CHAPTER III

CHARACTERIZATION OF INDUCED PAIRED DOMINATION NUMBER OF A GRAPH

In this chapter, we characterize the classes of all graphs whose sum of induced paired domination number and chromatic number equals to $2n - 6$ and $2n-7$, for any $n \geq 4$.

The concept of induced paired domination number of a graph was introduced by D.S. Studer, T.W. Haynes and L.M. Lawson [27], with the following application in mind.

In the guard application an induced paired dominating set represents a configuration of security guards in which each guard is assigned one other as a designated backup within (as in a paired dominating set), but to avoid conflicts (such as radio interference) between a guard and his/her backup, we require that the backup each guard be unique. Since among the guards only designated partners are adjacent to each other, we reduce the possibility of conflicts in communication.

A set $S \subseteq V$ is a induced -paired dominating set if $S$ is a dominating set of $G$ and the induced subgraph $<S>$ is a set of independent edges. The induced - paired domination number $\pi_p(G)$ is the minimum cardinality taken over all induced paired dominating sets in $G$.

In [4], the authors characterized the classes of all graphs whose sum of induced paired domination number and chromatic number of order up to $2n - 5$.

In this chapter, we characterize the classes of all graphs whose sum of induced paired domination number and chromatic number equals to $2n - 6$, and $2n - 7$ for any $n \geq 4$.

We use the following preliminary result for our consequent characterization:
Theorem 3.1[27] If \( G \) is a connected graph of order \( n \geq 3 \), then \( \gamma_{ip}(G) \leq n - 1 \) and equality holds if and only if \( G \) is isomorphic to \( P_3, C_3, P_5 \) or \( G' \) where \( G' \) is the graph as in the following figure 3.1.

![Figure 3.1](image)

**Figure 3.1(\ where \ s, t \geq 2)\)**

In [4], the authors characterized the classes of all graphs whose sum of induced paired domination number and chromatic number of order up to \( 2n - 5 \).

In this chapter, we characterize all graphs for which \( \gamma_{ip}(G) + \chi(G) = 2n - 6 \) and \( \gamma_{ip}(G) + \chi(G) = 2n - 7 \) for any \( n \geq 4 \).

**Theorem 3.2** For any connected graph \( G \) of order \( n, n \geq 4 \), \( \gamma_{ip} + \chi = 2n - 6 \) if only if \( G \cong K_8, K_4(P_4), P_6, C_6, K_{1,4}, S^*(K_{1,3}), C_4(P_2), K_4(P_3), K_6(1), K_6(2), K_6(3), K_6(4), K_6(5), K_4(u(P_3, P_3)), C_4(P_2, 0, P_2, 0), C_4(P_3), K_4(P_3, P_2, 0, 0), K_4(P_2, P_2, 0, 0), K_4(2P_2, 0, 0, 0) \) or any one of the graphs shown in figure 3.2.
Proof: If $G$ is any one the graphs given in the figure 3.2, then it can be verified that

$$\gamma_{ip}(G) + \chi(G) = 2n - 6.$$  

Conversely, let $\gamma_{ip}(G) + \chi(G) = 2n - 6$. Then the various possible cases are

(i) $\gamma_{ip}(G) = n - 1$ and $\chi(G) = n - 5$
(ii) $\gamma_{ip}(G) = n - 2$ and $\chi(G) = n - 4$
(iii) $\gamma_{ip}(G) = n - 3$ and $\chi(G) = n - 3$
(iv) $\gamma_{ip}(G) = n - 4$ and $\chi(G) = n - 2$
(iv) $\gamma_{ip}(G) = n - 5$ and $\chi(G) = n - 1$
(vi) $\gamma_{ip}(G) = n - 6$ and $\chi(G) = n$.

Case i. $\gamma_{ip}(G) = n - 1$ and $\chi(G) = n - 5$.

Since $\gamma_{ip}(G) = n - 1$, By theorem 3.1, $G$ is isomorphic to $P_3$, $C_3$, $P_5$, or $G'$. Since $\chi(G) = n - 5$, $G$ is not isomorphic to $P_3$, $C_3$, $P_5$, or $G'$. Hence no graph exists.

Case ii. $\gamma_{ip}(G) = n - 2$ and $\chi(G) = n - 4$.

Since $\chi(G) = n - 4$, $G$ contains a clique $K$ on $n - 4$ vertices or does not contain a clique $K$ on $n - 4$ vertices.

Let $G$ contains a clique $K$ on $n - 4$ vertices.

Figure 3.2
Let $S = \{v_1, v_2, v_3, v_4\}$. Then the induced subgraph $<S>$ has the following possible cases.

$<S> = K_4, K_3, P_4, C_4, K_{1,3}, K_2 \cup K_2, K_3 \cup K_1, K_4 - \{e\}, C_3(1, 0, 0), P_3 \cup K_1, K_3 \cup K_1, K_4$.

**Subcase i.** Let $<S> = K_4$.

Since $G$ is connected, there exists a vertex $u_i$ of $K_{n-4}$ which is adjacent to any one of $\{v_1, v_2, v_3, v_4\}$. Let $u_i$ be adjacent to $v_1$ for some $i$ in $K_{n-4}$. Then $\{v_1, u_i\}$ is an $\gamma_{ip}$ -set of $G$, so that $\gamma_{ip} = 2$ and $n = 4$, which is a contradiction. Hence no graph exists.

