Y)/3 \ldots \ldots (2)
\]

\( (1) -(2) \) gives

\[
B+C+D+ \ldots \ldots +Y+Z+Z/3 = A/3
\]

gives

\[
A+B+C+D+ \ldots \ldots +Z+Z/3 = 4A/3
\]

i.e \( A \left( 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots \cdots \cdots \cdots \right) = 4A/3 \)

\[
1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots \cdots \cdots \cdots = \frac{4}{3}
\]

*This is the first known example of summation of infinite series.*
2.8.2: Visual demonstration of sum of the above series

![Unit square diagram](image)

Fig.2.10: Unit square

Sum of areas of white squares = \(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} \ldots\)

Sum of areas of black squares = \(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} \ldots\)

Sum of areas of grey squares = \(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} \ldots\)

The area of the biggest square = \(3 \times (\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} \ldots) = 1\)

Hence, \(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} \ldots = \frac{4}{3}\)

2.8.3: Comments

Here, we have observed that above infinite series is a convergent series (geometric series with common ratio \(\frac{1}{4} < 1\)) whose sum is \(\frac{1}{1 - \frac{1}{4}} = \frac{4}{3}\). Convergent series permits one to get a finite sum of infinite number of summands. Here, luckily Archimedes faced no difficulty in extending sum of finite number of terms to infinity in this series. Geometric series played a crucial role to the development of calculus. The first example of a geometric series yet discovered is due to the Babylonians (2000 BC). In Egyptian mathematics, the first problem of geometric series is found in Ahmes Rhind papyrus (1500 BC). It is as follows:
Euclid gave one problem that may be expressed as

\[
\frac{a_{n+1} - a_1}{a_n + a_{n-1} + \cdots + a_1} = \frac{a_2 - a_1}{a_1} \quad \text{which amounts to saying} \quad \frac{ar^n - a}{s_n} = \frac{ar - a}{a}.
\]

During medieval times, Bhaskara (1150) gave a problem like this: “A person gave a mendicant a couple of cowry shells first; and promised a two fold increase of the alms daily. How many nischas does he give in a month?”

[1 nischas = 16 × 6 × 4 × 20 cowry shells]

It is equivalent to finding \(2 + 2^2 + 2^3 + \cdots + 2^{31}\).

Walli’s method of finding area under a hyperbola led Newton to deal with geometric series. The convergence of geometric series reveals that a sum involving an infinite number of summands can be finite. But Archimedes used ‘reductio ad absurdum’ method to avoid the summation of infinite series. Zeno’s assumption was that infinite number of finite steps cannot be finite. For this reason, Zeno’s paradoxes could not be resolved.

2.9: SOLIDS OF REVOLUTION

Archimedes used the method of compression that is most closely to the construction underlying the modern definite integral in his treatise “On Conoids and Spheroids”. A conoid is what we would call a paraboloid or hyperboloid of revolution and a spheroid is an ellipsoid of revolution.

He showed the volume of a paraboloid of revolution \(P\) inscribed in a cylinder \(C\) with radius \(R\) and height \(H\)

\[
\nu(P) = \frac{1}{2} \times \text{volume of the circumscribed cylinder}
\]

\[
= \frac{1}{2} \pi R^2 H
\]
To prove this result, he constructed $n$-slices of equal thickness $h = \frac{H}{n}$ that can be proved by modern integral calculus also.

Archimedes used this slicing construction method for finding the volumes of segments of ellipsoids and hyperboloids of revolution cut off by planes that are not necessarily perpendicular to the axes of the figures by employing a common general modus-operandi—a logical step to the formulation of a general concept of integration.

Dividing volumes and areas into the sum of a large number of individual pieces, he basically used the limiting process of integral calculus 2000 years before Newton (1643-1727) and Leibniz (1646-1716).

Works of Archimedes hardly survived during the so called Dark Ages (500AD-1000 A.D). Almost all translations of Archimedes’ work were ultimately vanished except Latin translation of *The Method*, without a trace in the 16th century. But the idea he planted, gave a glimpse to what may be called the technique of calculus. Many of his solutions can be viewed as computations of definite integral of the form $\int_{c}^{d}(ax + bx^2) \, dx$.

**2.10: FAILURE OF THE GREEKS**

The failure of the Greeks in inventing integral calculus can be thought mainly for two reasons:

(i) their discomfort with the concept of infinity.

(ii) their lack of sufficient knowledge of algebra due to lack of proper notation and inability to deal with equations.

**2.11: ERA OF TRANSLATIONS**

During the 7th and 8th centuries, the Muslim empire extended towards Persia, Syria in the east and to Spain, Morocco in the west. There were several mathematicians and astronomers working at Baghdad during the early 9th century. Among them Al- Khowarizmi was the most eminent astronomer and mathematician. He wrote text books on Arithmetic and Algebra viz *Al-jabar wa’l muqabalah*. 
During the 9th and 10th centuries, many books of Archimedes, Apollonius and Ptolemy including Euclid’s *Elements* were translated from Greek to Arabic. Christian scholars visited Spain and Sicily to acquire Moslem learning. During 11th century, Greek classics were translated from Arabic to Latin by the Christian scholars of Europe by the opening of Western European commercial relation with the Arabian world. 12th century became a *century of translations*. One of the most eminent translators of this period was Gherado of Cremona. He translated more than ninety Arabian books to Latin among which many books of Archimedes were included. In the meantime, eastern civilization where arithmetic and algebra were extensively studied came in touch with the Italian merchants. They played an important role in spreading Hindu-Arabic system of numeration in Europe.

By the 12th century, development of Arabic science began to turn down with the rise of Western Europe. During this era of translation and propagation; idea of motion, variability and the infinite together with the symbolic algebra and analytic geometry were merged. This amalgamation paved the way for invention and growth of mathematics related to infinitesimals in 17th century.

On the other hand, the Hindus in India were very much interested in numbers including irrationals and had little interest in geometry and deductive methods. This is one of the distinguishing features of Greek as well as Indian mathematics.

### 2.12: MEDIEVAL THOUGHTS

In the 13th century, universities at Paris, Oxford, Cambridge, Padua and Naples came into being and became the powerful factors in the development of mathematics. Campanus’ Latin translation of Euclid’s *Elements* printed in 1482 became the first printed version of Euclid’s great book. The impact of the era of translation on newly founded universities brought back quantitative science in Western Europe. Yet, one would be surprised to know that Archimedes’ works drew no significant attention from the scholars at that time also.

