CHAPTER 6
TEST FOR HOMOGENEITY OF LIFETIMES OF SEVERAL SYSTEMS
UNDER GENERALIZED INVERTED SCALE FAMILY OF DISTRIBUTIONS
BASED ON TYPE II CENSORED SAMPLING DESIGN

Introduction

In previous chapters, we had discussed the importance of reliability and life testing experiments for the study of systems. We had also studied problem of identifying, among several machines of different brand performing similar work, one which has better reliability. In this chapter, instead of using the Self Relocated Design given by Srivastva (1987), we consider a generalized Type II censoring design given by Srivastava (1987) for the comparison of more than two types of systems. On the line of Chapter 5, in this chapter we study generalized Type II censoring design when lifetime distribution is Generalized Inverted Family of Distributions (GIFD) proposed by Potdar and Shirke (2013). Further, we obtain maximum likelihood equations using the data from generalized Type II censoring design and provide algorithm for solving the equations numerically. The likelihood ratio test for homogeneity of life-times for several systems of several makes is also derived. The organization of the chapter is given below.

In Section 6.1, we give the probability density function, the survival function and the hazard rate of the generalized inverted scale family of distributions (GIFD), and develop the likelihood for generalized Type II censored sampling design under GIFD. In Section 6.2, we derive the expressions for maximum likelihood equations for the parameters and their asymptotic variance-covariance matrix for both cases of known and unknown shape parameters. Further, we obtain asymptotic confidence interval for $m+1$ parameters of the distributions. Section 6.3 discusses likelihood ratio test for simultaneous
testing of homogeneity of scale parameters when the shape parameter is known and when it is unknown. In Section 6.4, we consider GIED as a special case and obtain ML estimates of the parameters, confidence intervals for the parameters, reliability characteristics of the distribution, and efficiency measures like mean square error (MSE) and standard error (SE) for the parameters through Monte-Carlo simulations. Further, the likelihood ratio test for simultaneous testing of homogeneity of scale parameters, when the shape parameter is known, is discussed and the cut-off points for the test statistic are also obtained through Monte-Carlo simulation. In Section 6.5, we simulate various optimality criteria for the design. In Section 6.6, we discuss a cost function for the problem of planning such experiments. We further study the effect of proportion of censoring on the cost function. Some concluding remarks are given in Section 6.7.

6.1. Generalized Inverted Scale Family of Distributions (GIFD) and Likelihood Function for Type II Censoring Design

Consider an item whose life time is denoted by $T$. The random variable $T$ is said to have two-parameter generalized inverted scale family of distributions, if its distribution function is given by

$$F_T(t; \beta, \alpha) = P(T \leq t) = 1 - \left[H\left(\frac{1}{\beta t}\right)\right]^\alpha, \quad t \geq 0; \quad \alpha, \beta > 0,$$

(6.1.1)

where $H(t)$ is a parameter free survival function. If $T$ has the distribution function (6.1.1), then the corresponding density function, reliability function and hazard function are respectively given by

$$f(t; \beta, \alpha) = \frac{\alpha}{\beta t^2} h\left(\frac{1}{\beta t}\right) \left[H\left(\frac{1}{\beta t}\right)\right]^{\alpha-1}, \quad t \geq 0; \quad \alpha, \beta > 0,$$

(6.1.2)

$$F_T(t; \beta, \alpha) = \left[H\left(\frac{1}{\beta t}\right)\right]^\alpha,$$

(6.1.3)
and

\[ r_T(t; \beta, \alpha) = \frac{f_T(t)}{F_T(t)} = \frac{\alpha}{\beta^2} \frac{h\left(\frac{1}{\beta t}\right) H\left(\frac{1}{\beta t}\right)^{\alpha-1}}{[H\left(\frac{1}{\beta t}\right)]^\alpha}, \quad (6.1.4) \]

where \( h(x) \) is the pdf corresponding to the cdf \( H(x) \).

The extension of Type II censoring, called generalized Type II censoring design is discussed in Section 4.1.1 of Chapter 4. The marginal likelihood function for \( i \)-th type of system under generalized type II censoring based on observing \( G^* \) failures from \( u \) units is

\[ L_i = \frac{u!}{(u-G^*)!} \prod_{g=1}^{G^*} f_i(t_g)[\bar{F}_i(t_{G^*})]^{u-G^*}. \quad (6.1.5) \]

Therefore, the joint likelihood of the whole experiment is

\[ L = \prod_{i=1}^{m} L_i = \prod_{i=1}^{m} \left\{ \frac{u!}{(u-G^*)!} \left[ \prod_{g=1}^{G^*} f_i(t_g)\right]^{u-G^*} \right\}. \quad (6.1.6) \]

Substituting the density function in (6.1.2) and survival function in (6.1.3) in (6.1.6) we get

\[ L = \prod_{i=1}^{m} \left\{ \frac{u!}{(u-G^*)!} \left[ \prod_{g=1}^{G^*} \frac{\alpha}{\beta t_{g_i}} h\left(\frac{1}{\beta t_{g_i}}\right)\left[\frac{1}{\beta t_{g_i}}\right]^{\alpha-1}\left[H\left(\frac{1}{\beta t_{g_i}}\right)^{\alpha(u-G^*)}\right]\right\}. \quad (6.1.7) \]

### 6.2. Maximum Likelihood Estimation and Confidence Interval

The log likelihood function of (6.1.7) is

\[ l = m \ln \left(\frac{u!}{(u-G^*)!}\right) + mG^* \ln \alpha \]

\[-G^* \sum_{i=1}^{m} \ln \beta_i - 2 \sum_{i=1}^{m} \sum_{g=1}^{G^*} \ln t_{g_i} + \sum_{i=1}^{m} \sum_{g=1}^{G^*} \ln \left[ h\left(\frac{1}{\beta t_{g_i}}\right) \right] + (\alpha - 1) \sum_{i=1}^{m} \sum_{g=1}^{G^*} \ln \left[ H\left(\frac{1}{\beta t_{g_i}}\right)^{\alpha(u-G^*)} \right]. \quad (6.2.1) \]

Differentiating (6.2.1) with respect to \( \alpha \) and \( \beta_i (i = 1, 2, \cdots, m) \) we get

\[ \frac{\partial l}{\partial \alpha} = \frac{mg^*}{\alpha} + \sum_{i=1}^{m} \sum_{g=1}^{G^*} \ln \left[ H\left(\frac{1}{\beta t_{g_i}}\right)\right] + (u-G^*) \sum_{i=1}^{m} \ln \left[ H\left(\frac{1}{\beta t_{G^*}}\right)\right]. \quad (6.2.2) \]
\[ \frac{\partial l}{\partial \beta_i} = - \frac{G^*}{\beta_i} + \sum_{g=1}^{G^*} \left( \frac{1}{h} \left( \frac{1}{\tilde{\beta}_i t_{ggi}} \right) \left( - \frac{1}{t_{ggi} \beta_i^2} \right) + (\alpha - 1) \sum_{g=1}^{G^*} \left( \frac{1}{h} \left( \frac{1}{\tilde{\beta}_i t_{ggi}} \right) \left( - \frac{1}{t_{ggi} \beta_i^2} \right) \right) \right) + \alpha (u - G^*) \frac{H \left( \frac{1}{\tilde{\beta}_i t_{ggi}} \right)}{H \left( \frac{1}{t_{G^* i} \beta_i^2} \right)} \left( - \frac{1}{t_{G^* i} \beta_i^2} \right). \]

Therefore,

\[ \frac{\partial l}{\partial \beta_i} = - \frac{G^*}{\beta_i} - \frac{1}{\beta_i^2} \sum_{g=1}^{G^*} \left( \frac{1}{t_{ggi} \beta_i^2} \right) - (\alpha - 1) \sum_{g=1}^{G^*} \left( \frac{1}{t_{ggi} \beta_i^2} \right) - \alpha (u - G^*) \frac{H \left( \frac{1}{\tilde{\beta}_i t_{ggi}} \right)}{H \left( \frac{1}{t_{G^* i} \beta_i^2} \right)} \left( - \frac{1}{t_{G^* i} \beta_i^2} \right), \]

\[ i = 1, 2, \ldots, m. \]  

Equating equation (6.2.2) to zero we have

\[ \hat{\alpha} = - \frac{m G^*}{\sum_{i=1}^{m} \sum_{g=1}^{G^*} \ln \left[ H \left( \frac{1}{\tilde{\beta}_i t_{ggi}} \right) \right] + (u - G^*) \sum_{i=1}^{m} \ln \left[ H \left( \frac{1}{t_{G^* i} \beta_i^2} \right) \right]} \]  

(6.2.4)

The estimates of parameters \( \beta = (\beta_1, \ldots, \beta_m) \) can be obtained in two different cases: (i) shape parameter \( \alpha \) is known and (ii) shape parameter \( \alpha \) is unknown.

### 6.2.1. Maximum Likelihood Estimation When Shape Parameter \( \alpha \) is known

The solutions of ML equations (6.2.3) cannot be obtained explicitly. Hence, we use the Newton-Raphson method for the data \((u, G, t_{ggi}; g = 1, 2, \ldots, G^*; i = 1, 2, \ldots, m)\).