**Subcase ii.** Let $<S> = K_3$.

Let $\{v_1, v_2, v_3, v_4\}$ be the vertices of $K_3$.

Since $G$ is connected, two vertices of the $K_3$ are adjacent to one vertex say $u_i$ and the remaining two vertices of $K_3$ are adjacent to one vertex say $u_j$ for $i \neq j$. In this case $\{u_i, u_j\}$ for $i \neq j$ is a $\gamma_{ip}$ -set of $G$, so that $\gamma_{ip} = 2$ and $n = 4$, which is a contradiction. Hence no graph exists.
Since G is connected, one vertex of $K_4$ is adjacent to $u_i$ and the remaining three vertices of $K_4$ are adjacent to vertex say $u_j$ for $i \neq j$. In this case $\{u_i, u_j\}$ for $i \neq j$ forms a $\gamma_{ip}$-set of G, so that $\gamma_{ip} = 2$ and $n = 4$, which is a contradiction. Hence no graph exists.

Since G is connected, all the vertices of $K_n$ are adjacent to one vertex say $u_i$ in the vertices of $K_{n-4}$. In this case $\{u_i, u_j\}$ for $i \neq j$ is a $\gamma_{ip}$-set of G, so that $\gamma_{ip} = 2$ and $n = 4$, which is a contradiction. Hence no graph exists.

Since G is connected, two vertices of $K_n$ is adjacent to $u_i$ and one vertex is adjacent to $u_j$ for $i \neq j$ and the remaining one vertex is adjacent to a vertex say $u_k$ for $i \neq j \neq k$. In this case $\gamma_{ip}$-set does not exist. If $u_i$ is adjacent to $v_1$ and $u_j$ for $i \neq j$ is adjacent to $v_2$ and $u_k$ for $i \neq j \neq k$ is adjacent to $v_3$ and $u_i$ for $i \neq j \neq k \neq s$ is adjacent to $v_4$. In this case $\gamma_{ip}$-set does not exist.

**Subcase iii.** Let $<S> = P_4 = v_1v_2v_3v_4$.

Since G is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$ (or $v_4$) or $v_2$ (or $v_3$). If $u_i$ is adjacent to $v_1$, then $\{u_i, v_2, v_3\}$ forms a $\gamma_{ip}$ set of G, so that $\gamma_{ip} = 4$ and $n = 6$. Hence $K = K_2 = u_1u_2$. If $u_1$ is adjacent to $v_1$. If $\deg (v_1) = 2$, $\deg (v_2) = \deg (v_3) = 2$, $\deg (v_4) = 1$, then $G \cong P_6$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_4$. If $\deg (v_1) = 2 = \deg (v_2)$, $\deg (v_3) = 2 \neq \deg (v_4)$, then $G \cong C_6$. If $u_i$ is adjacent to $v_2$, then $\{u_j, v_k, v_2, v_3\}$ for some $u_j$ and $u_k$ for $i \neq j \neq k$ in $K_{n-4}$ forms a $\gamma_{ip}$-set of G, so that $\gamma_{ip} = 4$ and $n = 6$ and hence $K = K_2 = u_1u_2$. Hence no graph exists.

**Subcase iv.** Let $<S> = K_2 \cup K_2$.

Let $v_1$, $v_2$ be the vertices of $K_2$ and $v_3$, $v_4$ be the vertices of $K_2$. Since G is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to any one of $\{v_1, v_2\}$ and any one of $\{v_3, v_4\}$.
Let $u_i$ be adjacent to $v_1$ and $v_3$. In this case \{$u_i, u_k, v_1, v_2, v_3, v_4\}$ for some $u_i$ and $u_k$ for $i \neq j \neq k$ in $K_{n-4}$ forms an $\gamma_{ip}$-set of $G$ so that $\gamma_{ip} = 6$ and $n = 8$ and hence $K = K_4 = \langle u_1, u_2, u_3, u_4 \rangle$.

Let $u_3$ be adjacent to $v_1$ and $v_3$. If $\text{deg} (v_1) = 2 = \text{deg} (v_3)$, $\text{deg} (v_2) = 1$, $\text{deg} (v_4) = 1$, then $G \cong K_4 (u(P_3, P_3))$.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_3$ and $u_j$ for $i \neq j$ is adjacent to $v_1$ and $u_k$ for $i \neq j \neq k$ and $u_s$ for $i \neq j \neq k \neq s$. In this case \{$v_1, v_3, v_4, u_j\}$ is an $\gamma_{ip}$-set of $G$ so that $\gamma_{ip} = 4$ and $n = 6$ and which is a contradiction. Hence no graph exists.

Subcase v. \langle S \rangle = K_2 \cup P_2.

Let $v_1, v_2$ be the vertices of $K_2$ and $v_3, v_4$ be the vertices of $K_2$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$, which is adjacent to $v_1$ and $v_2$ and any one of $\{$$v_3, v_4$$\}$. 

Let $u_i$ be adjacent to $v_1, v_2, v_3$. In this case \{$u_i, v_3\}$ is a $\gamma_{ip}$-set of $G$ so that $\gamma_{ip} = 2$ and $n = 4$, which is a contradiction. Hence no graph exists.