#### 2.12.1: Concept of changeable things and Merton Scholars

Aristotle’s *Treatise on Physics* explored the nature of infinite and the existence of indivisibles or infinitesimals and the divisibility of continuous
quantities—time, motion and geometric magnitudes. Aristotle observed that motion was itself continuous and infinite was present in it. The definition of continuity was given as “what is infinitely divisible is continuous” [26].

The problem of changeable things was first successfully studied by a group of logicians and natural philosophers at Merton College in Oxford in 14th century. During 13th century, Aristotelian thoughts influenced the western intellectuals. William Heytesbury (before 1313-1372), Richard Kelvington, John Dumbleton, Thomas Bradwardine (later Archbishop of Canterbury) and Richard Swineshead were also the Mertonian scholars. They sought to study variations in the intensity of a quality, intensive or local (such as hotness, density) and extensive or global (such as heat, weight).

2.12.2: Mean Speed Theorem (MST)

The Merton scholars investigated velocity as a measure of motion. According to Aristotle, all the qualities of a body (temperature, size and place) could change. But the central issue relating to the motion as interpreted by the Merton scholars was the ‘change of place’. They defined uniform motion (or constant speed) as the equal distances described in equal times and uniform acceleration as that for which equal increments of velocity acquired in equal intervals [19]. In spite of this, ‘instantaneous velocity’ was not perfectly defined due to the lack of notion of ‘limits of ratios’. They could work only in the light of Euclid i.e with proportions of equivalent units.

William Heytesbury is well known for developing the rule of uniform acceleration i.e. the Mean Speed Theorem which was stated in 6th chapter of his work *Regulae Solvendi Sophismata* in the following way:

‘If a moving body is uniformly accelerated during a given time interval, then the total distance $S$ traversed is that which it would move during the same time interval with a uniform velocity equal to the average of its initial velocity $v_0$ and its final velocity $v_f$ (namely, its instantaneous velocity at the mid pt. of the time interval)’

i.e $S = \frac{1}{2} \left( v_0 + v_f \right) t$ where $t$ is the length of time - interval.
In this way, emergence of kinematics took place at Merton College. Merton’s studies became popular and was spread to France and Italy in the mid-14\textsuperscript{th} century. These medieval scholars demonstrated this theorem as the foundation of the “law of falling bodies” long before Galileo. The mathematical physicist and historian of science Clifford A. Truesdell in his book *Essays in the History of Mechanics* wrote, “The now published sources prove to us, beyond contention, that the main kinematical properties of uniformly accelerated motions, still attributed to Galileo by the physics texts, were discovered and proved by scholars of Merton college…….. In principle, the qualities of Greek physics were replaced, at least for motions, by the numerical quantities that have ruled western science ever since.”

2.12.3: Nicole Oresme and Graphical representation

Nicole Oresme, a Persian scholar in 1350 wrote his *Treatise on the Configurations of Qualities and Motions* and introduced the concept of graphical representations of intensities of qualities connecting geometry and the physical world which became a “second characteristic habit” of western thought.

He discussed mainly in case of a ‘linear’ quality whose ‘extension’ was measured by an interval (alien segment) of either space or time. He measured the intensity of quality at each point of the reference interval by a perpendicular line segment at that point, thereby constructing a graph with the reference interval as its base. Reference interval and its intensity at a point were said to be *longitude* and *latitude* respectively which we designate as the abscissa and ordinate today. This graphical presentation produced by Oresme marked his demonstration, a simple but influential step towards the development of calculus.

![Graphical representation of MST](image)
In case of uniformly accelerated motion during a time interval \([0,t]\) corresponding to the longitude \(AB\), the time-velocity graph is shown in Fig. 2.11. The latitude \(PQ\) at each point \(P\) of \(AB\) represents the velocity acquired in time \(AP\). So \(CD\) represents the time-velocity graph and the configuration found is a trapezoid \(ABCD\).

Latitude \(AD = v_0 = \) the initial velocity \\
Latitude \(BC = v_f = \) the final velocity.

Oresme assumed that the area of this trapezoid \(S = \) the total distance traversed \(= \frac{1}{2} (v_0 + v_f) t\)

His assumption perhaps based on the fact that area of the trapezoid \(ABCD = \frac{1}{2} (AD + BC) \times AB\)

We may think it as the geometrical interpretation of Merton Rule. The following innovative ideas may be found in Oresme’s works:

- the measurement of diverse types of physical variables (like temperature, density, velocity) by means of line segments instead of real numbers by the Greeks
- some notion of a functional relationship between variables (here velocity as a function of time)
- a graphical representation of such a functional relationship
- a conceptual process of ‘integration’ or continuous summation to calculate distance as area under a velocity-time graph.

These ideas played a pivotal role in the development of calculus in the 17th century. Galileo (1638) started with the mean speed theorem with a proof and geometric diagram similar to those of Oresme and proceeds to the distance formula. It was written in his Discourses on Two New Sciences as

\[ S = \frac{1}{2}a.t^2 \] when the motion is uniformly accelerated with acceleration \(a\) during time interval \([0,t]\).
2.12.4: The Infinite and Merton Scholars

One of the Merton scholars, Swineshead studied the infinite following the path of Aristotle who lived in 4th century BC. He solved a problem stated as:

“if a point moves throughout the first half of a certain time interval with a constant velocity, throughout the next quarter of the interval at double the initial velocity, throughout the following eighth at triple the initial velocity, and so on ad. infinitum; then the average velocity during the whole time interval will be double the initial velocity.” [26]

Taking both time interval and initial velocity as unity, he got the following series from the above problem

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots = \infty$$ \hspace{1cm} (1)

This is the second infinite series appeared in the history of mathematics after the infinite series

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^n} + \ldots = \frac{4}{3}$$

deduced by Archimedes used in the *Quadrature of the Parabola*.

To find the summation of the series (1), Oresme gave a geometrical proof in his *Treatise on Configuration* which is explained in Fig. 2.12(a) and 2.12(b). The area covered in Fig.2.12(a) is same as area covered in Fig 2.12(b).
Calculus before Newton and Leibniz

So the series (1) can be written

\[
\sum_{n=1}^{\infty} a(A_n) = \sum_{n=0}^{\infty} a(B_n) = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \ldots + \frac{n}{2^n} + \ldots \infty
\]

[From Fig.2.12(a)]

\[
= 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} + \ldots \infty
\]

\[
= 2 \quad \text{[From Fig.2.12(b)]}
\]

During the 15\textsuperscript{th} and 16\textsuperscript{th} century also, the study of infinite was done following the path of Swineshead and Oresme. These studies and the results opened the door for

- an atmosphere of free acceptance of the Infinite and the infinite process into mathematics that was already ignored or set aside by the earlier mathematicians.