The method requires some initial values of \( \beta \). The initial values are obtained using the least square method as suggested in Ng (2005). The method uses the empirical distribution function for \( i \)-th population as given below

\[ \tilde{F}_i(t_g) = 1 - \prod_{j=1}^{\theta_g} (1 - \hat{p}_{ij}), \]

with

\[ \hat{p}_{ij} = \frac{1}{u-j+1}, \quad j = 1, 2, \ldots, g; \quad g = 1, 2, \ldots, G^*; \quad i = 1, 2, \ldots, m. \]
The parameters $\beta_i$'s are estimated by fitting the linear regression through the method of least square. Here we consider

$$y_{gi} = \beta_i t_{gi},$$

where

$$y_{gi} = \frac{1}{H^{-1}\left[\frac{1}{2(1-F_i(t_{g-1}))^{1/\alpha} + (1-F_i(t_g))^{1/\alpha}}\right]}, \ g = 1, 2 \ldots, G^*; \ i = 1, 2, \ldots, m,$$

and $t_0$ is such that

$$\hat{F}_i(t_0) = 0; \ i = 1, 2, \ldots, m.$$

The least square estimates of $\beta_i; \ i = 1, 2, \ldots, m$ are given by

$$\hat{\beta}_{0i} = \frac{\sum_{g=1}^{G^*} y_{gi} t_{gi}}{\sum_{g=1}^{G^*} t_{gi}^2}, \ i = 1, 2, \ldots, m. \quad (6.2.5)$$

These values are used as initial solution for the ML estimation by the Newton-Raphson method. The MLEs of reliability ($\hat{F}_i(t_i); \ i = 1, 2, \ldots, m$) and hazard rate ($\hat{r}_i(t_i); \ i = 1, 2, \ldots, m$) can be evaluated using invariance property of MLEs as

$$\hat{F}_i(t_i) = \left[H\left(\frac{1}{\beta_i}\right)\right]^{a}, \quad \frac{\hat{r}_i(t_i; \hat{\beta}, \alpha)}{\hat{\beta}_i t_i^2} \left[\frac{h\left(\frac{1}{\beta_i}\right)}{H\left(\frac{1}{\beta_i}\right)}\right] \quad i = 1, 2, \ldots, m. \quad (6.2.7)$$

### 6.2.1.1 Observed Fisher Information Matrix under the Design

The Fisher information matrix is

$$V = \left((v_{ij})\right),$$

where

$$v_{ij} = -E\left(\frac{\partial^2 l}{\partial \beta_i \partial \beta_j}\right), \ i, j = 1, 2, \ldots, m.$$
Now for \( i = j, \)

\[
\frac{\partial^2 l}{\partial \beta_i^2} = \frac{G^*}{\beta_i^2} - \sum_{g=1}^{G^*} \frac{1}{t_{gi}} \left\{ \frac{1}{\beta_i^2} \left[ h\left( \frac{1}{\beta_i t_{gi}} \right) h'\left( \frac{1}{\beta_i t_{gi}} \right) - \frac{1}{t_{gi} \beta_i^2} h\left( \frac{1}{\beta_i t_{gi}} \right) \right] \right\} + \frac{h\left( \frac{1}{\beta_i t_{gi}} \right)}{n} \left( \frac{1}{\beta_i t_{gi}} \right)^{-2} \left[ \beta_i \right] \]

\[
-(\alpha - 1) \sum_{g=1}^{G^*} \frac{1}{t_{gi} \beta_i^3} \left\{ \frac{1}{\beta_i^2} \left[ H\left( \frac{1}{\beta_i t_{gi}} \right) H'\left( \frac{1}{\beta_i t_{gi}} \right) - \frac{1}{t_{gi} \beta_i^2} H\left( \frac{1}{\beta_i t_{gi}} \right) \right] \right\} + \frac{H\left( \frac{1}{\beta_i t_{gi}} \right)}{n} \left( \frac{1}{\beta_i t_{gi}} \right)^{-2} \left[ \beta_i \right] \]

Therefore,

\[
\frac{\partial^2 l}{\partial \beta_i^2} = \frac{G^*}{\beta_i^2} + \sum_{g=1}^{G^*} \frac{1}{t_{gi} \beta_i^4} \left\{ \frac{1}{\beta_i^2} \left[ h\left( \frac{1}{\beta_i t_{gi}} \right) h'\left( \frac{1}{\beta_i t_{gi}} \right) - \frac{1}{t_{gi} \beta_i^2} h\left( \frac{1}{\beta_i t_{gi}} \right) \right] \right\}

+ (\alpha - 1) \sum_{g=1}^{G^*} \frac{1}{t_{gi} \beta_i^3} \left\{ \frac{1}{\beta_i^2} \left[ H\left( \frac{1}{\beta_i t_{gi}} \right) H'\left( \frac{1}{\beta_i t_{gi}} \right) - \frac{1}{t_{gi} \beta_i^2} H\left( \frac{1}{\beta_i t_{gi}} \right) \right] \right\} \]

\[
+ \frac{\alpha(u-G^*)}{t_{G^*} \beta_i^2} \left\{ \frac{1}{\beta_i^2} \left[ H\left( \frac{1}{\beta_i t_{G^*}} \right) H'\left( \frac{1}{\beta_i t_{G^*}} \right) - \frac{1}{t_{G^*} \beta_i^2} H\left( \frac{1}{\beta_i t_{G^*}} \right) \right] \right\}, \quad i = 1, 2, ..., m; \tag{6.2.8}
\]

where

\[
h'() = \frac{\partial}{\partial \beta_i} h(), \quad H'() = \frac{\partial}{\partial \beta_i} H(), \quad h''() = \frac{\partial^2}{\partial \beta_i^2} h(), \quad H''() = \frac{\partial^2}{\partial \beta_i^2} H(), \quad i = 1, 2, ..., m.
\]

Since, systems are independently functioning in the experimental set up, we have

\[
\frac{\partial^2 l}{\partial \beta_i \partial \beta_j} = 0, \forall j \neq i = 1, 2, ..., m. \tag{6.2.9}
\]

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**Theorem 6.1**: Let us assume that all the usual regularity conditions needed for the asymptotic distribution of mle hold true. Then, the limiting distribution of \( \hat{\beta} \), as \( u \to \infty \) such that \( \frac{c^*}{u} \) remains constant, is \( m \)-variate normal with mean vector \( \beta \) and dispersion matrix \( V^{-1} \).

**6.2.2. Maximum Likelihood Estimation When Shape Parameter \( \alpha \) is Unknown**

The solutions of ML equations (6.2.2) and (6.2.3) cannot be obtained explicitly. Hence, we use the Newton-Raphson iterative method for the data \((u, G, t_{gi}; g = 1, 2, ..., G^*; i = 1, 2, ..., m)\). The initial values of the parameters of \( \beta_i; i = 1, 2, ..., m \) are obtained through the method of least square as discussed in Section 6.2.1 and initial value \( \alpha \) is obtained by using equation (6.2.4). These initial values are used in the Newton-Raphson method to obtain the MLEs of \((\alpha, \beta)\). The MLE of reliability \( \tilde{F}_i(t_i) \) and hazard rate \( \tilde{r}_i(t_i), i = 1, 2, ..., m \) can be evaluated using invariance property of MLEs as

\[
\tilde{F}_i(t_i) = [H \left( \frac{1}{\tilde{\beta}_{ti}} \right)]^{\tilde{\alpha}}.
\]

\[
\tilde{r}_i(t_i; \tilde{\beta}, \tilde{\alpha}) = \tilde{\alpha} \frac{\tilde{\alpha}}{\hat{\beta} t^2} \left[ h \left( \frac{1}{\tilde{\beta}_{ti}} \right) \right], \quad i = 1, 2, ..., m.
\]

**6.2.2.1. Observed Fisher Information Matrix under the Design**

The Fisher information matrix is

\[
W = -E \begin{bmatrix}
\frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta}
\end{bmatrix}^T \begin{bmatrix}
\Delta
\end{bmatrix}^T
\]

\[
V = \begin{pmatrix}
v_{ij}
\end{pmatrix},
\]

where

\[
v_{ij} = \frac{\partial^2 l}{\partial \beta_i \partial \beta_j} \quad \text{for} \quad i, j = 1, 2, ..., m
\]
and
\[ \Delta' = \left( \frac{\partial^2 l}{\partial \alpha \partial \beta_1}, \frac{\partial^2 l}{\partial \alpha \partial \beta_2}, \ldots, \frac{\partial^2 l}{\partial \alpha \partial \beta_m} \right). \]

\[ \frac{\partial^2 l}{\partial \alpha^2} = -\frac{mc^*}{a^2}. \quad (6.2.12) \]

\[ \frac{\partial^2 l}{\partial \alpha \partial \beta_i} = \sum_{g=1}^{G^*} \frac{1}{H\left(\frac{1}{\beta_i t_{gi}}\right)} - \frac{1}{t_{gi} \beta_i^2} \left( u - G^* \right) \frac{H\left(\frac{1}{\beta_i t_{G^* i}}\right)}{H\left(\frac{1}{\beta_i t_{gi}}\right)} - \frac{1}{t_{G^* i} \beta_i^2}, \quad i = 1, 2, \ldots, m. \quad (6.2.13) \]

\[ \frac{\partial^2 l}{\partial \beta_i^2} = \frac{G^*}{\beta_i^2} + \sum_{g=1}^{G^*} \frac{1}{t_{gi} \beta_i^4} \left\{ h\left(\frac{1}{\beta_i t_{gi}}\right) \frac{1}{h\left(\frac{1}{\beta_i t_{gi}}\right)} - \left( h\left(\frac{1}{\beta_i t_{gi}}\right) \right)^2 + 2\beta_i t_{gi} h\left(\frac{1}{\beta_i t_{gi}}\right) \right\} \]

\[ + (\alpha - 1) \sum_{g=1}^{G^*} \frac{1}{t_{gi} \beta_i^4} \left\{ H\left(\frac{1}{\beta_i t_{gi}}\right) H\left(\frac{1}{\beta_i t_{gi}}\right) - \left( H\left(\frac{1}{\beta_i t_{gi}}\right) \right)^2 + 2\beta_i t_{gi} h\left(\frac{1}{\beta_i t_{gi}}\right) H\left(\frac{1}{\beta_i t_{gi}}\right) \right\} \]

\[ \cdots \cdots \cdots \]

\[ + \alpha (u - G^*) \left\{ H\left(\frac{1}{\beta_i t_{gi}}\right) H\left(\frac{1}{\beta_i t_{gi}}\right) - \left( H\left(\frac{1}{\beta_i t_{gi}}\right) \right)^2 + 2\beta_i t_{gi} h\left(\frac{1}{\beta_i t_{gi}}\right) H\left(\frac{1}{\beta_i t_{gi}}\right) \right\}, \quad i = 1, 2, \ldots, m; \quad (6.2.14) \]

where, \( h'(\cdot) = \frac{\partial}{\partial \beta_i} h(\cdot), \ H'(\cdot) = \frac{\partial}{\partial \beta_i} H(\cdot), \ h''(\cdot) = \frac{\partial^2}{\partial \beta_i^2} h(\cdot), \]

\[ H''(\cdot) = \frac{\partial^2}{\partial \beta_i^2} H(\cdot), \quad i = 1, 2, \ldots, m. \]

Since, the systems are independently functioning in the experimental set up, we have

\[ \frac{\partial^2 l}{\partial \beta_i \partial \beta_j} = 0, \forall j \neq i = 1, 2, \ldots, m. \quad (6.2.15) \]
**Theorem 6.2:** Let us assume that all the usual regularity conditions needed for the asymptotic distribution of mle hold true. Then the limiting distribution of \((\hat{\alpha}, \hat{\beta})\), as \(u \to \infty\) such that \(\frac{G^*}{u}\) remains constant, is \((m + 1)\)-variate normal with mean vector \((\alpha, \beta)\) and dispersion matrix \(W^{-1}\) under regularity conditions.