Since $G$ is connected there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$ and there exists a vertex $u_j$ for $i \neq j$ in $K_{n-4}$ is adjacent to $v_2$ and $v_3$. In this case $\gamma_{ip}$-does not exist.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent $v_1$ and $u_j$ for $i \neq j$ and $u_k$ for $i \neq j \neq k$ is adjacent to $v_3$. In this case \{$u_i, u_j, v_3, v_4\}$ forms a $\gamma_{ip}$-set of $G$. So that $\gamma_{ip} = 4$ and $n = 6$, which is a contradiction. Hence no graph exists.

Subcase vi. \langle S \rangle = P_3 \cup K_1.

Let $v_1, v_2, v_3$ be the vertices of $P_3$ and $v_4$ be the vertex of $K_1$.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to any one of \{$v_1, v_2, v_3\}$ and $v_4$. In this case \{$u_i, v_2, v_3, v_4\}$ is a $\gamma_{ip}$-set of $G$ so that $\gamma_{ip} = 4$ and $n = 6$. Hence $K = K_2 = u_1u_2$. 

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Let \( u_1 \) be adjacent to \( v_1 \) and \( v_4 \). If \( \deg (v_1) = 2 = \deg (v_2) \), \( \deg (v_4) = 1 = \deg (v_3) \) then \( G \cong G_1 \). Let \( u_1 \) be adjacent to \( v_1 \) and \( v_4 \) and \( u_2 \) be adjacent to \( v_2 \). If \( \deg (v_1) = 2 \), \( \deg (v_2) = 3 \), \( \deg (v_3) = 1 = \deg (v_4) \), then \( G \cong C_4 (P_2, 0, P_2, 0) \).

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_1 \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_4 \). In this case \( \{u_i, u_j, v_2, v_3\} \) is a \( \gamma_{ip} \)-set of \( G \), so that \( \gamma_{ip} = 4 \) and \( n = 6 \) and hence \( K = K_2 = u_1 u_2 \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( v_3 \) and \( u_2 \) be adjacent to \( v_4 \). If \( \deg (v_1) = 2 \), \( \deg (v_2) = 2 \), \( \deg (v_3) = 2 \), \( \deg (v_4) = 1 \), then \( G \cong C_4 (P_3) \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( u_2 \) be adjacent to \( v_4 \). If \( \deg (v_1) = 2 = \deg (v_2) \), \( \deg (v_3) = 1 \), \( \deg (v_4) = 1 \), then \( G \cong P_5 \).

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_2 \) and \( v_4 \). In this case \( \{u_i, v_2\} \) is a \( \gamma_{ip} \)-set of \( G \), so that \( \gamma_{ip} = 2 \) and \( n = 4 \), which is a contradiction.

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \), which is adjacent to \( v_2 \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_4 \). In this case \( \{u_j, u_k, v_2, v_3\} \) for some \( u_j \) for \( i \neq j \neq k \) in \( K_{n-4} \) is a \( \gamma_{ip} \)-set of \( G \), so that \( \gamma_{ip} = 4 \) and \( n = 6 \) and hence \( K = K_2 = u_1 u_2 \). Let \( u_1 \) be adjacent to \( v_2 \) and \( u_2 \) be adjacent to \( v_4 \). If \( \deg (v_1) = 1 \), \( \deg (v_4) = \deg (v_3) = 1 \), \( \deg (v_2) = 3 \), then \( G \cong G_1 \).

**Subcase vii.** \( <S> = K_3 \cup K_1 \).

Let \( v_1, v_2, v_3 \) be the vertices of \( K_3 \) and \( v_4 \) be the vertices of \( K_1 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) is adjacent to any one of \( \{v_1, v_2, v_3\} \) and \( v_4 \). In this case \( \{u_i, v_2\} \) is a \( \gamma_{ip} \)-set of \( G \), so that \( \gamma_{ip} = 2 \) and \( n = 4 \), which is a contradiction. Hence no graph exists.

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_2 \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_4 \). In this case \( \{u_j, u_k, v_1, v_2\} \) for some \( u_k \) for \( i \neq j \neq k \) in \( K_{n-4} \) is a \( \gamma_{ip} \)-set of \( G \), so that \( \gamma_{ip} = 4 \) and \( n = 6 \) and hence \( K = K_2 \), which is a contradiction. Hence no graph exists.
Subcase viii. $\langle S \rangle = K_4 - \{e\}$

Let $v_1, v_2, v_3, v_4$ be the vertices of $K_4$. Let $\{e\}$ be any one the edge inside the cycle $C_4$.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$ of degree 3. In this case $\{u_i, v_1\}$ is a $\gamma_{ip}$-set of $G$, so that $\gamma_{ip} = 2$ and $n = 4$, which is a contradiction. Hence no graph exists.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_2$ of degree 2. In this case $\{u_i, u_j, v_1, v_3\}$ is a $\gamma_{ip}$-set of $G$. So that $\gamma_{ip} = 4$ and $n = 6$, which is a contradiction. Hence no graph exists.

Subcase ix. $\langle S \rangle = C_3 (1, 0, 0)$.

Let $v_1, v_2, v_3$ be the vertices of $C_3$ and $v_4$ is adjacent to $v_1$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_2$ and $u_j$ for $i \neq j$ and $u_k$ for $i \neq j \neq k$. In this case $\{u_i, u_k, v_1, v_2\}$ is a $\gamma_{ip}$-set of $G$, so that $\gamma_{ip} = 4$ and $n = 6$, which is a contradiction. Hence no graph exists.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$. In this case $\{u_i, v_1\}$ is a $\gamma_{ip}$-set of $G$, so that $\gamma_{ip} = 2$ and $n = 4$, which is a contradiction. Hence no graph exists.