- Invention of other infinite series and related works.

2.13: ANALYTIC GEOMETRY

The invention of analytic geometry by Rene Descartes (1596 - 1650) and Pierre de Fermat (1601 - 1665) was a gigantic step towards the preparation for the new infinitesimal mathematics due to the following reasons.

- The correspondence between an equation \( f(x, y) = 0 \) and its locus consisting of all points \((x, y)\) relative to fixed perpendicular axes. For both of them, the unknown quantities \(x\) and \(y\) were line segments instead of numbers. \(x\) is measured from a reference point along a horizontal axis, and the second was placed as a vertical ordinate at the end point of the first line segment.

- Descartes began to study with a curve and derived its algebraic equation, whereas Fermat began with an algebraic equation and derived from it the geometric properties of the corresponding curve.

- The notion of a variable played important and epoch making role to the development of calculus.
And thus analytic geometry opened up a vast field for the use of \textit{infinitesimal technique} by using variety of curves.

\textbf{2.14: CAVALIERI AND KEPLER}

The systematic study and use of infinitesimal techniques for area and volume computations was popularized by two outstanding books written by Bonaventura Cavalieri (1598 -1647), a countryman and pupil of Galileo (1564 -1642). These books were \textit{Geometria Indivisibilibus continuorum} (Geometry of indivisibles) and \textit{Exercitationes Geometricae Sex} (six geometrical exercises) of 1647. Brief English translations of these books may be found in Struik’s source book [71]. These publications exerted enormous influence upon the development of infinitesimal calculus of later period. Johan Kepler (1571 - 1630) studied area and volume problems needed for his astronomical studies.

Cavalieri’s method differs from that of Kepler in the following aspects:

- Kepler considered a given geometrical figure to be decomposed into infinitesimal figures and summing up the areas and volumes of these infinitesimal figures obtained the area and volume of the whole figure. Kepler in his work on planetary motion had to find the area of sectors of an ellipse.

- Kepler’s consideration of infinitesimal figures are of the same dimension as that of the geometrical figure. To find area (volume) of a figure, he needed area (volume) of indivisibles.

Cavalieri’s method of indivisibles consists of two approaches viz. collective and distributive.

- In collective approach, Cavalieri found by considering a one - to one correspondence between the indivisible elements of two given geometrical figures. If \(\sum l_1\) and \(\sum l_2\) be the sums of the line (surface) indivisibles for two figures \(P_1\) and \(P_2\) and if \(\frac{\sum l_1}{\sum l_2} = \frac{\alpha}{\beta}\) then \(\frac{P_1}{P_2} = \frac{\alpha}{\beta}\).

Knowing the area or volume of one figure, the area or volume of the other will be found out.
In distributive approach, Cavalieri’s consideration of indivisibles are of lower dimension than that of the geometrical figure. To find the area (volume) of two enclosed figures \( C_1 \) and \( C_2 \), Cavalieri considered indivisibles parallel and equidistant line segments (plane sections) respectively. For corresponding line segments (plane sections) \( l_1 \) and \( l_2 \), if \( l_1 = l_2 \) then \( C_1 = C_2 \) [Fig-2.13].

Cavalieri’s method showed greater generality and a more abstractness in the method of treatment than that of Kepler.

For example, to derive the volume of a circular cone \( C \) whose base is \( r \) and height is \( h \), Cavalieri proceeded by comparing with Pyramid \( P \) with unit square base and height \( h \). If \( C_x \) be the section of \( C \) with radius \( r' \) and \( P_x \) be the section of \( P \) with side \( y \) at equal distances, then

\[
\text{Area of } C_x = \frac{\pi r'^2 x^2}{h^2} \quad \text{and area of } P_x = y^2 = \frac{x^2}{h^2} \quad \therefore \frac{x}{h} = \frac{r'}{r}
\]

\[
\therefore \text{area of } C_x \ / \ \text{area of } P_x = \pi. r'^2
\]

Hence, Cavalieri concluded that

\[
\text{Volume of } C \ / \ \text{Volume of } P = \pi. r^2
\]

Since, volume of \( P = l^2 \times \frac{h}{3} = \frac{h}{3} \quad \therefore \text{volume of } C = \frac{\pi r^2 h}{3}
\]

Important fact to be noticed in Cavalieri’s method is that the indivisible units may have thickness or not.
2.14.1: Comments

In Cavalieri’s method, a geometrical figure is considered to be composed of infinitely large number of indivisibles of lower dimension such as

(i) area as composed of parallel and equidistant line segments

(ii) volume as composed of parallel and equidistant plane sections.

These important features differ from that of Kepler.

2.14.2: Cavalieri’s new method

Cavaliere devised a new method of calculating the volume of a single solid in terms of its cross-sections.

![Fig. 2.14](image)

For example, let us consider a triangle ABC with base AB and height BC = a and a typical vertical section PQ of height x.

According to Cavaliere’s sense of indivisibles

Area of $\Delta ABC = \sum_A x$

We consider a pyramid P whose vertex is A and base is the square drawn on BC.

The volume of the pyramid $P = v(P) = \sum_A x^2$ thinking of the pyramid as the sum of its cross-sections.

Also the area under the parabola $y = x^2$ is $\sum_A x^2$
If Q is the solid obtained by the revolution of the parabola about AB, then its cross-section at a distance \( x \) from the vertex A = \( \sum_A^B \pi (x^2)^2 = \sum_A^B \pi x^4 \)

\[ \therefore \text{Volume of the solid of revolution} = V(Q) = \sum_A^B \pi x^4 = \pi \sum_A^B x^4 \]

As an outline of the method for computing these sums, we start with a square ABCD with one side equal to \( a \).