6.2.3. Confidence Interval

Assuming asymptotic normal distribution for the MLEs, CIs for \((\alpha, \beta_1, \beta_2, \beta_3, ..., \beta_m)\) are constructed. Let \((\hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, ..., \hat{\beta}_m)\) are MLE of \((\alpha, \beta_1, \beta_2, \beta_3, ..., \beta_m)\) respectively. Let \(\hat{\sigma}^2(\hat{\alpha})\) and \(\hat{\sigma}^2(\hat{\beta}_i); i = 1, 2, ..., m\) are the estimated variances of \(\hat{\alpha}\) and \(\hat{\beta}_i; i = 1, 2, ..., m\) respectively. Then, \(100(1 - \xi)\)% asymptotic CI for \(\alpha\) and \(\beta_i; i = 1, 2, ..., m\) are respectively given by

\[
\left(\hat{\alpha} - Z_{\xi/2} \sqrt{\hat{\sigma}^2(\hat{\alpha})}, \hat{\alpha} + Z_{\xi/2} \sqrt{\hat{\sigma}^2(\hat{\alpha})}\right) \quad \text{and} \quad \left(\hat{\beta}_i - Z_{\xi/2} \sqrt{\hat{\sigma}^2(\hat{\beta}_i)}, \hat{\beta}_i + Z_{\xi/2} \sqrt{\hat{\sigma}^2(\hat{\beta}_i)}\right),
\]

\(i = 1, 2, ..., m\).  \hspace{1cm} (6.2.16)

where \(Z_{\xi/2}\) is the upper \(100(1 - \xi)^{th}\) percentile of standard normal distribution.

6.3. Testing of Hypotheses

The proposed design will have significance only when we are able to ascertain that the \(m\) type of systems are not all have identical life times. This can be done by developing ANOVA approach for the proposed design. However, we will utilize likelihood approach to develop a test. The testing of hypothesis problem is to test

\[H_0: \beta_1 = \beta_2 = \cdots = \beta_m = \beta \quad \text{against} \quad H_1: \beta_i \neq \beta_j \quad \text{for at least one pair} \quad (i, j), \ i \neq j.\]

\(6.3.1\)

As we are considering maximum likelihood estimation, the use likelihood ratio test is much convenient. The test statistic is
Differentiate (6.3.2) with respect to \( \alpha \) and \( \beta \) to have

\[
\lambda_{LR} = \frac{\max L(t, \beta, \alpha)}{\max L(t, \beta, \alpha)}.
\]

The test based on \(-2ln(\lambda_{LR})\) rejects \( H_0 \) in support of \( H_I \) if it is larger than upper \( \xi \)th cut point of chi-square distribution \((m - 1)\) degrees of freedom.

### 6.3.1. Computation of Likelihood under \( H_\theta \)

The log likelihood equation (6.2.1) can be written under to as

\[
l = m \ln \left( \frac{u^l}{(u - G^*)^l} \right) + mG^* \ln \alpha - mG^* \ln \beta - 2 \sum_{i=1}^{m} \sum_{g=1}^{G^*} \ln t_{gi} + \sum_{i=1}^{m} \sum_{g=1}^{G^*} \ln h \left( \frac{1}{\beta t_{gi}} \right)
\]

\[+ (\alpha - 1) \sum_{i=1}^{m} \sum_{g=1}^{G^*} \ln H \left( \frac{1}{\beta t_{gi}} \right) + \alpha (u - G^*) \sum_{i=1}^{m} \ln H \left( \frac{1}{\beta t_{G^*i}} \right). \tag{6.3.2}
\]

Differentiate (6.3.2) with respect to \( \alpha \) and \( \beta \) we have

\[
\frac{\partial l}{\partial \alpha} = \frac{mg^*}{\alpha} + \sum_{i=1}^{m} \sum_{g=1}^{G^*} \ln H \left( \frac{1}{\beta t_{gi}} \right) + (u - G^*) \sum_{i=1}^{m} \ln H \left( \frac{1}{\beta t_{G^*i}} \right), \tag{6.3.3}
\]

\[
\frac{\partial l}{\partial \beta} = \frac{mg^*}{\beta} - \frac{1}{\beta^2 \sum_{i=1}^{m} \sum_{g=1}^{G^*} \frac{\partial g^*}{\partial i} \frac{h}{1 \left( \frac{1}{\beta t_{gi}} \right)} - (\alpha - 1) \sum_{i=1}^{m} \sum_{g=1}^{G^*} \left( \frac{1}{t_{gi}} - \frac{H}{H} \right) \frac{h}{1 \left( \frac{1}{\beta t_{gi}} \right)} - \frac{\alpha (u - G^*)}{\beta^2} \sum_{i=1}^{m} \left( \frac{1}{t_{G^*i}} \frac{H}{H} \right) \frac{h}{1 \left( \frac{1}{\beta t_{gi}} \right)} \tag{6.3.4}
\]

Differentiate (6.3.3) and (6.3.4) with respect to \( \alpha \) and \( \beta \) we have

\[
\frac{\partial^2 l}{u \partial \alpha^2} = -\frac{mg^*}{\alpha^2}. \tag{6.3.5}
\]

\[
\frac{\partial^2 l}{\partial \alpha \partial \beta} = -\frac{1}{\beta^2} \sum_{g=1}^{G^*} \left( \frac{1}{t_{gi}} \frac{H}{H} \right) - \frac{\alpha (u - G^*)}{\beta^2} \sum_{i=1}^{m} \left( \frac{1}{t_{G^*i}} \frac{H}{H} \right) \frac{h}{1 \left( \frac{1}{\beta t_{gi}} \right)} \tag{6.3.6}
\]

\[
\frac{\partial^2 l}{\partial \beta^2} = \frac{mg^*}{\beta^2} + \frac{1}{\beta^4} \sum_{i=1}^{m} \sum_{g=1}^{G^*} \frac{\partial^2 g^*}{\partial i^2} \frac{h}{1 \left( \frac{1}{\beta t_{gi}} \right)} - \frac{\alpha (u - G^*)}{\beta^2} \sum_{i=1}^{m} \left( \frac{1}{t_{G^*i}} \frac{H}{H} \right) \frac{h}{1 \left( \frac{1}{\beta t_{gi}} \right)} \tag{6.3.6}
\]
Since, the likelihood equations (6.3.3) and (6.3.4) are not mathematically tractable, we use the Newton-Raphson method to obtain the estimates of parameters $\alpha$ and $\beta$.

6.4. Application of generalized inverted exponential distribution (GIED)

Consider a member of generalized inverted family of distribution, namely generalized inverted exponential distribution suggested by Abouammoh and Alshingti (2009). Let, $T$ be generalized inverted exponential random variable. The cdf and pdf of $T$ are respectively,

$$F_T(t) = 1 - \left[1 - e\left(-\frac{1}{\beta t}\right)\right]^\alpha, \quad t \geq 0, \alpha, \beta > 0, \quad (6.4.1)$$

and

$$f_T(t) = \frac{\alpha}{\beta t^2} e\left(-\frac{1}{\beta t}\right) \left[1 - e\left(-\frac{1}{\beta t}\right)\right]^{\alpha-1}, \quad t \geq 0, \alpha, \beta > 0. \quad (6.4.2)$$

Here $\alpha$ and $\beta$ are scale and shape parameters respectively.

The reliability function of GIED ($\beta, \alpha$) is given by

$$\overline{F}(t) = \left[1 - e\left(\frac{-1}{\beta t}\right)\right]^{\alpha}, \quad t \geq 0, \alpha, \beta > 0. \quad (6.4.3)$$

The failure rate function of GIED ($\beta, \alpha$) is given by

$$r_T(t) = \frac{f(t)}{R(t)} = \frac{\alpha}{\beta t^2} e\left(\frac{-1}{\beta t}\right) \frac{1}{1 - e\left(\frac{-1}{\beta t}\right)} , \quad t \geq 0, \alpha, \beta > 0. \quad (6.4.4)$$
Since the mean of the distribution has no closed form and it is finite only if \( \alpha > 1 \), we shall consider here the median time to system failure \( M_{dTSF} \) which is given by

\[
M_{dTSF} = -\frac{1}{\beta \ln \left[ 1 - (0.5)^{\alpha} \right]}, \quad \alpha, \beta > 0.
\]  

(6.4.5)

It is clearly seen that the distribution belongs to GIFD if we take

\[
H \left( \frac{1}{\beta t} \right) = \left[ 1 - e^{\left( -\frac{1}{\beta t} \right)} \right],
\]

(6.4.6)

and

\[
h \left( \frac{1}{\beta t} \right) = e^{\left( -\frac{1}{\beta t} \right)},
\]

(6.4.7)

and some other results

\[
h' \left( \frac{1}{\beta t} \right) = \frac{d}{d \beta} h \left( \frac{1}{\beta t} \right) = \frac{d}{d \beta} e^{\left( -\frac{1}{\beta t} \right)} = -e^{\left( -\frac{1}{\beta t} \right)} h \left( \frac{1}{\beta t} \right) = e^{\left( -\frac{1}{\beta t} \right)},
\]

\[
H' \left( \frac{1}{\beta t} \right) = e^{\left( -\frac{1}{\beta t} \right)},
\]

(6.4.8)

Substituting, the equations (6.4.6) to (6.4.8) in equations (6.1.1) to (6.1.13) we obtain the log likelihood function, maximum likelihood equations for parameters and Fisher information of parameters for generalized inverted exponential distribution.