Since $G$ is connected there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_4$ and $u_j$ for $i \neq j$. In this case $\{u_i, u_j, v_1, v_2\}$ is a $\gamma_{ip}$-set of $G$, so that $\gamma_{ip} = 4$ and $n = 6$, which is a contradiction. Hence no graph exists.

Subcase x. $\langle S \rangle = K_{1,3}$.

Let $v_1$ be the root vertex and $v_2, v_3, v_4$ are adjacent to $v_1$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$. In this case $\{u_i, v_1\}$ is a $\gamma_{ip}$-set of $G$, so that $\gamma_{ip} = 2$ and $n = 4$, which is a contradiction. Hence no graph exists.
Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to any one of 
\{v_2, v_3, v_4\}.

Let $u_i$ be adjacent to $v_2$. In this case \{u_j, u_k, v_1, v_2\} is a $\gamma_{ip}$-set of $G$, so that $\gamma_{ip} = 4$ and 
n = 6, and hence $K = K_2 = u_1u_2$. Let $u_1$ be adjacent to $v_2$. If $\deg(v_1) = 3$, $\deg(v_2) = 2$, 
$\deg(v_3) = 1 = \deg(v_4)$, then $G \cong G_1$.

Let $u_1$ be adjacent to $v_2$ and $v_3$. If $\deg(v_1) = 3$, $\deg(v_2) = 2 = \deg(v_3)$, $\deg(v_4) = 1$, then 
$G \cong C_4(P_2, 0, P_2, 0)$ . Let $u_1$ be adjacent to $v_2$ and $v_4$. If $\deg(v_1) = 3$, $\deg(v_2) = 2 = \deg(v_4)$, 
$\deg(v_3) = 1$, then $G \cong C_4(P_2, 0, P_2, 0)$.

Subcase xi. $\langle S \rangle = C_4$.

In this case it can be verified that no graph exists.

If $G$ does not contain a clique $K$ on $n - 4$ vertices, then it can be verified that no new graph exits.

Case iii. $\gamma_{ip} = n - 3$ and $\chi = n - 3$.

Since $\chi = n - 3$, $G$ contains a clique $K$ on $n - 3$ vertices or does not contain a clique $K$ on 
n - 3 vertices.

Let $G$ contains a clique $K$ on $n - 3$ vertices.
Let $S = V(G) - V(K) = \{v_1, v_2, v_3\}$. Then the induced subgraph $<S>$ has the following possible cases. $<S> = K_3, P_3, K_2 \cup K_1$.

**Subcase i.** $<S> = K_3 = <v_1, v_2, v_3>$.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ which is adjacent to any one of $\{v_1, v_2, v_3\}$. Let $u_i$ be adjacent to $v_1$, then $\{u_i, v_1\}$ is an $\gamma_{ip}$-set of $G$, so that $\gamma_{ip} = 2$ and $n = 5$, which is a contradiction. Hence no graph exists.

**Subcase ii.** $<S> = P_3 = <v_1, v_2, v_3>$.

Since $G$ is connected, one of the vertices of $K_{n-3}$ say $u_i$ is adjacent to all the vertices of $S$. If $u_j$ be adjacent to $v_1$, $v_2$ and $u_k$ be adjacent to $v_3$ for $i \neq j$ (or) $u_i$ be adjacent to $v_1$ and $u_j$ be adjacent to $v_2$ and $u_k$ be adjacent to $v_3$ for $i \neq j \neq k$. If $u_i$ for some $i$ is adjacent to all the vertices of $S$, then $\{u_i, u_j\}$ for some $u_i$ for $i \neq j$ in $K_{n-3}$ is a $\gamma_{ip}$-set of $G$, so that $\gamma_{ip} = 2$ and $n = 5$ and hence $K = K_2 = u_1u_2$. If $u_1$ is adjacent to $v_1$, $v_2$ and $v_3$, then $G \cong K_{1,4}$. If $u_i$ is adjacent to $v_1$ and $u_j$ for
i ≠ j is adjacent to v₂ and v₃, then \{uᵢ, uⱼ\} is an \(\gamma_{ip}\) -set of G, so that \(\gamma_{ip} = 2\) and \(n = 5\). Hence K = \(K₂ = u₁u₂\). If \(u₁\) is adjacent to v₁ and v₂ and \(u₂\) is adjacent to v₃, then G \(\cong S^*(K_{1,3})\).

Since G is connected, if \(uᵢ\) is adjacent to v₁, uⱼ for \(i ≠ j\) in \(K_{n-3}\) is adjacent to v₂ and uₖ for \(i ≠ j ≠ k\) in \(K_{n-3}\) is adjacent to v₃. Then \(\gamma_{ip}\) -set does not exist.

**Subcase iii.** \(<P₃> = v₁v₂v₃\).

Since G is connected, there exists a vertex \(uᵢ\) in \(K_{n-3}\) which is adjacent to v₁ (or equivalently v₃) or v₂.

If \(uᵢ\) is adjacent to v₂, then \{uᵢ, v₂\} is a \(\gamma_{ip}\) -set of G, so that \(\gamma_{ip} = 2\) and \(n = 5\). Hence K = \(K₂ = u₁u₂\). If \(u₁\) is adjacent to v₂ then G \(\cong S^*(K_{1,3})\).