If PQ and QR denote the length of a typical section such that PQ = \( x \) and QR = \( y \), then \( x + y = a \)

\[ \therefore \text{area of the square} = a^2 = \sum_A^B a = \sum_A^B (x + y) \]

\[ = \sum_A^B x + \sum_A^B y = 2 \sum_A^B x \quad \text{[by symmetry]} \]

\[ \therefore \sum_A^B x = \frac{a^2}{2} \quad \text{.................................................. (1)} \]
Again volume of a cube = $a^3$

$$\Sigma^B_A a^2 = \Sigma^B_A (x + y)^2 = \Sigma^B_A (x^2 + y^2 + 2xy)$$

$$= \Sigma^B_A x^2 + \Sigma^B_A y^2 + 2 \Sigma^B_A xy = 2 \Sigma^B_A x^2 + 2 \Sigma^B_A \left(\frac{a}{2} + z\right) \left(\frac{a}{2} - z\right)$$

$$= 2 \Sigma^B_A x^2 + 2 \Sigma^B_A \left(\frac{a^2}{4} - z^2\right)$$

$$= 2 \Sigma^B_A x^2 + \frac{1}{2} \Sigma^B_A a^2 - 2 \Sigma^B_A z^2$$

i.e. $\Sigma^B_A a^2 = 4 \Sigma^B_A x^2 - 4 \Sigma^B_A z^2$ .......................... (2)

$\Sigma^B_A z^2$ represents the sum of squares of lines in two triangles AEF and CFG.

Also $\Sigma z^2$ represents the volume of a pyramid with dimensions equal to half of those of the pyramid whose volume is $\Sigma^B_A x^2$.

$$\therefore \Sigma^B_A z^2 = 2 \cdot \frac{1}{6} \Sigma^B_A x^2 = \frac{1}{4} \Sigma^B_A x^2$ ............................ (3)

By (2) and (3)

$$\Sigma^B_A a^2 = 4 \Sigma^B_A x^2 - \Sigma^B_A z^2 = 3 \Sigma^B_A x^2$$

$$\therefore \Sigma^B_A x^2 = \frac{1}{3} a^3$ .......................... (4)

Proceeding in this way, we will get

$$\Sigma^B_A x^3 = \frac{1}{3} a^4$ .......................... (5)

(1), (4) are the first two examples of the general formula given by Cavalieri

$$\Sigma^B_A x^n = \frac{a^{n+1}}{n+1}$$

He verified the result for $n = 1, 2, \ldots, 9$.

From this, he found the area under the general parabola $y = x^n$ ($n$ is a + ve integer) over the unit interval as $A = \Sigma^1_0 x^n = \frac{1}{n+1}$

And the volume of the solid by revolving this area around $X$ - axis as

$$V = \pi \Sigma(x^n)^2 = \pi \Sigma^1_0 x^{2n} = \frac{\pi}{2n+1}$$

This procedure, though far from a rigorous proof, led him to a correct result equivalent to the integral $\int_0^a x^n dx = \frac{a^{n+1}}{n+1}$ [26].
This is a big leap towards the development of algorithmic procedures of calculus.

Archimedes proved that the area of the region $S$ bounded by turn of a spiral and the line segment joining its initial and final points is one third of that of the circle $C$ centered at the initial point and passing through the final point

\[ a(S) = \frac{1}{3} \pi (2\pi a)^2 \]

To prove this he made use of the formulae

\[ 1 + 2 + 3 + \ldots + n = \frac{n^2}{2} + \frac{n}{2} \quad (1) \]

\[ 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \quad (2) \]

These two formulae play the key role to prove the limits

\[ \lim_{n \to \infty} \frac{1 + 2 + \ldots + n}{n^2} = \frac{1}{2} \]

\[ \lim_{n \to \infty} \frac{1^2 + 2^2 + \ldots + n^2}{n^3} = \frac{1}{3} \]

that was later used to prove the quadrature results

\[ \int_0^a x \, dx = \frac{a^2}{2} \quad \text{and} \quad \int_0^a x^2 \, dx = \frac{a^3}{3} \]

Arabian mathematical science reached its peak in the 11th century. Al-Haitham (ca. 965-1039) known as Alhazen in the west extended some of Archimedes’ work on volume results. He showed that if a segment of a parabola is revolved about its base (rather than about its axis as in Archimedes’ On Conoids) then the volume of the solid obtained is $\frac{8}{15}$ of that of the circumscribed cylinder. For this computation, the formulas for the sums of the first $n$ cubes and fourth powers are necessary.

Alhazen proved the formula

\[ (n + 1) \sum_{i=1}^n i^k = \sum_{i=1}^n i^{k+1} + \sum_{p=1}^n (\sum_{i=1}^p i^k) \quad (3) \]

With the help of a geometric derivation shown in the following figure
Putting $k = 1$ in (3) and with the help of (1)

$$(n + 1)(1 + 2 + ... + n) = (1^2 + 2^2 + ... + n^2) + \sum_{p=1}^{n}(1 + 2 + ... + p)$$

$$= (1^2 + 2^2 + ... + n^2) + [1 + (1 + 2) + (1 + 2 + 3) + ... + (1 + 2 + 3 + ... + n)]$$

Putting $k = 2$ in (3) and with the help of (1) and (2)

$$\therefore 1^2 + 2^2 + ... + n^2 = \frac{n}{6}(n + 1)(2n + 1) \text{ which is eqn. (2)}$$

Putting $k = 2$ in (3) and with the help of (1) and (2) we get

$$1^3 + 2^3 + 3^3 + ... + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \ldots \ldots \ldots \ldots \ldots \ldots (4)$$

Putting $k = 3$ in (3) and with the help of (1), (2) and (4) we get

$$1^4 + 2^4 + 3^4 + ... + n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

From these above two results, we get

$$\lim_{n \to \infty} \frac{1^2 + 2^2 + \ldots + n^2}{n^4} = \frac{1}{4}$$

$$n \to \infty$$

$$\lim_{n \to \infty} \frac{1^4 + 2^4 + \ldots + n^4}{n^5} = \frac{1}{5}$$

$$n \to \infty$$
In general we can write
\[
\lim_{n \to \infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} = \frac{1}{k+1}
\] (5)

Three French mathematicians Fermat, Pascal and Roberval gave more or less rigorous proofs of the formula \( \int_0^a x^k \, dx = \frac{a^{k+1}}{k+1} \) which was known as Cavalieri’s (conjectured) general formula for the area under the general parabola \( y = x^k \) (\( k \) being a +ve integer) i.e. quadrature of the generalized parabola \( y = x^k \).

In proving this, each of them used the concept of limit in (5). It required arithmetical computations only. By simpler arithmetical computations, Cavalieri’s intuitive ideas of geometrical indivisibles were replaced.