6.4.1. Log Likelihood Function

The log likelihood function is

\[
l = \ln \left( \frac{u!}{(u - G^*)!} \right) + m G^* \ln \alpha - G^* \sum_{i=1}^{m} \ln \beta_i - 2 \sum_{i=1}^{m} \sum_{g=1}^{G^*} \ln t_{gi} - \sum_{i=1}^{m} \sum_{g=1}^{G^*} \left( \frac{1}{\beta_i t_{gi}} \right) \\
+ (\alpha - 1) \sum_{i=1}^{m} \sum_{g=1}^{G^*} \ln \left[ 1 - e^{\left( -\frac{1}{\beta_i t_{gi}} \right)} \right] + \alpha (u - G^*) \sum_{i=1}^{m} \ln \left[ 1 - e^{\left( -\frac{1}{\beta_i G^* t_{gi}} \right)} \right].
\]

(6.4.9)
6.4.2. Maximum Likelihood Estimation

\[
\frac{\partial \ln l}{\partial \alpha} = \frac{mc^*}{\alpha} + \sum_{i=1}^{m} \sum_{g=1}^{G^*} \ln \left(1 - e^{-\frac{1}{\beta_i t_{gi}}}\right) + (u - G^*) \sum_{i=1}^{m} \ln \left(1 - e^{-\frac{1}{\beta_i t_{Gi}^*}}\right). \quad (6.4.10)
\]

Substitute (6.4.8) into (6.2.2) and (6.2.3) we have,

\[
\frac{\partial l}{\partial \beta_i} = -\frac{G^*}{\beta_i} - \frac{1}{\beta_i^2} \sum_{g=1}^{G^*} \frac{1}{t_{gi}} \left[1 + \frac{e^{-\frac{1}{\beta_i t_{gi}}}}{1 - e^{-\frac{1}{\beta_i t_{gi}}}}\right] - \frac{1}{\beta_i^2} \sum_{g=1}^{G^*} \frac{1}{t_{gi}} \left[1 - \frac{1}{t_{gi}} \frac{e^{-\frac{1}{\beta_i t_{gi}}}}{1 - e^{-\frac{1}{\beta_i t_{gi}}}}\right] - \frac{\alpha(u - G^*)}{t_{Gi}^* \beta_i^2} \left[\frac{e^{-\frac{1}{\beta_i t_{Gi}^*}}}{1 - e^{-\frac{1}{\beta_i t_{Gi}^*}}}\right]
\]

\[
= -\frac{G^*}{\beta_i} + \frac{1}{\beta_i^2} \sum_{g=1}^{G^*} \frac{1}{t_{gi}} \left[1 + \frac{e^{-\frac{1}{\beta_i t_{gi}}}}{1 - e^{-\frac{1}{\beta_i t_{gi}}}}\right] - \frac{1}{\beta_i^2} \sum_{g=1}^{G^*} \frac{1}{t_{gi}} \left[\frac{e^{-\frac{1}{\beta_i t_{gi}}}}{1 - e^{-\frac{1}{\beta_i t_{gi}}}}\right] - \frac{\alpha(u - G^*)}{t_{Gi}^* \beta_i^2} \left[\frac{e^{-\frac{1}{\beta_i t_{Gi}^*}}}{1 - e^{-\frac{1}{\beta_i t_{Gi}^*}}}\right]
\]

\[
= -\frac{G^*}{\beta_i} + \frac{1}{\beta_i^2} \sum_{g=1}^{G^*} \frac{1}{t_{gi}} \left[1 + \frac{1}{1 - e^{-\frac{1}{\beta_i t_{gi}}}}\right] - \frac{1}{\beta_i^2} \sum_{g=1}^{G^*} \frac{1}{t_{gi}} \left[\frac{1}{1 - e^{-\frac{1}{\beta_i t_{gi}}}}\right] - \frac{\alpha(u - G^*)}{t_{Gi}^* \beta_i^2} \left[\frac{1}{1 - e^{-\frac{1}{\beta_i t_{Gi}^*}}}\right]
\]

\[
i = 1, 2, \cdots, m. \quad (6.4.11)
\]

Equation (6.4.10) equated to zero we have,

\[
\hat{\alpha} = -\frac{mc^*}{\sum_{i=1}^{m} \sum_{g=1}^{G^*} \ln \left(1 - e^{-\frac{1}{\beta_i t_{gi}}}\right) + (u - G^*) \sum_{i=1}^{m} \ln \left(1 - e^{-\frac{1}{\beta_i t_{Gi}^*}}\right)} \quad (6.4.12)
\]

The estimates of parameters \(\hat{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)\) are obtained in two cases when (i) shape parameter \(\alpha\) is known and (ii) shape parameter \(\alpha\) is unknown.
6.4.2.1. Maximum Likelihood Estimation when $\alpha$ is Known

The solutions of equations (6.4.11) can be evaluated numerically by some suitable iterative procedure such as the Newton-Raphson method, for given values of $\left(u, G, t_{gi}; g = 1, 2, \ldots, G^*; i = 1, 2, \ldots, m\right)$. The method requires some initial value of parameters. We obtain initial value of parameters using method discussed in Section 6.2.1. The MLEs of Reliability $(\bar{F}_i(t_i); i = 1, 2, \ldots, m)$ and hazard rate $(\bar{r}_i(t_i); i = 1, 2, \ldots, m)$ can be evaluated using invariance property of MLEs as

\[
\bar{F}_i(t_i) = 1 - \left(1 - e^{-\beta_i t_i}ight)^\alpha .
\] (6.4.13)

\[
\bar{r}_i(t_i) = \alpha \beta_i \left[\frac{e^{-\beta_i t_i}}{1 - e^{-\beta_i t_i}}\right] \left[\frac{\left(1 - e^{-\beta_i t_i}\right)^\alpha}{1 - \left(1 - e^{-\beta_i t_i}\right)^\alpha}\right], \quad i = 1, 2, \ldots, m .
\] (6.4.14)

6.4.2.1.1. Observed Fisher Information Matrix under Design

The Fisher information matrix is

\[
V = \left(\begin{array}{c}
\left(v_{ij}\right)
\end{array}\right),
\]

where

\[
v_{ij} = -E\left(\frac{\partial^2 l}{\partial \beta_i \partial \beta_j}\right), i, j = 1, 2, \ldots, m.
\]

Now for $i = j$, To obtain $v_{ij}$ we take derivatives of equation (6.4.11) with respect to $\beta_i; i = 1, 2, \ldots, m$.

Therefore we have,

\[
\frac{\partial^2 l}{\partial \beta_i^2} = g^* \frac{\gamma^*}{\beta_i^2} + \sum_{g = 1}^{G^*} \frac{1}{\tau_{gi}^2 \beta_i^2} \left\{ e^{-\frac{1}{\tau_{gi}^2 \beta_i}} e^{-\frac{1}{\tau_{gi}^2 \beta_i}} - \left(-e^{-\frac{1}{\tau_{gi}^2 \beta_i}}\right)^2 + 2\beta_i t_{gi} \left(e^{-\frac{1}{\tau_{gi}^2 \beta_i}}\right) \left(-e^{-\frac{1}{\tau_{gi}^2 \beta_i}}\right)\right\} \left(e^{-\frac{1}{\tau_{gi}^2 \beta_i}}\right)^2
\]
Hence, we use the Newton-Raphson iterative method for the data (6.4.2.2 Maximum Likelihood Estimation When Shape Parameter α is Unknown)

Also,

\[ \frac{\partial^2 l}{\partial \beta_i \partial \beta_j} = 0, \forall j \neq i = 1, 2, \ldots, m. \]  

(6.4.16)

6.4.2.2 Maximum Likelihood Estimation When Shape Parameter α is Unknown

The solutions of ML equations (6.4.11) and (6.4.12) cannot be obtained explicitly. Hence, we use the Newton-Raphson iterative method for the data \((u, G, t_{gi}; g = 1, 2, \ldots, G^*; i = 1, 2, \ldots, m)\). The initial values of the parameters of \(\beta_i; i = 1, 2, \ldots, m\) are obtained through the method of least square as discussed in Section 6.2.1 and initial value α is obtained by using equation (6.4.12). These initial values are used in the Newton-Raphson method to obtain the MLEs of \((\alpha, \beta)\). The MLE of reliability \(\overline{F}_i(t_i)\) and hazard rate \(r_i(t_i), i = 1, 2, \ldots, m\) can be evaluated using invariance property of MLEs as
6.4.2.2.1. Observed Fisher Information Matrix under the Design

The Fisher information matrix is

\[
W = -E \left[ \frac{\partial^2 l}{\partial \alpha^2} \frac{\Lambda'}{V} \right]
\]

\[
V = \left( (v_{ij}) \right),
\]

where,

\[
v_{ij} = \frac{\partial^2 l}{\partial \beta_i \partial \beta_j}, i, j = 1, 2, ..., m,
\]

and

\[
\Delta' = \left( \frac{\partial^2 l}{\partial \alpha \partial \beta_1}, \frac{\partial^2 l}{\partial \alpha \partial \beta_2}, ..., \frac{\partial^2 l}{\partial \alpha \partial \beta_m} \right).
\]