If \(u₁\) is adjacent to v₂ and \(u₂\) is adjacent to v₃. If \(deg (v₁) = 1\), \(deg (v₂) = 3\), \(deg (v₃) = 2\), then G \(\cong C₄(P₂)\).

If \(u₁\) is adjacent to v₂ and \(u₂\) is adjacent to v₁ and v₃. If \(deg (v₁) = 2\), \(deg (v₂) = 3\), \(deg (v₃) = 2\), then G \(\cong G₂\).

Since G is connected, there exists a vertex \(uᵢ\) in \(K_{n-3}\) which is adjacent to v₁. Then \{uᵢ, uⱼ, v₂, v₃\} and uⱼ for \(i ≠ j\) is a \(\gamma_{ip}\) -set of G, so that \(\gamma_{ip} = 4\) and \(n = 7\) and hence K = \(K₄ = <u₁, u₂, u₃, u₄>\).

Let \(u₁\) be adjacent to v₁. If \(deg (v₁) = 2 = deg (v₂)\), \(deg (v₃) = 1\), then G \(\cong K₄(P₄)\).

Let \(u₁\) be adjacent to v₁ and \(u₃\) be adjacent to v₁. If \(deg (v₁) = 3\), \(deg (v₂) = 2\), \(deg (v₃) = 1\), then G \(\cong G₃\).
Subcase iv. \(<S> = K_2 \cup K_1\).

Let \(v_1, v_2\) be the vertices of \(K_2\) and \(v_3\) be the isolated vertex.

Since \(G\) is connected, there exists a vertex \(u_i\) in \(K_{n-3}\) which is adjacent to any one of \(\{v_1, v_2\}\) and \(\{v_3\}\) (or) \(u_i\) is adjacent to any one of \(\{v_1, v_2\}\) and \(u_j\) for \(i \neq j\) is adjacent to \(v_3\). In this case \(\{v_1, v_2, v_3, u_j\}\) is a \(\gamma_{ip}\) -set of \(G\), so that \(\gamma_{ip} = 4\) and \(n = 7\) and hence \(K = K_4 = \langle u_1, u_2, u_3, u_4 \rangle\).

Let \(u_1\) be adjacent to \(v_1\) and \(u_2\) be adjacent to \(v_3\). If \(\deg (v_1) = 2, \deg (v_2) = 1 = \deg (v_3)\), then \(G \cong K_d(P_3, P_2, 0, 0)\).

Let \(u_1\) be adjacent to \(v_1\) and \(u_3\) be adjacent to \(v_1\) and \(u_2\) be adjacent to \(v_3\). If \(\deg (v_1) = 3, \deg (v_2) = 1 = \deg (v_3)\), then \(G \cong G_d\). If a vertex \(u_i\) in \(K_{n-3}\) is adjacent to \(v_1\) and \(v_3\) then \(\{u_i, v_1\}\) is a \(\gamma_{ip}\) -set of \(G\), so that \(\gamma_{ip} = 2\) and \(n = 5\) and hence \(K = K_2 = u_1u_2\).

Let \(u_1\) be adjacent to \(v_1\) and \(v_3\). If \(\deg (v_1) = 2, \deg (v_2) = 1 = \deg (v_3)\) then \(G \cong S^*(K_{1,3})\).

If \(G\) does not contain a clique \(K\) on \(n - 3\) vertices, then it can be verified that no new graph exists.

Case v. \(\gamma_{ip} = n - 4\) and \(\chi = n - 2\).

Since \(\chi = n - 2\), \(G\) contains a clique \(K\) on \(n - 2\) vertices or does not contain a clique \(K\) on \(n - 2\) vertices.

Let \(G\) contains a clique \(K\) on \(n - 2\) vertices.
Let $S = V(G) - V(K) = \{v_1, v_2\}$. Then $<S> = K_{n-2}$.

**Subcase i.** $<S> = K_2$.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-2}$ is adjacent to any one of $\{v_1, v_2\}$ then $\{u_i, v_1\}$ is a $\gamma_{ip}$-set of $G$, so that $\gamma_{ip} = 2$ and $n = 6$ and hence $K = K_4 = <u_1, u_2, u_3, u_4>$.

Let $u_1$ be adjacent to $v_1$. If $\deg (v_1) = 2$, $\deg (v_2) = 1$, then $G \cong K_4(P_3)$.

Let $u_1$ be adjacent to $v_1$ and $u_3$ be adjacent to $v_1$. If $\deg (v_1) = 3$, $\deg (v_2) = 1$, then $G \cong G_5$.

Let $u_1$ be adjacent to $v_1$ and $u_3$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$. If $\deg (v_1) = 4$, $\deg (v_2) = 1$ then $G \cong G_6$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg (v_1) = 2 = \deg (v_2)$, then $G \cong G_7$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_1$. If $\deg (v_1) = 3$, $\deg (v_2) = 2$, then $G \cong G_8$. 

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Let \( u_1 \) be adjacent to \( v_1 \) and \( u_2 \) be adjacent to \( v_2 \) and \( u_3 \) be adjacent to \( v_2 \) and \( u_4 \) be adjacent to \( v_2 \). If \( \deg(v_1) = 2, \deg(v_2) = 4 \), then \( G \cong G_9 \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( u_2 \) be adjacent to \( v_1 \) and \( v_2 \). If \( \deg(v_1) = 3, \deg(v_2) = 2 \), then \( G \cong G_{10} \).