For computation of the area under the curve \( y = x^k \) over \([0, a]\), the interval \([0, a]\) is subdivided into \( n \)-equal subintervals of length \( \frac{a}{n} \).

Constructing inscribed polygons \( P_n \) and circumscribed polygons \( Q_n \) with base \( \frac{a}{n} \) and height \( \left( \frac{a}{n} \right)^k, \left( \frac{2a}{n} \right)^k, \left( \frac{3a}{n} \right)^k, \ldots, \left( \frac{(n-1)a}{n} \right)^k \) and \( \left( \frac{a}{n} \right)^k, \left( \frac{2a}{n} \right)^k, \left( \frac{3a}{n} \right)^k, \ldots, \left( \frac{na}{n} \right)^k \) respectively and making use of the concept of limit, Fermat, Pascal and Roberval proved that \( S = \frac{a^{k+1}}{k+1} \) which coincides with the result \( \int_0^a x^k \, dx = \frac{a^{k+1}}{k+1} \).

**2.15: PROBLEM OF RECTIFICATION**

Rectification of an arc of a curve is the construction of a straight line segment that has equal length of the arc. “Rectification, in geometry, is the finding of a right line equal to a curve” [18]

Fig. 2.18
It was thought that rectification of an algebraic curve could not be possible as one would not be able to find the same length of a curve a constructible straight segment.

A plane curve can be approximated by joining a finite number of points on the curve using straight line segment to make a polygonal path. Calculating each linear segment with the help of Pythagoras Theorem one can approximate the length of the curve.

But it is not always possible for all curves.

Archimedes’ pioneering attempts to find the area beneath a curve by Method of Exhaustion broke the ground and explored that the approximation of the length of a curve could be done. In the 17th century, Method of Exhaustion led to rectification of curves by geometrical methods of several transcendental curves, the logarithmic spirals by Evangelista Toricelli in 1645 (some sources say John Wallis(1616-1703) in 1650 ), the cycloid by Christopher Warren in 1658 and the catenary by Leibniz in 1691. Neil’s rectification of semi-cubical parabola $y^2 = x^3$ was published in July or August, 1657. It was the first algebraic curve to be rectified. Wallis published the Method in 1659 giving credit to Neil for his invention [53].

In 1687, Leibniz raised a question whether the semi-cubical parabola is a curve along which a particle under gravity may descend so that it traverses equal vertical distance in equal times? Huygens proved that it is true for semi-cubical parabola. So the curve is known as isochronous curve.

**2.15.1: Neil’s rectification of algebraic curve**

It was commonly believed that an algebraic curve could never have the same length as a constructible straight line segment. William Neil in 1657 found the rectification of the algebraic curve known as semi cubical parabola whose equation given by $y^2 = x^3$ when he was nineteen. His procedure was as follows:

Let the curve be defined on the interval $[0, a]$. Subdivide $[0, a]$ into an indefinitely large number of infinitesimal subintervals. The $i$-th subinterval is $[x_{i-1}, x_i]$. If $s_i$ denotes the length (almost straight) of the curve joining
Calculus before Newton and Leibniz

\[ s_i \approx \left[ (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 \right]^{1/2} \]

\[ \therefore \text{Length of the curve} = S \approx \sum_{i=1}^{n} \left[ (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 \right]^{1/2} \]

Fig. 2.19

Here, Neil introduced an auxiliary curve, the parabola \( z = x^{3/2} \) in \([0, a]\). If \( A_i \) denotes the area under the parabola \( z = x^{3/2} \) over \([0, x_i]\), then from the general quadrature formula \( A_i = \frac{2}{3} \cdot x_i^{3/2} \).

The approximate area of the rectangle over \([x_{i-1}, x_i]\) with a height \( z_i = x_i^{3/2} \) is \( z_i (x_i - x_{i-1}) \).

Now, \( y_i - y_{i-1} = x_i^{3/2} - x_{i-1}^{3/2} \)

\[ = \frac{3}{2} (A_i - A_{i-1}) \]

\[ \approx \frac{3}{2} z_i (x_i - x_{i-1}) \]

\[ \therefore S \approx \sum_{i=1}^{n} \left[ 1 + \left( \frac{y_i - y_{i-1}}{x_i - x_{i-1}} \right)^2 \right]^{1/2} \cdot (x_i - x_{i-1}) \]

\[ \approx \sum_{i=1}^{n} \left[ 1 + \frac{9}{4} \cdot x_i^{3/2} \right]^{1/2} \cdot (x_i - x_{i-1}) \]

\[ \approx \sum_{i=1}^{n} \frac{3}{2} \left[ x_i^{3/2} + \frac{4}{9} \right]^{1/2} \cdot (x_i - x_{i-1}) \]

\[ \approx \sum_{i=1}^{n} \frac{3}{2} \left[ x_i^{3/2} + \frac{4}{9} \right]^{1/2} \cdot (x_i - x_{i-1}) \]
which is equal to the area of the segment of the parabola \( y = \frac{3}{2} \left( x + \frac{4}{9} \right)^{\frac{3}{2}} \) lying over the interval \([0, a]\). By translating the area will be same as that of the segment of the parabola \( y = \frac{3}{2} x^{\frac{3}{2}} \) over \([\frac{4}{9}, a + \frac{4}{9}]\).

Hence, \( S = \frac{3}{2} \left[ \frac{2}{3} (a + \frac{4}{9})^{\frac{3}{2}} - \frac{2}{3} (\frac{4}{9})^{\frac{3}{2}} \right] = \frac{(9a+4)^{\frac{3}{2}}-8}{27} \)

**2.15.2: Comments**

In Neil’s rectification of a curve \( y = f(x) \) over \([0, a]\), we need an auxiliary curve \( z = g(x) \) for which the area \( A_i \) over \([0, x_i]\) in modern notation is

\[
A_i = \int_0^{x_i} g(x) \, dx = f(x_i) = y_i
\]

This gives

\[
y_i - y_{i-1} = A_i - A_{i-1} \equiv g(x_i)(x_i - x_{i-1})
\]

So,

\[
S \equiv \sum_{i=1}^{n} \left[ 1 + (g(x))^2 \right] \cdot (x_i - x_{i-1})
\]

\[
= \int_0^a \sqrt{1 + [g(x)]^2} \, dx
\]

Choosing \( g(x) = f'(x) \), we get proper auxiliary curve and

\[
S = \int_0^a \sqrt{1 + [f'(x)]^2} \, dx
\]

This is probably the first instance of the interplay between differential and integral calculus or tangent and quadrature problem.