To obtain Fisher information matrix we take derivatives of equation (6.4.11) and (6.4.12) with respect to \( \alpha, \beta_i; i = 1, 2, ..., m \). Therefore we have,

\[
\frac{\partial^2 l}{\partial \alpha^2} = -\frac{m G^*}{\alpha^2}.
\]

\[
\frac{\partial^2 l}{\partial \alpha \partial \beta_i} = -\frac{1}{\beta_i^2} \sum_{g=1}^{G^*} 1 \left\{ \frac{\frac{\beta_{igi}}{g^{1/t_{igi}}} e^{-\frac{1}{G^*_{igi}}} 1}{1 - e^{-\frac{1}{G^*_{igi}}}} \right\} t_{igi} \left( \frac{1}{1 - e^{-\frac{1}{G^*_{igi}}}} \right), \quad i = 1, 2, ..., m. \tag{6.4.17}
\]

\[
\frac{\partial^2 l}{\partial \beta_i^2} = \frac{G^*}{\beta_i^2} - 2 \sum_{g=1}^{G^*} \frac{1}{t_{igi} \beta_i^3}
\]

\[
- (\alpha - 1) \sum_{g=1}^{G^*} \frac{1}{t_{igi} \beta_i^2} \left\{ \left( e^{-\frac{1}{\beta_{igi}}} - 2 \beta_i t_{igi} \left( 1 - e^{-\frac{1}{\beta_{igi}}} \right) \left( e^{-\frac{1}{\beta_{igi}}} \right) \right) \right\}
\]

\[
\left[ 1 - e^{-\frac{1}{\beta_{igi}}} \right]^2
\]
\[ - \frac{\alpha(u-G^*)}{t^{2}_{G^*} \beta^{3}} \left\{ \left( e^{-\frac{1}{\beta_{1}G^*}} \right)^{2} - 2\beta_{1}t_{G^*} \left( 1 - e^{-\frac{1}{\beta_{1}G^*}} \right) \left( e^{-\frac{1}{\beta_{2}G^*}} \right) \right\} , i = 1,2, \ldots, m. \] (6.4.19)

Also,

\[ \frac{\partial^2 I}{\partial \beta_i \partial \beta_j} = 0, \forall j \neq i = 1,2, \ldots m. \] (6.4.20)

### 6.4.3. Algorithm, Numerical Exploration and Conclusions

In this Section, a Monte Carlo simulation study is conducted to compare the performance of the estimates. Maximum likelihood estimates are obtained for observations generated through the Type II censoring design when the numbers of systems to be compared are 2 and 3 for known as well as unknown shape parameter. All calculations are performed with the aid of R-language version R.2.12.0. The simulation study is conducted for both known as well as unknown shape parameter.

#### 6.4.3.1. Known Shape Parameter

In this section, we carry out simulation study for two sets of parameter values

\[ m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3 \] and for \( m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4 \)

for different values of \( u \) and \( G^* \). Here, we keep the total number of failures in the whole experiment \( G = mG^* \) fixed. We simulate 1000 samples for each case. The simulation results are summarized in Table 6.4.1 and Table 6.4.2. To carry out our objective we proceed through following algorithm.

**Step 1:** Taking \( m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3 \) we generate \( u \) random numbers from \( GIED(\alpha, \beta_1, \beta_2, \ldots, \beta_m) \) for each type of systems. The same is repeated for the parameters \( m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4 \).
Step 2: Generate $G^*$ Type II censored observations for each type of systems. The generated $G^*$ failure times are $(t_{1i}, t_{2i}, ..., t_{G^*i}); i = 1,2, ..., m$ for each type of systems.

Step 3: Obtain initial estimate of parameters $\beta_i; i = 1,2, ..., m$ using the least method discussed in Section 6.2.1 i.e. evaluate initial estimates using formulae

$$
\hat{\beta}_i = \frac{\sum_{g=1}^{G^*} y_{gi} t_{gi}}{\sum_{g=1}^{G^*} t_{gi}^2}; i = 1,2, ..., m.
$$

Step 4: Obtain initial value for sample information matrix $\hat{W}$ using the value obtained in Step 3 and also obtain the score vector $S = \left( \frac{\partial l}{\partial \beta_1}, \frac{\partial l}{\partial \beta_2}, ..., \frac{\partial l}{\partial \beta_m} \right)$.

Step 5: Use Newton-Raphson iterative method

$$
\hat{\beta}_{\text{New}} = \hat{\beta}_{\text{Old}} + \hat{W}^{-1} \left( \hat{\beta}_{\text{Old}} \right) * S.
$$

Step 6: Repeat the Step 5 until the $\sum_{i=1}^{m} |\hat{\beta}_{i,\text{New}} - \hat{\beta}_{i,\text{Old}}| < \epsilon$ where $\epsilon$ is very small predefined quantity.

Step 7: Repeat the procedures in Step 1 to Step 6 for $n = 1000$ times and obtain the following quantities.

(a) $EV_i = \frac{\sum_{j=1}^{n} \hat{\beta}_{ij}}{n}$

(b) Mean Squared Error, $MSE_i = \frac{\sum_{j=1}^{n} (\hat{\beta}_{ij} - \beta_i)^2}{n}$ where $\beta_i; i = 1,2, ..., m$ the values of parameters given in Step 1.

(c) Average of variance-covariance matrices computed for different simulated samples, say $\nu^{*-1}$.
(d) Evaluate Reliability functions $\widehat{F}_{ij}(t_i)$ and hazard rate $\widehat{r}_{ij}(t_i); i = 1, 2, ..., m; j = 1, 2, ..., n$ using equations (6.4.3) and (6.4.4) for each simulated sample and obtain the average of all these values.

(e) Obtain $\text{MSE} (\text{Reliability}) = \frac{\sum_{j=1}^{n}(\widehat{F}_{ij}(t_i) - \bar{F}_i(t_i))^2}{n}$ and $\text{MSE} (\text{hazard rate}) = \frac{\sum_{j=1}^{n}(\widehat{r}_{ij}(t_i) - \bar{r}_i(t_i))^2}{n}$ respectively at median given in equation (6.4.5).

Step 8: Obtain Standard Error (SE) of estimates by taking square root of diagonal elements $V^{*-1}$. 
Table 6.4.1
Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and Their Efficiency Measures

\[ m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3 \text{ and } t = (0.4700, 0.5424), \bar{F}(t) = (0.5, 0.5), r(t) = (2.4099, 2.0886) \]

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<th>( G^* )</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>( \bar{F}_1(t_1) )</th>
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Table 6.4.2
Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and Their Efficiency Measures

\[
\begin{align*}
\bar{F}(t) &= (0.5, 0.5, 0.5), \quad r(t) = (2.4099, 2.0886, 2.2493)
\end{align*}
\]

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<th>( G^* )</th>
<th>( \hat{\beta}_1 )</th>
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The conclusions for these studies are given below.

We observe that for the known shape parameter \( \alpha \), averages of estimated values of the parameters are very close to their true values and the averages of mean square errors are relatively small. Further, we observe that the estimates and standard/mean square errors are decreasing functions of number of systems \( u \) of each type put on test.

### 6.4.3.2. Unknown Shape Parameter

To carry out objective of estimating scale parameters, reliability characteristics, hazard rates and their performance measures MSE and SE when shape parameter is unknown, we proceed with following algorithm.
Step 1: Perform Step 1 to Step 3 of Algorithm given in Section 6.4.3.1.

Step 2: Obtain initial estimate of parameters $\beta_i; i = 1, 2, ..., m$ and $\alpha$ using the least method discussed in Section 6.4.3.1 i.e. evaluate initial estimates $\beta_i; i = 1, 2, ..., m$ and $\alpha$ using formulae 

$$\hat{\beta}_{i0} = \frac{\Sigma_{g=1}^{G} y_{g} t_{gi}}{\Sigma_{g=1}^{G} t_{gi}^2}; i = 1, 2, ..., m.$$ 

and

$$\hat{\alpha}_0 = -\frac{\Sigma_{b=1}^{B} \Sigma_{i=1}^{m} \ln \left[ 1 - e^{-\left( \frac{1}{\hat{\beta}_{io} t_{ib}} \right)} \right]}{\Sigma_{b=1}^{B} \Sigma_{i=1}^{m} \ln \left[ 1 - e^{-\left( \frac{1}{\hat{\beta}_{io} t_{ib}} \right)} \right]}$$ 

respectively.

Step 3: Obtain initial value of sample information matrix $\hat{W}$ using the value obtained in Step 2 and also obtains the score vector $S' = (\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \beta_1}, \frac{\partial l}{\partial \beta_2}, ..., \frac{\partial l}{\partial \beta_m})^t$. Here consider, 

$$\underline{\beta} = (\alpha, \beta_1, \beta_2, ..., \beta_m)'$$

Step 5: Use Newton-Raphson iterative method

$$\hat{\beta}_{New} = \hat{\beta}_{Old} + \hat{W}^{-1} \left( \hat{\beta}_{Old} \right) * S$$

Step 6: Repeat the Step 5 until the $\sum_{i=1}^{m+1} |\hat{\beta}_{New} - \hat{\beta}_{Old}| < \varepsilon$ where $\varepsilon$ is very small predefined quantity.

Step 7: Repeat the procedures in Step 1 to Step 6 for $n = 1000$ times and obtain the following quantities.

(a) $EV_\hat{\alpha} = \frac{\sum_{j=1}^{n} \hat{\alpha}_j}{n}$ and $EV_i = \frac{\sum_{j=1}^{n} \hat{\beta}_{ij}}{n}; i = 1, 2, ..., m$

(b) $MSE_\hat{\alpha} = \frac{\sum_{j=1}^{n} (\hat{\alpha}_j - \alpha)^2}{n}$ and $MSE_i = \frac{\sum_{j=1}^{n} (\hat{\beta}_{ij} - \beta_i)^2}{n}$ where $(\alpha, \beta_i); i = 1, 2, ..., m$

the values of parameters given in Step 1.

(c) Average of variance-covariance matrices computed for different simulated samples, say $W^{*+1}$.
(d) Reliability functions $\hat{F}_{ij}(t_i)$ and hazard rate $\hat{r}_{ij}(t_i); i = 1, 2, ..., m; j = 1, 2, ..., n$ evaluate using equations (6.4.13-6.4.14) and corresponding MSE's are $\frac{\sum_{i=1}^{n}(\hat{F}_{ij}(t_i) - F_i(t_i))^2}{n}$ and $\frac{\sum_{i=1}^{n}(\hat{r}_{ij}(t_i) - r(t_i))^2}{n}$ respectively.

Step 8: Obtain Standard Error (SE) of estimates by taking square root of diagonal elements of $W^{-1}$.