**Subcase (ii)** Let \( <S> = \mathbb{R}_2 \).

Since \( G \) is connected, \( v_1 \) and \( v_2 \) are adjacent to a common vertex say \( u_i \) of \( K_{n-2} \) (or) \( v_1 \) is adjacent to \( u_i \) for some \( i \) and \( v_2 \) is adjacent to \( u_i \) for some \( i \neq j \) in \( K_{n-2} \). In both cases \( \{ u_i, u_j \} \) is a \( \gamma_p \)-set of \( G \), so that \( \gamma_p = 2 \) and \( n = 6 \) and hence \( K = K_4 = <u_1, u_2, u_3, u_4> \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( u_2 \) be adjacent to \( v_2 \). If \( \deg(v_1) = 1 = \deg(v_2) \), then \( G \cong K_4(P_2, P_2, 0, 0) \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( v_2 \). If \( \deg(v_1) = 1 = \deg(v_2) \), then \( G \cong K_4(2P_2, 0, 0, 0) \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( u_2 \) be adjacent to \( v_2 \). If \( \deg(v_1) = 1, \deg(v_2) = 2 \), then \( G \cong G_{11} \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( u_2 \) be adjacent to \( v_1 \) and \( v_2 \). If \( \deg(v_1) = 2 = \deg(v_2) \), then \( G \cong G_{12} \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( u_2 \) be adjacent to \( v_2 \) and \( u_3 \) be adjacent to \( v_1 \) and \( u_4 \) be adjacent to \( v_2 \). If \( \deg(v_1) = 2 = \deg(v_2) \), then \( G \cong G_{13} \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( u_3 \) be adjacent to \( v_1 \). If \( \deg(v_1) = 2, \deg(v_2) = 1 \), then \( G \cong G_{14} \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( v_2 \) and \( u_2 \) be adjacent to \( v_2 \) and \( u_4 \) be adjacent to \( v_2 \). If \( \deg(v_1) = 1, \deg(v_2) = 3 \), then \( G \cong G_{15} \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( v_2 \) and \( u_2 \) be adjacent to \( v_1 \) and \( v_2 \) and \( u_3 \) be adjacent to \( v_1 \) and \( u_4 \) be adjacent to \( v_2 \). If \( \deg(v_1) = 3, \deg(v_2) = 3 \), then \( G \cong G_{16} \).
If $G$ does not contain a clique $K$ on $n - 2$ vertices, then it can be verified that no new graph exits.

**Case v.** $\gamma_{ip} = n - 5$ and $\chi = n - 1$.

Since $\chi = n - 1$, $G$ contains a clique $K$ on $n - 1$ vertices.

![Figure 3.2 (d)](image)

Let $v_1$ be the vertex not on $K_{n-1}$. Since $G$ is connected, there exists a vertex $v_1$ is adjacent to one vertex $u_i$ of $K_{n-1}$.

In this case $\{u_i, v_1\}$ is a $\gamma_{ip}$-set of $G$, so that $\gamma_{ip} = 2$ and $n = 7$ and hence $K = K_6 = \langle u_1, u_2, u_3, u_4, u_5, u_6 \rangle$.

Let $u_1$ be adjacent to $v_1$. If $\deg(v_1) = 1$, then $G \cong K_6(1)$.

Let $u_1$ be adjacent to $v_1$ and $u_6$ be adjacent to $v_1$. If $\deg(v_1) = 2$, then $G \cong K_6(2)$.

Let $u_1$ be adjacent to $v_1$ and $u_6$ be adjacent to $v_1$ and $u_5$ be adjacent to $v_1$. If $\deg(v_1) = 3$, then $G \cong K_6(3)$.

Let $u_1$ be adjacent to $v_1$ and $u_6$ be adjacent to $v_1$ and $u_5$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 4$, then $G \cong K_6(4)$. 

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Let \( u_1 \) be adjacent to \( v_1 \) and \( u_6 \) be adjacent to \( v_1 \) and \( u_5 \) be adjacent to \( v_1 \) and \( u_4 \) be adjacent to \( v_1 \) and \( u_3 \) be adjacent to \( v_1 \). If \( \deg(v_1) = 5 \), then \( G \cong K_6(5) \).

If \( G \) does not contain clique \( K \) on \( n - 1 \) vertices, then it can be verified that no new graph exists.

**Case vi.** \( \gamma_{ip} = n - 6 \) and \( \chi = n \).

Since \( \chi = n \), \( G = K_n \). But for \( K_n \), \( \gamma_{ip} = 2 \), so that \( n = 8 \). Hence \( G \cong K_8 \).

**Theorem 3.3** For any connected graph \( G \) of order \( n (n \geq 4) \), \( \gamma_{ip}(G) + \chi(G) = 2n - 7 \) if and only if \( G \cong C_3(P_3), C_3(3P_3), C_3(P_6), C_3(K_{1,3}), C_3(P_3, P_3, P_2), C_3(2P_2, P_2, 0), C_3(P_3, P_3, 0), C_3(P_3, P_2, P_2), C_3(2P_3, P_2, 0), C_3(P_4, P_2, 0), C_3(u(P_4, P_2)), C_3(u(P_3, P_2)), K_5(P_4), K_5(P_3), K_5(2P_3), K_5(2P_2), K_7(P_2), K_7(2), K_7(3), K_7(4), K_7(5), K_7(6), K_5(P_3, P_2, 0, 0, 0), K_5(P_2, P_2, 0, 0, 0), K_9 \) or any one of the graphs shown in figure 3.3.
Proof: If \( G \) is any one of the graphs stated in the theorem, then it can be verified that
\[ \gamma_{ip}(G) + \chi(G) = 2n - 7. \]