**2.16: TANGENT PROBLEM**

**2.16.1 Importance of tangent problem**

Finding a tangent to a curve was a leading mathematical question of the early seventeenth century. Because, in optics, the tangent determines the angle at which a ray of light entered a lens. In mechanics, the tangent determines the direction of the motion of a body at every point along its path. In geometry, the tangents to two curves at a point of intersection determines the angle at which the curves intersected.
Historically, the calculation of areas dates back to many civilizations like Egyptian, Babylonian, Greek etc. But the problem of finding tangent line to a curve has long been studied by the Greeks.

**2.16.2 Tangent to a circle (Greek view)**

The Greeks in particular, constructed tangent line to a circle at a point on the circumference. They did it by drawing a radius to a point P on the circumference and then drawing a line perpendicular to the radius. A tangent line to a circle at a point indicates the direction of the circle at that point. Important property of the tangent line to a circle is that it touches the circle, but does not cut through it. With this property it is not easy to draw a tangent line to most curves.

**2.16.3: Tangent to other curves**

Apollonius proved many theorems relating to tangent lines to conic sections. Archimedes determined the tangent line to the spiral. The method of Archimedes was applied to other curves like cycloid, cissoid etc. by Gilles de Roberval and Evangelista Torricelli, a student of Galileo in between 1630 and 1640.

Tangent construction to different curves was a major problem to the 17th century mathematicians. Before Newton and Leibniz, many brilliant mathematicians like Pierre de Fermat, Rene Descartes, Sluse, Huddes etc. worked on it. They came upon a different approach to the problem.

**2.16.4: Tangent construction by kinematic method**

In between 1630 and 1640, Galileo, Evangelista Torricelli (1608 - 1647), Gilles Personne de Roberval (1602 - 1675), introduced an approach to construct tangent lines from the intuitive concept of instantaneous motion of a moving point on a curve. For Archimedean spiral, the motion of the point along the spiral was considered as the resultant of radial motion (away from the origin) and an angular motion. For the cycloid, the motion of the point along the cycloid was considered as the resultant of uniform translation with constant speed and clockwise rotation with unit angular speed. Roberval was successful in obtaining the tangent lines to the parabola and ellipse by this method.
For a freely falling body, Galileo established that in $t$ seconds the body would traverse a distance $= \frac{1}{2}at^2 = f(t) = y$.

Average velocity over an interval from $t$ to $t+h$,

\[
= \frac{1}{2}a[(t+h)^2 - t^2] = at + \frac{ah}{2}
\]

= difference quotient of $f(t) = at$

= slope of the secant joining $(t, f(t))$ and its nearby point $(t+h, f(t+h))$.

With smaller and smaller $h$, the difference quotient can be interpreted as the instantaneous velocity or slope of the tangent at a point to a curve.

Thus, calculus revealed the profound association between geometry and physics that led to the process of transforming physical realities to mathematical entities.

2.16.5: Secant approach and its shortcomings

In this approach, a secant line joining two neighboring points P and Q on the curve are drawn. If Q comes closer to the point P, the slope of the secant PQ comes closer to the slope of the desired tangent line at P. But how much ‘closer’ would the secant come? Any point Q we choose cannot be the closest possible. So, choosing point closer and closer to P is a never ending process. How many secant lines are to be drawn to get the desired tangent line? The answer is infinitely many.

2.16.6: Analytic geometry and Tangent construction

Using analytic geometry, the problem of finding tangent line at a point to a curve can be translated to equivalent algebraic problem: given an arbitrary value of one quantity (i.e. time) what is the rate of change (i.e. velocity) of the other quantity (i.e. position). Rene Descartes cited the problem of finding a tangent to a curve was “the most useful and general problem I know but even that I have ever desired to know in geometry” [66].
Systematic methods for the rate of change (i.e. derivative) were developed by Newton and Leibniz independently with the use of analytic geometry to more curves [Ch-3].

2.16.7: Principle of Elimination

The principle of elimination states that “when you have eliminated the impossible, whatever remains, however improbable, must be the truth” [56].

Sherlock Holmes (1858-1930) used the principle of elimination fruitfully in his detective novels. Applying this principle in respect of tangent construction by secant method, when all non-tangent lines to a curve have been eliminated, the line left would be the tangent line to the curve. For instance, we examine one problem to understand the principle of elimination and its drawbacks.

Let us consider the simplest form of quadratic function \( y = x^2 \).

We want to find the slope of the tangent to the curve at \( P (3, 3^2) \) by the principle of elimination.

Let \( Q (3+h, (3+h)^2) \) be another point on the curve near to \( P (3, 3^2) \) if \( h \neq 0 \).

The slope of the line \( PQ \) which is a “non tangent” to the curve at \( P (3, 3^2) \) is

\[
\frac{(3+h)^2-3^2}{3+h-3} = \frac{9+6h+h^2-9}{h} = 6 + h, h \neq 0
\]

For slope of the tangent, \( 6 + h \) is the wrong answer. To get the right answer, we have to eliminate the condition \( h \neq 0 \). Putting \( h = 0 \), the slope of the tangent to the curve \( y = x^2 \) at \( (3, 3^2) \) is 6.

2.16.8: Comments

The method of elimination contains serious drawbacks. Whether a curve possesses a tangent line at a certain point or not, cannot be known by it. Hence a clear definition is needed.

2.17: Fermat’s method and the Derivative

Pierre de Fermat (1601-1665) introducing the notion of variables, developed a method for finding tangent during 1630’s. His method was more or
less exactly the method used by Newton and Leibniz. He used the notion of *limit* in a crude style to invent a workable definition of the slope of a tangent line to a curve after slight modification of the method of elimination thereby removing the drawbacks of it.

He first found the slope of the line joining the points \( P (c, f(c)) \) and \( Q (c+h, f(c+h)) \) where \( h \neq 0 \) The slope of PQ is \( \frac{f(c+h)-f(c)}{h} \) which is not our desired slope of the tangent line. So to define the slope of the tangent at \((c, f(c))\) to be the number (if there is one) that the above expression tends to that number as \( h \) approaches zero. Fermat’s idea is easy. The right answer is the limiting value of the wrong answer.

He discovered the formula for finding tangent to the curve \( y = x^n \), the slope of such a tangent line is \( nx^n \).