We make simulation studies for the set of parameter values $m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, t = (0.4700, 0.5424)$ and $m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4, t = (0.4700, 0.5424, 0.5036)$ by taking n=1000.

### Table 6.4.3

Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and Their Efficiency Measures

$m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, t = (0.4700, 0.5424), F(t) = (0.5, 0.5), r(t) = (2.4099, 2.0886)$

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<th>$\hat{\beta}_2$</th>
<th>$\hat{F}_{ij}(t_i)$</th>
<th>$\hat{F}<em>{ij}(t</em>{i2})$</th>
<th>$\hat{r}<em>{ij}(t</em>{i1})$</th>
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Table 6.4.4
Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and Their Efficiency Measures

\[ r(t) = (2.4099, 2.0886, 2.2493) \]

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<th>( \hat{\alpha} )</th>
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<th>( \hat{\beta}_2 )</th>
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<th>( \hat{F}_1(t_1) )</th>
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<th>( \hat{F}_3(t_3) )</th>
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| 36  | 12  | 3.7464 | 1.3992 | 1.1971 | 1.2955 | 0.4617 | 0.4673 | 0.4660 | 3.0371 | 2.1688 | 2.8132 |
|     |     | 6.0686 | 0.0969 | 0.0683 | 0.0829 | 0.0110 | 0.0113 | 0.0102 | 1.5790 | 1.1968 | 1.2909 |
|     |     | 2.2504 | 0.2875 | 0.2451 | 0.2657 |       |       |       |       |       |       |

| 48  | 16  | 3.3883 | 1.4173 | 1.2222 | 1.3097 | 0.4696 | 0.4740 | 0.4761 | 2.8825 | 2.4756 | 2.6624 |
|     |     | 3.2677 | 0.0671 | 0.0550 | 0.0625 | 0.0079 | 0.0074 | 0.0077 | 0.9885 | 0.6801 | 0.8306 |
|     |     | 1.5922 | 0.2503 | 0.2165 | 0.2315 |       |       |       |       |       |       |

| 60  | 20  | 3.1061 | 1.4421 | 1.2493 | 1.3409 | 0.4804 | 0.4800 | 0.4824 | 2.7300 | 2.3677 | 2.5397 |
|     |     | 1.8389 | 0.0574 | 0.0416 | 0.0484 | 0.0056 | 0.0060 | 0.0058 | 0.6293 | 0.4656 | 0.5389 |
|     |     | 1.2094 | 0.2286 | 0.1979 | 0.2123 |       |       |       |       |       |       |

| 72  | 24  | 3.0412 | 1.4359 | 1.2477 | 1.3457 | 0.4828 | 0.4814 | 0.4851 | 2.6981 | 2.3424 | 2.5212 |
|     |     | 1.3220 | 0.0448 | 0.0336 | 0.0419 | 0.0047 | 0.0047 | 0.0047 | 0.4747 | 0.3555 | 0.4222 |
|     |     | 1.0363 | 0.2065 | 0.1795 | 0.1939 |       |       |       |       |       |       |

| 84  | 28  | 2.8649 | 1.4628 | 1.2744 | 1.3692 | 0.4894 | 0.4869 | 0.4874 | 2.5988 | 2.2585 | 2.4355 |
|     |     | 0.8869 | 0.0395 | 0.0311 | 0.0337 | 0.0036 | 0.0041 | 0.0337 | 0.2440 | 0.3134 |       |
|     |     | 0.8748 | 0.1956 | 0.1706 | 0.1832 |       |       |       |       |       |       |

| 96  | 32  | 2.8502 | 1.4554 | 1.2650 | 1.3650 | 0.4908 | 0.4898 | 0.4883 | 2.5903 | 2.2454 | 2.4244 |
|     |     | 0.7773 | 0.0329 | 0.0260 | 0.0307 | 0.0035 | 0.0032 | 0.0032 | 0.3053 | 0.2125 | 0.2584 |
|     |     | 0.8057 | 0.1816 | 0.1579 | 0.1705 |       |       |       |       |       |       |

In the presence of unknown shape parameter \( \alpha \), from Table 6.4.3 and Table 6.4.4 it is seen that the MLEs of scale parameters \( \beta_i \); \( i = 1, 2, \ldots, m \), the reliability characteristics and hazard rate are getting estimated closely to their true values. However the convergence rate is slow compared to the convergence rate when shape parameter \( \alpha \) is known. Perhaps, it may be the effect of estimate of unknown shape parameter \( \alpha \). Further we can say, slightly large sample size is required than what we consider for the estimates to reach their true values.

6.4.4. Testing of Hypothesis under GIED

The hypothesis given in equation (6.3.1) can be tested for GIED by simply substituting equations (6.4.6-6.4.7) in (6.3.2-6.3.8) we have required equations.
6.4.4.1. Computation of Likelihood under $H_0$

The log likelihood equation (6.4.9) can be written under to as

$$l = m \ln \left( \frac{u}{(u - G^\ast)^l} \right) + mG^\ast \ln \alpha - mG^\ast \ln \beta - 2 \sum_{l=1}^{m} \sum_{g=1}^{G^\ast} \ln t_{gl} - \frac{1}{\beta} \sum_{l=1}^{m} \sum_{g=1}^{G^\ast} \ln t_{gl}$$

$$+ \alpha(u - G^\ast) \sum_{l=1}^{m} \ln \left[ 1 - e^{-\left(\frac{1}{\beta t_{gl}}\right)^l} \right] + (\alpha - 1) \sum_{l=1}^{m} \sum_{g=1}^{G^\ast} \ln \left[ 1 - e^{-\left(\frac{1}{\beta t_{gl}}\right)} \right].$$  \hfill (6.4.21)

$$\frac{\partial l}{\partial \alpha} = \frac{mG^\ast}{\alpha} + \sum_{l=1}^{m} \sum_{g=1}^{G^\ast} \ln \left[ 1 - e^{-\left(\frac{1}{\beta t_{gl}}\right)^l} \right] + (u - G^\ast) \sum_{l=1}^{m} \ln \left[ 1 - e^{-\left(\frac{1}{\beta t_{gl}}\right)} \right].$$ \hfill (6.4.22)

$$\frac{\partial l}{\partial \beta} = -\frac{mG^\ast}{\beta} - \frac{1}{\beta^2} \sum_{l=1}^{m} \sum_{g=1}^{G^\ast} \left( \frac{1}{t_{gl}} \left[ 1 - e^{-\left(\frac{1}{\beta t_{gl}}\right)} \right] - \frac{\alpha(u - G^\ast)}{\beta^2} \sum_{l=1}^{m} \sum_{g=1}^{G^\ast} \left( \frac{1}{t_{gl}} \left[ e^{-\left(\frac{1}{\beta t_{gl}}\right)} \right] \right) \right)$$

$$+ \frac{\alpha(u - G^\ast)}{\beta^2} \sum_{l=1}^{m} \left( \frac{1}{t_{g^\ast l}} \left[ e^{-\left(\frac{1}{\beta t_{gl}}\right)} \right] \right).$$ \hfill (6.4.23)

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{mG^\ast}{\alpha^2}.$$ \hfill (6.4.24)

$$\frac{\partial^2 l}{\partial \alpha \partial \beta} = -\frac{1}{\beta^2} \sum_{l=1}^{m} \sum_{g=1}^{G^\ast} \left( \frac{1}{t_{gl}} \left[ 1 - e^{-\left(\frac{1}{\beta t_{gl}}\right)} \right] \right) - \frac{(u - G^\ast)}{\beta^2} \sum_{l=1}^{m} \sum_{g=1}^{G^\ast} \left( \frac{1}{t_{g^\ast l}} \left[ e^{-\left(\frac{1}{\beta t_{gl}}\right)} \right] \right).$$ \hfill (6.4.25)

$$\frac{\partial^2 l}{\partial \beta^2} = \frac{mG^\ast}{\beta^2} - \frac{2}{\beta^3} \sum_{l=1}^{m} \sum_{g=1}^{G^\ast} \left[ \frac{1}{t_{gl}} \sum_{l=1}^{m} \sum_{g=1}^{G^\ast} \left[ e^{-\left(\frac{1}{\beta t_{gl}}\right)} + 2 \beta t_{gl} \left( 1 - e^{-\left(\frac{1}{\beta t_{gl}}\right)} \right) e^{-\left(\frac{1}{\beta t_{gl}}\right)} \right] \right]$$

$$- \frac{\alpha(u - G^\ast)}{\beta^4} \sum_{l=1}^{m} \sum_{g=1}^{G^\ast} \left[ \frac{1}{t_{g^\ast l}} \left[ e^{-\left(\frac{1}{\beta t_{gl}}\right)} + 2 \beta t_{g^\ast l} \left( 1 - e^{-\left(\frac{1}{\beta t_{gl}}\right)} \right) e^{-\left(\frac{1}{\beta t_{gl}}\right)} \right] \right].$$ \hfill (6.4.26)
Since the likelihood equations (6.4.22) and (6.4.23) are not mathematically tractable for known as well as unknown shape parameter, we use the Newton-Raphson method to obtain the estimate of parameter $\beta$. Here, we deal with only known shape parameter.

### 6.4.4.2 Algorithm for Likelihood Ratio Test

We demonstrate the test procedure for $m = 2$ and $m = 3$. We generate data under our design for the parameter values under $H_0: \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1$ and $H_1: \alpha = 2.5, \beta_1 = 1.9, \beta_2 = 1.5, \beta_3 = 1$ respectively. Then carryout the test procedure as suggested above. The procedure is repeated for the different choices of $u$ and $G^*$. The results are produced in the Table 6.4.5 and Table 6.4.6 respectively. To carry out our objective we proceed through following algorithm.

**Step 1:** Repeat Step 1 to Step 6 of algorithm given in Section 6.4.2 and obtain the estimates of $\beta_i; i = 1, 2, ..., m$ under alternative hypothesis.

**Step 2:** Calculate log likelihood using estimates obtained in Step 1 Say $LL_{H_1}$.

**Step 3:** Using the data generated in Step 1 algorithm given in Section 6.4.2, obtain initial estimate of parameters $\beta$ by average out initial estimates obtained in algorithm given in Section 6.4.3 and call it $\hat{\beta}_0$.