Conversely, let \( \gamma_{ip}(G) + \chi(G) = 2n - 7. \)

Then the various possible cases are,

(i) \( \gamma_{ip}(G) = n - 1 \) and \( \chi(G) = n - 6 \)

(ii) \( \gamma_{ip}(G) = n - 2 \) and \( \chi(G) = n - 5 \)

(iii) \( \gamma_{ip}(G) = n - 3 \) and \( \chi(G) = n - 4 \)
(iv) $\gamma_{ip}(G) = n - 4$ and $\chi(G) = n - 3$

(v) $\gamma_{ip}(G) = n - 5$ and $\chi(G) = n - 2$

(vi) $\gamma_{ip}(G) = n - 6$ and $\chi(G) = n - 1$

(vii) $\gamma_{ip}(G) = n - 7$ and $\chi(G) = n$.

Case (i): $\gamma_{ip} = n - 1$ and $\chi = n - 6$.

Since $\gamma_{ip} = n - 1$, by Theorem, 3. 1 $G \cong P_3$, $C_3$, $P_3$ or $G'$ where $G'$ is the graph as in figure 3.3. Since $\chi = n - 6$, $G \cong G'$. But for $G'$, $\chi = 3$, $n = 9$ so that $G \cong C_3(3P_3), G_1$.

Case (ii): $\gamma_{ip} = n - 2$ and $\chi = n - 5$.

Since $\chi(G) = n - 5$, $G$ contains a clique $K$ on $n - 5$ vertices (or) does not contains a clique $K$ on $n - 5$ vertices.

Let $S = \{v_1, v_2, v_3, v_4, v_5\}$. Then $\langle S \rangle$ has the following possible cases.
That is, \( \langle S \rangle = K_5, K_5, W_5, P_5, C_5, F_2, K_{1,4}, K_5 - \{e\}, P_4 \cup K_1, P_3 \cup K_2, P_3 \cup K_2, P_2 \cup K_3, P_2 \cup P_2 \cup K_1, K_4 \cup K_1, K_{1,3} \cup K_1, K_4(P_2), K_3 \cup K_2, C_3(P_2) \cup K_1, C_4 \cup K_1, K_3 \cup K_2, C_4(P_2), C_3(P_3), C_3(2P_2), C_3(P_2, P_2, 0) \) and the following graphs in the figure 3.3 (b).

**Figure 3.3 (b)**

**Subcase (i):** If \( \langle S \rangle = K_5 \).

Let \( v_1, v_2, v_3, v_4, v_5 \) be the vertices of \( K_5 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n,5} \) which is adjacent to any one of \( \{v_1, v_2, v_3, v_4, v_5\} \). In this case \( \{u_i, v_1\} \) forms a \( \gamma_{ip} \) -set of \( G \) so that \( \gamma_{ip} = 2 \) and \( n = 4 \), which is a contradiction. Hence no graph exists.

**Subcase (ii):** If \( \langle S \rangle = \overline{K}_5 \).

Let \( v_1, v_2, v_3, v_4, v_5 \) be the vertices of \( \overline{K}_5 \). Since \( G \) is connected, all the vertices of \( \overline{K}_5 \) are adjacent to a vertex \( u_i \) in \( K_{n,5} \). In this case \( \{u_i, v_1\} \) forms a \( \gamma_{ip} \) -set of \( G \) so that \( \gamma_{ip} = 2 \) and \( n = 4 \), which is a contradiction. Hence no graph exists. Since \( G \) is connected, one vertex of \( \overline{K}_5 \) is adjacent to a vertex \( u_i \) in \( K_{n,5} \) and one vertex is adjacent to \( u_j \) and one vertex is adjacent to \( u_k \) and one vertex is adjacent to \( u_l \) and one vertex is adjacent to \( u_m \) for \( i \neq j \neq k \neq l \neq m \). In this case \( \gamma_{ip} \) -set of \( G \) does not exists. Since \( G \) is connected, there exists a vertex \( u_i \) in \( \overline{K}_5 \) is adjacent to \( v_1, v_2, v_3, v_4 \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_5 \). In this case \( \{u_i, u_j\} \) for \( i \neq j \) forms a \( \gamma_{ip} \) -set of \( G \) so that
\( \gamma_{ip} = 2 \) and \( n = 4 \), which is a contradiction. Hence no graph exists. Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_5 \) is adjacent to \( v_1, v_2, v_3 \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_4 \) and \( v_5 \). In this case \( \{u_i, u_j\} \) for \( i \neq j \) forms a \( \gamma_{ip} \)-set of \( G \) so that \( \gamma_{ip} = 2 \) and \( n = 4 \), which is a contradiction. Hence no graph exists. Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_5 \) is adjacent to \( v_1, v_2 \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_3 \) and \( v_4 \) and \( u_k \) for \( i \neq j \neq k \) is adjacent to \( v_5 \). In this case \( \gamma_{ip} \)-set of \( G \) does not exists.

**Subcase (iii):** If \( <S> = P_5 \).

Let \( v_1, v_2, v_3, v_4, v_5 \) be the vertices of \( P_5 \).