For this and other results, Fermat is considered by Lagrange and many, to be the inventor of differential calculus, although his work did not approach the generality of Newton or Leibniz [66].

**2.17.1: Fermat’s pseudo-equality* (adequality) method**

The problem of construction of tangent lines to general curves was not studied until the middle decades of the 17th century. It was one of the major problems facing the mathematicians of 17th century. The tangent or slope of a curve is so important because it tells us the rate at which something changes. The rate at which things change, is fundamental in science. Ultimately this problem was solved by Newton and Leibniz at the end of the century.

In 1635, a few methods for the construction of tangent lines to general curves were discovered. These methods combined with the area problems and techniques produced calculus as a new unified method of mathematical analysis. Fermat’s pseudo- equality method first dealt the solution of maximum- minimum of a function near an extreme value. His method was formulated in the late 1620’s and published in 1637.

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* Pseudo-equality- The term pseudo-equality (adequality) was introduced by Fermat to mean an approximate equality. Fermat said that he had borrowed it from Diophantus.
2.17.2. Fermat’s maxima (minima) problem

The first problem of maxima or minima dealt by Fermat was to divide a length $b$ into two segments $x$ and $b-x$ whose product $x(b-x)$ is maximum.

Let $f(x) = x(b-x)$

For this purpose, he substituted $x+E$ for $x$ and wrote the ‘pseudo-equality’ to compare the resulting expression with the original one i.e $f(x+E) = f(x)$. Because, from intuition or pictorial ground, $f(x)$ changes very slowly when it attains maximum or minimum.

Dividing throughout the result by $E$ and canceling the terms containing $E$, i.e setting $E = 0$, he obtained $x = b/2$.

2.17.3:. Fermat’s tangent construction

To find tangent line to a curve at a point, Fermat developed his own method. His idea was to find subtangent for the point that is described as the segment on the x-axis between the foot of the ordinate drawn to the point of contact and the intersection of the tangent line with the x-axis.

From the figure, $TM = s$ is the subtangent at P to the curve. Triangles PTM and RTS are similar where R is a point neighboring to P on the curve.

\[
\therefore \frac{TS}{TM} = \frac{s+E}{s} = \frac{RS}{PQ} = \frac{k}{f(x)} = \frac{f(x+E)}{f(x)}\text{ where } s = TM\text{ is the subtangent.}
\]

\[
\therefore s = \frac{Ef(x)}{f(x+E) - f(x)}
\]
2.17.4: Comments

Here, we would like to make a comment that

\[ E \frac{f(x)}{f(x+E)-f(x)} = \frac{f(x)}{f(x)} \frac{f(x)}{[f(x+E)-f(x)]/E} \approx \frac{f(x)}{E} \]

which would give \( f'(x) \) as \( E \to 0 \).

\[ \therefore s = \frac{f(x)}{f'(x)}. \]

In this method, Newton saw the means of defining a derived function by which measurement of instantaneous rate of change can be found and applied in studying physics of motion. Newton arrived at \( s = \frac{f(x)}{f'(x)} \) as slope of the tangent line at P.

2.18: DESCARTES’ CIRCLE METHOD

Descartes discovered a method known as Descartes’ circle method. The method was purely an algebraic method. He found in 1630 the following method for finding tangent line for the curves that are not circles. This method was published in Book II of *Lageometrie* written by Descrates.

His method can be summarized as follows:

(i) Choose a point on the curve

(ii) Find a circle that is tangent to the curve at the chosen point, so that the circle and the curve have the same tangent line at that point.

(iii) Find the slope of the radius and thereby the slope of the tangent.

The beauty of this method is that once the circle is known, tangent line is known, because by nature the tangent line to a circle is perpendicular to the line of radius of the circle.

2.18.1: Descartes inverse problem

After the invention of analytic geometry, several procedures for determining the tangent line to certain class of functions have been described. The converse of it is the problem of deriving the equation of the function itself where certain characteristic properties of the tangent are known. The inverse problem of tangents had been studied in a special case by Descartes in 1638-39.
2.19: HUDDE’S RULE

Hudde discovered a simpler method known as Hudde’s rule. Basically it involves the derivatives. Descartes’ method and Hudde’s rule were important because they influenced Newton.

For a given polynomial \( F(x) = \sum_{i=0}^{n} a_i x^i \), consider another polynomial \( F^*(x) \) where \( F^*(x) \) is constructed by arranging the terms of \( F(x) \) in order of increasing degree and multiplied by the terms of an arithmetic progression \( a, a+b, a+2b, \ldots, a+nb \)

Then, \( F^*(x) = a F(x) + b x F'(x) \) where \( F'(x) = \sum ia_i x^{i-1} \), the well known derivative of a polygon.

Hudde’s rule states that any double root of \( F(x) = 0 \) must be a root of \( F^*(x) = 0 \), i.e., any double root of the polynomial \( F(x) \) must be a root of its derivative \( F'(x) \).

In particular, Hudde’s rule can be applied to extremum problems also. The combination of Hudde’s rule and Descartes’ circle method applied only to algebraic curves written in explicit form \( y = f(x) \).

But the Sluse’s method is applicable even to those algebraic curves that can be written in implicit form \( f(x, y) = 0 \) where \( f(x, y) = \sum c_{ij} x^i y^j \) is a Polynomical in \( x \) and \( y \).

Previously, Leibniz accepted Sluse’s rule of tangent without proof. In November 1676 manuscript, he showed that Sluse’s rule of tangent can be derived from his calculus also.

2.20: INFINITESIMAL METHOD (BARROW’S METHOD)

The infinitesimal tangent method came into light just after the invention of algebraic rules of Hudde and Sluse.

Sir Isaac Barrow (1630 - 1677), the first Lucasian professor at Cambridge and the teacher of Sir Isaac Newton, published *Optical and Geometrical Lectures* in 1669. In his *Geometrical Lectures*, he discussed the tangent and quadrature problem from a classical and geometrical point of view. He used the Greek’s view of tangent line as a straight line touching the curve at a single point.
Barrow developed a technique that came closest to solving the tangent line problem. Later, the problem was solved finally by Newton and Leibniz.

![Figure 2.21: Barrow’s tangent construction](image)

Let MN be an ‘infinitely small’ arc of the curve given by \( f(x, y) = 0 \)

The points are given by \( M(x, y) \) and \( N(x + e, y + a) \).