**Step 4:** Obtain initial value of $-\frac{\partial^2 l}{\partial \beta^2}$ using the value obtained in Step 3 and also obtain $\frac{\partial l}{\partial \beta}$.

**Step 5:** Use Newton-Raphson iterative method

$$\hat{\beta}_{New} = \hat{\beta}_{Old} - \left[ \frac{\partial l}{\partial \beta} / \left( \frac{\partial^2 l}{\partial \beta^2} \right)_{\hat{\beta}_{Old}} \right]_{\beta_{Old}}$$
Step 6: Repeat the Step 5 until the \( |\hat{\beta}_{New} - \hat{\beta}_{Old}| < \epsilon \) where \( \epsilon \) is very small predefined quantity.

Step 7: Calculate log likelihood using estimates obtained in Step 6 Say \( LL_{H_0} \).

Step 8: Evaluate \( \chi^2_{cal} = -2 \times [LL_{H_0} - LL_{H_1}] \) and using table obtain \( \chi^2_{(\xi, m-1)} \).

Step 9: If \( \chi^2_{cal} > \chi^2_{(\xi, m-1)} \) or \( p \)-value corresponds to \( \chi^2_{cal} \) is less than \( \xi \), predefined level of significance then reject the null hypothesis otherwise accept it.

Table 6.4.5

<table>
<thead>
<tr>
<th>( u )</th>
<th>( G^* )</th>
<th>( \beta )</th>
<th>( \hat{\beta}_1 )</th>
<th>( \hat{\beta}_2 )</th>
<th>( LL_{H_0} )</th>
<th>( LL_{H_1} )</th>
<th>( \chi^2 )</th>
<th>( p )-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>12</td>
<td>1.3358</td>
<td>1.6602</td>
<td>1.0435</td>
<td>19.89</td>
<td>21.17</td>
<td>2.58</td>
<td>0.1085</td>
</tr>
<tr>
<td>24</td>
<td>18</td>
<td>1.2892</td>
<td>1.5217</td>
<td>1.0778</td>
<td>107.65</td>
<td>109.76</td>
<td>4.22</td>
<td>0.0399</td>
</tr>
<tr>
<td>36</td>
<td>24</td>
<td>1.2828</td>
<td>1.4946</td>
<td>1.0838</td>
<td>146.05</td>
<td>148.55</td>
<td>4.99</td>
<td>0.0255</td>
</tr>
<tr>
<td>48</td>
<td>30</td>
<td>1.2782</td>
<td>1.5000</td>
<td>1.0698</td>
<td>188.27</td>
<td>191.75</td>
<td>6.95</td>
<td>0.0083</td>
</tr>
<tr>
<td>60</td>
<td>36</td>
<td>1.2719</td>
<td>1.4907</td>
<td>1.0679</td>
<td>258.68</td>
<td>262.65</td>
<td>7.95</td>
<td>0.0048</td>
</tr>
<tr>
<td>72</td>
<td>42</td>
<td>1.2668</td>
<td>1.5241</td>
<td>1.0299</td>
<td>301.63</td>
<td>313.60</td>
<td>13.00</td>
<td>0.0003</td>
</tr>
<tr>
<td>96</td>
<td>48</td>
<td>1.2579</td>
<td>1.5139</td>
<td>1.0208</td>
<td>355.89</td>
<td>355.89</td>
<td>15.02</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

From the Table 6.4.5, we infer that the power the test is poor for small sizes. However as the sample size becomes 36 (total number failures observed irrespective type of systems) it exhibits its power in identifying the alternative. It requires more sample size to detect small departure from homogeneity. From Table 6.4.6, it can reveal that for comparing homogeneity of three systems, as sample size becomes 24 it exhibits its power.
to identify magnitude of departure from homogeneity. The consistency of the test is also inferred as sample size tends to 96 the p value becomes almost zero up to three digits in both Table 6.4.5 and Table 6.4.6.

6.5 Design Optimality Criteria

In this Section, three optimality criteria namely; “A-optimality”, “D-optimality” and “E-optimality” for SRD21GED are discussed. These criteria defined in Section 6 of Chapter 3 (i.e. in Section 3.6). Since the variance-covariance matrices of the estimates are not mathematically tractable, we simulate them using Monte-Carlo technique for both cases when shaper parameter is known and when it is unknown. Further, we obtain optimality criterion (A, D and E) for the number of units changing from 24 to 96. The simulated results are given in Table 6.5.1 and Table 6.5.2 for known shape parameter and in Table 6.5.3 and Table 6.5.4 for unknown shape parameter.

6.5.1. Algorithm for Obtaining Optimality Criteria when Shape Parameter is Known

Using Average of the variance-covariance matrices computed for different simulated samples, say \( V^{*-1} \) obtained in Section 6.4.2.1 of Step 7(c), evaluate trace of \( V^{*-1} \), determinant of \( V^{*-1} \) and minimum eigen value of \( V^* \), gives A-optimality, D-optimality and E-optimality of the design respectively when shape parameter is known and unknown. The results are summarized in Table 6.5.1 and Table 6.5.2.

<table>
<thead>
<tr>
<th>( u )</th>
<th>( G' )</th>
<th>( A )</th>
<th>( D )</th>
<th>( E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>12</td>
<td>0.0848</td>
<td>0.001760</td>
<td>20.6609</td>
</tr>
<tr>
<td>36</td>
<td>18</td>
<td>0.0552</td>
<td>0.000745</td>
<td>31.7558</td>
</tr>
<tr>
<td>48</td>
<td>24</td>
<td>0.0411</td>
<td>0.000413</td>
<td>42.6348</td>
</tr>
<tr>
<td>60</td>
<td>30</td>
<td>0.0329</td>
<td>0.000265</td>
<td>53.6333</td>
</tr>
<tr>
<td>72</td>
<td>36</td>
<td>0.0270</td>
<td>0.000184</td>
<td>64.0913</td>
</tr>
<tr>
<td>84</td>
<td>42</td>
<td>0.0233</td>
<td>0.000133</td>
<td>75.3147</td>
</tr>
<tr>
<td>96</td>
<td>48</td>
<td>0.0205</td>
<td>0.000103</td>
<td>85.2022</td>
</tr>
</tbody>
</table>
Table 6.5.2
Optimality Criteria for Type II Censoring Design for Known Shape Parameter

\( m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4 \)

<table>
<thead>
<tr>
<th>( u )</th>
<th>( G^* )</th>
<th>( A )</th>
<th>( D )</th>
<th>( E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>12</td>
<td>0.1259</td>
<td>0.000072</td>
<td>20.9889</td>
</tr>
<tr>
<td>36</td>
<td>18</td>
<td>0.0827</td>
<td>0.000020</td>
<td>31.7852</td>
</tr>
<tr>
<td>48</td>
<td>24</td>
<td>0.0619</td>
<td>0.000008</td>
<td>42.5005</td>
</tr>
<tr>
<td>60</td>
<td>30</td>
<td>0.0491</td>
<td>0.000004</td>
<td>53.5013</td>
</tr>
<tr>
<td>72</td>
<td>36</td>
<td>0.0409</td>
<td>0.000002</td>
<td>63.8464</td>
</tr>
<tr>
<td>84</td>
<td>42</td>
<td>0.0351</td>
<td>0.000001</td>
<td>74.4041</td>
</tr>
<tr>
<td>96</td>
<td>48</td>
<td>0.0300</td>
<td>0.000000</td>
<td>88.1218</td>
</tr>
</tbody>
</table>

6.5.2 Algorithm for Obtaining Optimality Criteria when Shape Parameter is Unknown

Using Average of variance-covariance matrices computed for different simulated samples, say \( V^{-1} \) obtained in Section 6.4.2.2 of Step 7(c), evaluate trace of \( W^{-1} \), determinant of \( W^{-1} \) and minimum eigen value of \( W \), gives A-optimality, D-optimality and E-optimality of Design respectively when shape parameter is known. The results are summarized in Table 6.5.3 and Table 6.5.4.

Table 6.5.3
Optimality Criteria for Type II Censoring Design for unknown Shape Parameter

\( m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3 \)

<table>
<thead>
<tr>
<th>( u )</th>
<th>( G^* )</th>
<th>( A )</th>
<th>( D )</th>
<th>( E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>12</td>
<td>7.7434</td>
<td>0.036826</td>
<td>0.1319</td>
</tr>
<tr>
<td>36</td>
<td>18</td>
<td>2.6302</td>
<td>0.004346</td>
<td>0.3941</td>
</tr>
<tr>
<td>48</td>
<td>24</td>
<td>1.3947</td>
<td>0.001092</td>
<td>0.7502</td>
</tr>
<tr>
<td>60</td>
<td>30</td>
<td>0.8968</td>
<td>0.000395</td>
<td>1.1743</td>
</tr>
<tr>
<td>72</td>
<td>36</td>
<td>0.7189</td>
<td>0.000201</td>
<td>1.4637</td>
</tr>
<tr>
<td>84</td>
<td>42</td>
<td>0.5812</td>
<td>0.000105</td>
<td>1.8092</td>
</tr>
<tr>
<td>96</td>
<td>48</td>
<td>0.4605</td>
<td>0.000063</td>
<td>2.2946</td>
</tr>
</tbody>
</table>

Table 6.5.4
Optimality Criteria for Type II Censoring Design for Unknown Shape Parameter

\( m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4 \)

<table>
<thead>
<tr>
<th>( u )</th>
<th>( G^* )</th>
<th>( A )</th>
<th>( D )</th>
<th>( E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>12</td>
<td>31.6875</td>
<td>0.009770</td>
<td>0.02767</td>
</tr>
<tr>
<td>36</td>
<td>18</td>
<td>5.2778</td>
<td>0.000324</td>
<td>0.1949</td>
</tr>
<tr>
<td>48</td>
<td>24</td>
<td>2.6985</td>
<td>0.000060</td>
<td>0.3847</td>
</tr>
<tr>
<td>60</td>
<td>30</td>
<td>1.5991</td>
<td>0.000015</td>
<td>0.6563</td>
</tr>
<tr>
<td>72</td>
<td>36</td>
<td>1.1865</td>
<td>0.000006</td>
<td>0.8860</td>
</tr>
<tr>
<td>84</td>
<td>42</td>
<td>0.8630</td>
<td>0.000003</td>
<td>1.2239</td>
</tr>
<tr>
<td>96</td>
<td>48</td>
<td>0.7326</td>
<td>0.000000</td>
<td>1.4385</td>
</tr>
</tbody>
</table>
From the Table 6.5.1 to Table 6.5.4, we observe that for both cases when the shape parameter is known and when it is unknown, A-optimal and D-optimal criteria decreases with increase in $G$ while E-optimality criteria increases with increase in $G$.

6.6. The Cost Function

The cost function of the experiment under generalized Type II censoring is $C^u_a$ which was discussed in detail at Section 4.7.1 of Chapter 4 is

$$C^u_a = C_0 + C_1G + C_2t_2 + C_3t_{max} + \gamma(N).$$  \hspace{1cm} (6.6.1)

We carry out Monte-Carlo simulation study for the choice of cost $C_0 = 100$, $C_1 = 5$, $C_2 = 10$, $C_3 = 10$ and $\gamma(N) = 0.5N$ for the cost effectiveness of Type II censoring scheme with different censoring proportions. We consider $m = 2$ and $m = 3$ and results are given in Table 6.5.1 and Table 6.5.2 respectively. Further, we fix total number of failures $G$ and change the number of units to be tested at each sub experiment. The results are given in Table 6.5.3.

6.6.1 Algorithm for Evaluation of Total Time under Experiment, Duration of Experiment and Total Cost

Step 1: Generate the $m$ subset of Type II censoring observations using Step 1 and Step 2 of algorithm given in Section 6.2.1.

Step 2: Repeat Step 1, $n = 1000$ times and obtain average failure times and store it as $(t_{1i}, t_{2i}, ..., t_{G'i})$; $i = 1,2, ..., m$.

Step 3: Evaluate total time on test and duration of test using formulae by

$$t_2 = \sum_{l=1}^{n} t_{1l} + t_{2l} + \cdots + (u - G^* + 1)t_{G'i} + t_{max}$$

Step 4: Use the value of $C_0, C_1, C_2, C_3$ and $\gamma(N) = 0.5N$, where $N = um$ in cost function
\[ C_{a}^{II} = C_0 + C_1 G + C_2 t_2 + C_3 t_{\text{max}} + \gamma(N) \] and obtain the different value of Total cost for different combinations of \((u, G)\).

**Table 6.6.1**
Comparative Table for Total Time under Experiment and Duration of Experiment when no censoring and 1/2 Censoring

\[
m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3
\]

<table>
<thead>
<tr>
<th>(u)</th>
<th>(G)</th>
<th>(t_2)</th>
<th>(t_{\text{max}})</th>
<th>(C)</th>
<th>(1/2) Censoring ((G/m))</th>
<th>(t_2)</th>
<th>(t_{\text{max}})</th>
<th>(C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>12</td>
<td>18.0732</td>
<td>2.9559</td>
<td>442.291</td>
<td>9.9583</td>
<td>0.5272</td>
<td>382.291</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>24</td>
<td>35.4111</td>
<td>3.5768</td>
<td>753.879</td>
<td>19.8305</td>
<td>0.5302</td>
<td>633.879</td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>36</td>
<td>53.5503</td>
<td>4.3074</td>
<td>1074.577</td>
<td>29.9269</td>
<td>0.539</td>
<td>894.577</td>
<td></td>
</tr>
<tr>
<td>48</td>
<td>48</td>
<td>71.3685</td>
<td>4.8756</td>
<td>1390.441</td>
<td>39.833</td>
<td>0.5388</td>
<td>1150.441</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>89.5124</td>
<td>5.4885</td>
<td>1710.009</td>
<td>49.7208</td>
<td>0.5372</td>
<td>1410.009</td>
<td></td>
</tr>
<tr>
<td>72</td>
<td>72</td>
<td>107.6014</td>
<td>5.9442</td>
<td>2027.456</td>
<td>59.7131</td>
<td>0.538</td>
<td>1667.456</td>
<td></td>
</tr>
<tr>
<td>84</td>
<td>84</td>
<td>125.1195</td>
<td>6.209</td>
<td>2337.285</td>
<td>69.8531</td>
<td>0.5423</td>
<td>1917.285</td>
<td></td>
</tr>
<tr>
<td>96</td>
<td>96</td>
<td>143.2841</td>
<td>6.7334</td>
<td>2656.175</td>
<td>79.4327</td>
<td>0.5393</td>
<td>2176.175</td>
<td></td>
</tr>
</tbody>
</table>

**Table 6.6.2**
Comparative Table for Total Time under Experiment and Duration of Experiment when No Censoring and 1/3 Censoring

\[
m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4
\]

<table>
<thead>
<tr>
<th>(u)</th>
<th>(G)</th>
<th>(t_2)</th>
<th>(t_{\text{max}})</th>
<th>(C)</th>
<th>(2/3) Censoring ((G/m))</th>
<th>(t_2)</th>
<th>(t_{\text{max}})</th>
<th>(C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>12</td>
<td>26.6254</td>
<td>2.6323</td>
<td>590.577</td>
<td>12.1862</td>
<td>0.3952</td>
<td>303.78</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>24</td>
<td>53.7196</td>
<td>3.607</td>
<td>1069.266</td>
<td>24.4007</td>
<td>0.4008</td>
<td>509.386</td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>36</td>
<td>80.7912</td>
<td>4.2592</td>
<td>1544.504</td>
<td>36.8109</td>
<td>0.3974</td>
<td>706.117</td>
<td></td>
</tr>
<tr>
<td>48</td>
<td>48</td>
<td>106.9302</td>
<td>5.1795</td>
<td>2013.097</td>
<td>48.9774</td>
<td>0.3999</td>
<td>903.276</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>134.4159</td>
<td>5.5035</td>
<td>2489.194</td>
<td>61.1037</td>
<td>0.4023</td>
<td>1105.037</td>
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</tr>
<tr>
<td>72</td>
<td>72</td>
<td>161.7802</td>
<td>6.293</td>
<td>2968.732</td>
<td>73.2893</td>
<td>0.4041</td>
<td>1304.934</td>
<td></td>
</tr>
<tr>
<td>84</td>
<td>84</td>
<td>187.9244</td>
<td>6.6052</td>
<td>3431.296</td>
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<td>3904.561</td>
<td>97.9054</td>
<td>0.4036</td>
<td>1707.09</td>
<td></td>
</tr>
</tbody>
</table>
### Table 6.6.3

Total Time under Experiment, Duration of Experiment and Cost of Experiment for Fixed Failures

\( m = 3, G = 12, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4 \)

<table>
<thead>
<tr>
<th>( u )</th>
<th>( t_2 )</th>
<th>( t_{\text{max}} )</th>
<th>( C )</th>
<th>( u )</th>
<th>( t_2 )</th>
<th>( t_{\text{max}} )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8.718</td>
<td>1.574</td>
<td>228.92</td>
<td>10</td>
<td>11.0219</td>
<td>0.4387</td>
<td>249.606</td>
</tr>
<tr>
<td>5</td>
<td>8.6535</td>
<td>0.8871</td>
<td>222.906</td>
<td>11</td>
<td>11.6579</td>
<td>0.4155</td>
<td>257.234</td>
</tr>
<tr>
<td>6</td>
<td>8.9423</td>
<td>0.6879</td>
<td>225.302</td>
<td>12</td>
<td>12.1182</td>
<td>0.3897</td>
<td>263.079</td>
</tr>
<tr>
<td>7</td>
<td>9.4497</td>
<td>0.5848</td>
<td>230.845</td>
<td>13</td>
<td>12.7585</td>
<td>0.377</td>
<td>270.855</td>
</tr>
<tr>
<td>8</td>
<td>10.0057</td>
<td>0.5146</td>
<td>237.203</td>
<td>14</td>
<td>13.1882</td>
<td>0.3582</td>
<td>276.465</td>
</tr>
<tr>
<td>9</td>
<td>10.6023</td>
<td>0.4851</td>
<td>244.374</td>
<td>15</td>
<td>13.8141</td>
<td>0.3506</td>
<td>284.147</td>
</tr>
</tbody>
</table>

### Table 6.6.4

Total time under experiment, duration of experiment and cost of experiment for fixed failures

\( m = 3, G = 24, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4 (p = 1 - \frac{c}{m+1}: \text{proportion of censoring}) \)

<table>
<thead>
<tr>
<th>( u )</th>
<th>( p )</th>
<th>( G )</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>( \beta_3 )</th>
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It can be seen from Table 6.6.1 and Table 6.6.2, the total time under the experiment, duration of the test and the cost associated with the experiment under generalized Type II censoring design are comparatively less than those under complete sample for $m=2$ as well as $m=3$. Therefore, in terms of cost, Type II censoring is recommended for comparison of several systems.

From Table 6.6.3 and Table 6.6.4 we can see that as proportion of censoring increases the total time on test also increases linearly and duration time of the experiment decreases. The fact can be seen from Figure 6.6.1 and Figure 6.6.2.

6.6. Concluding remarks

In this chapter, we obtained maximum likelihood estimates of parameters of $m$ lifetimes distributions using a generalization of Type II censoring scheme, proposed by Srivastava (1989), under the assumption of generalized inverted scale family of distributions. We considered generalized inverted family of exponential distribution (GIED) as an example for the more general procedure we discussed. Then we obtained the estimates of parameters involved in the distributions, reliability and hazard rate at a given point. The efficiency measures like MSE and SE were simulated. The likelihood ration test was discussed to test homogeneity of lifetimes of several brands. Finally we
studied the cost effectiveness of the Type II censored sampling scheme, through Monte-Carlo simulation. Expressions given in this chapter can also be used for generalized inverted half-logistic distribution and generalized inverted Rayleigh distribution etc.

**Paper Published**

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