Now, \( (x + e, y + a) = f(x, y) = 0 \) .......................... (1)

Canceling all terms containing powers of \( a \) and \( e \) and their products and then ignoring the distinction between the arc MN and segment MN and using the similarity of the \( \Delta TQM \) and ‘characteristic triangle’ originally known as ‘Barrow triangle’ \( \Delta MRN \) solved the above equation (1) for the

\[
\text{Slope} = m = \frac{MQ}{TQ} = \frac{NR}{MR} = \frac{a}{e}
\]

Barrow’s method can be summarized as

(i) Insert \( x + e \) and \( y + a \) into the equation
(ii) Neglect squares and higher powers and the product of \( a \) and \( e \)
(iii) Solve for \( \frac{a}{e} \)

Barrow used his method to compute the tangents to a wide variety of curves. His techniques of discarding the higher order terms is equivalent to the modern process of taking a limit i.e. the tangent was considered as the limit of a chord. The ratio \( \frac{a}{e} \) is now written as \( \frac{dy}{dx} \).

It is likely that Newton knew of Barrow’s method in 1664 or 23 years before the publication of *Principia*. Thus the analytical derivation of Sluse’s
rule was the spirit of Barrow’s approach. Barrow used the concept of ‘characteristic triangle’ which coincides the idea that $a \to 0$, $e \to 0$ (neglecting the ‘higher order differentials’) and the tangent becomes the limiting position of the secant line.

2.21: A COMPARATIVE DISCUSSION OF DESCARTES, FERMAT AND BARROW’S METHODS

Before 1665, a few mathematicians could perform tangent construction to a curve at a point. Descartes, Fermat and Issac Barrow were among them.

Descartes’ method of constructing tangent lines to curves was algebraic rather than infinitesimal one in character. For this, knowledge of theory of equations is necessary and it leads to tedious algebraic computations even for a simple curve. If we compare both the methods of Fermat and Descartes, we find that the first is much closer to that we are using today. Though, Descartes technique was not popular due to the complexity involved in it, as much more precise and fully stuck to geometric techniques; yet the method is considered mathematically sound method of computation of tangent line.

Fermat’s method of finding tangents seems to have been a bi-product of his method of finding maxima and minima. His idea was set forth in 1629 in a letter written to a certain M. Despagnet. Fermat was of the view that finding the tangent did not mean finding its equation, but finding the sub-tangent. In Fermat’s method, the point R neighboring to P lies on the tangent line drawn at P on the curve [Fig.2.20]. Fermat used sub-tangent, abscissa, and ordinates of the points P and R to find sub-tangent. He introduced a ‘small’ or ‘infinitesimal’ element $E$ to the abscissa. Fermat gave several examples of the application of his method. Some of his results were published by Pierre Herigone in his Supplementum Curses mathematici (1642). He could find tangents to all curves given by polynomial equations $y = p(x)$ and probably to all algebraic curves.

Issac Barrow, in his book Lectiones Opticae et geometricae written apparently in 1663, 1664 and published in 1669, 1670, gave his method of tangents. Before him, Roberval and Torricelli found tangent lines to a curve compounding two velocities in the direction of the axes of $x$ and $y$ to get a resultant along the tangent line.
In Barrow’s method, the point N neighboring to P lies on the curve itself [Fig.2.21]. The increments MR and NR of the abscissa and ordinate were denoted by $e$ and $a$ and the ratio $a:e$ was determined by substituting $x+e$ for $x$ and $y+a$ for $y$ in the equation of the curve, rejecting all terms of order higher than the first in $a$ and $e$. He introduced two infinitesimal quantities $a$ and $e$. Their ratio is equivalent to Leibniz’ symbol $\frac{dy}{dx}$. This process is equivalent to differentiation.\[ \Delta MNR \] is sometimes called Barrow’s differential triangle.

Barrow went further than Fermat in the theory of differentiation as he compared two increments. But Fermat’s trick may be noted as the fundamental trick in infinitesimal calculus. The formal algorithms for the construction of tangents were invented by the Dutch mathematicians Hudde and Rene Francois de Sluse in 1650.

### 2.22: AREA AS ANTIDERIVATIVE

![Fig. 2.22: Equality of the area of the circle and the triangle](image)

In the above figure, we may have three cases

(i) the area $A$ of the circle is greater than the area of the triangle

(ii) the area $A$ of the circle is smaller than the area of the triangle

(iii) the area $A$ of the circle is equal to the area of the triangle

Archimedes proved (iii) by ‘reductio ad absurdum’ method as (i) and (ii) give rise to contradictions.

Then $A = \pi r^2 = \frac{1}{2} C \cdot r$, where $C$ is the circumference and $r$ is the radius of the circle.

$\Rightarrow C = 2\pi r$
Calculus before Newton and Leibniz

\[ C = 2\pi r \]

represents a straight line of slope \(2\pi\) passing through the origin. The variables \(A, C, r\) are indicated as in the figure below:

For a change \(\Delta r\) of \(r, \Delta C\) of \(C,\)

\[ \Delta A = C\Delta r + \frac{1}{2}(\Delta C).\Delta r \]

\[ \therefore \frac{\Delta A}{\Delta r} = C + \frac{1}{2}(\Delta C) \]

As \(\Delta r \to 0, \Delta C \to 0\) and \(\Delta r \to 0, \frac{\Delta A}{\Delta r} \to C\)

i.e. the area beneath a curve (a continuous function) is related to the function i.e. area be an antiderivative of \(y\) where \(y\) is a function of \(x\).

Leibniz guessed it in about 1670 and Newton guessed it in 1666, but he kept it secret.

2.23: INTERPLAY BETWEEN AREA AND TANGENT

Toricelli and Issac Barrow were the first to make an association between tangent and area problems with the help of time and motion concepts. Both of them considered the problem of the motion of a point along a straight line with varying velocity. It can be represented pictorially by a velocity-time graph. It was first suggested by Galileo. The total distance covered by the point will be equal to the area under the velocity - time graph.

For the velocity –time graph of \(v = t^n\) in the interval \([0, t]\), then by the method of indivisibles, area under \(v = t^n\) in \([0, t]\) would be
\[ y = \frac{t^{n+1}}{n+1} \] = distance traversed by the point in time \( t \).

Also, the slope of the tangent line to \( y = \frac{t^{n+1}}{n+1} \) is \( t^n \).

This example may be cited as an example of interplay between area and tangent problem.