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to any one of \( \{v_1, v_5\} \) or \( \{v_2, v_4\} \) or \( \{v_3\} \).

If \( u_i \) is adjacent to \( v_1 \), then \( \{u_j, u_k, v_1, v_2, v_4, v_5\} \) for \( i \neq j \) forms a \( \gamma_{ip} \)-set of \( G \) so that \( \gamma_{ip} = 6 \) and \( n = 8 \). Hence \( K = K_3 = < u_1, u_2, u_3 > \). If \( u_1 \) is adjacent to \( v_1 \), then \( G \cong C_3(P_5) \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_2 \). In this case \( \{u_i, v_2, v_4, v_5\} \) forms a \( \gamma_{ip} \)-set of \( G \) so that \( \gamma_{ip} = 4 \) and \( n = 6 \), which is a contradiction. Hence no graph exists.

**Subcase (iv):** If \( <S> = P_3 \cup K_2 \).

Let \( v_1, v_2, v_3 \) be the vertices of \( P_3 \) and \( v_4, v_5 \) be the vertices of \( K_2 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to any one of \( \{v_1, v_3\} \) and \( \{v_4, v_5\} \). In this case \( \{u_i, v_2, v_3, v_4\} \) forms a \( \gamma_{ip} \)-set of \( G \) so that \( \gamma_{ip} = 4 \) and \( n = 6 \), which is a contradiction. Hence no graph exists.

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_2 \) and \( v_4 \). In this case \( \{u_j, u_k, v_2, v_3, v_4, v_5\} \) for \( i \neq j \neq k \) forms a \( \gamma_{ip} \)-set of \( G \) so that \( \gamma_{ip} = 6 \) and \( n = 8 \). Hence
\( K = K_3 = \langle u_1, u_2, u_3 \rangle \). Let \( u_i \) be adjacent to \( v_2 \) and \( v_4 \). If \( \text{deg} (v_1) = 1 = \text{deg} (v_3), \text{deg} (v_2) = 3, \text{deg} (v_4) = 2, \text{deg} (v_5) = 1 \), then \( G \cong G_2 \). In all the other cases not graph exists.

**Subcase (v):** If \( <S> = P_3 \cup K_2 \).

Let \( v_1, v_2, v_3 \) be the vertices of \( P_3 \) and \( v_4, v_5 \) be the vertices of \( K_2 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_1 \) and \( v_4 \). In this case \( \{u_i, v_2, v_3, u_j\} \) for \( i \neq j \) forms a \( \gamma_{ip} \) -set of \( G \) so that \( \gamma_{ip} = 4 \) and \( n = 6 \), which is a contradiction. Hence no graph exists.

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_1 \) and \( v_5 \). In this case \( \gamma_{ip} \) -set of \( G \) does not exist. Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_1 \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_2 \) and \( v_3 \). In this case \( \{u_i, v_1, v_2, u_k\} \) for \( i \neq j \neq k \) is adjacent to \( v_4 \) and \( v_5 \). In this case \( \{u_i, v_1, v_2, v_4\} \) for \( i \neq j \) forms a \( \gamma_{ip} \) -set of \( G \) so that \( \gamma_{ip} = 4 \) and \( n = 6 \), which is a contradiction. Hence no graph exists.

**Subcase (vi):** If \( <S> = K_3 \cup K_2 \).

Let \( v_1, v_2, v_3 \) be the vertices of \( K_3 \) and \( v_4, v_5 \) be the vertices of \( K_2 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to any one of \( \{v_1, v_2, v_3\} \) and \( u_j \) for \( i \neq j \) is adjacent to any one of \( \{v_4, v_5\} \). In this case \( \{v_i, v_1, v_2, v_4\} \) forms a \( \gamma_{ip} \) -set of \( G \) so that \( \gamma_{ip} = 4 \) and \( n = 6 \), which is a contradiction. Hence no graph exists.

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to any one of \( \{v_1, v_2, v_3\} \) and \( \{v_4, v_5\} \). In this case \( \{u_i, v_1, v_3, v_4\} \) forms a \( \gamma_{ip} \) -set of \( G \) so that \( \gamma_{ip} = 4 \) and \( n = 6 \), which is a contradiction. Hence no graph exists.
Subcase (vii): If $\langle S \rangle = C_5$.

Let $v_1, v_2, v_3, v_4, v_5$ be the vertices of $C_5$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to $v_1$. In this case $\{u_i, v_1, v_3, v_4\}$ forms a $\gamma_{ir}$-set of $G$ so that $\gamma_{ir} = 4$ and $n = 6$, which is a contradiction. Hence no graph exists.

Subcase (viii): If $\langle S \rangle = K_4 \cup K_1$.

Let $v_1, v_2, v_3, v_4$ be the vertices of $K_4$ and $v_5$ be the vertex of $K_1$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to any one of $\{v_1, v_2, v_3, v_4\}$ and $\{v_5\}$. In this case $\{u_i, v_1\}$ forms a $\gamma_{ir}$-set of $G$ so that $\gamma_{ir} = 2$ and $n = 4$, which is a contradiction. Hence no graph exists. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to any one of $\{v_1, v_2, v_3, v_4\}$ and $u_j$ for $i \neq j$ is adjacent to $v_5$. In this case $\{u_j, v_1, v_2, v_3\}$ forms a $\gamma_{ir}$-set of $G$ so that $\gamma_{ir} = 4$ and $n = 6$, which is a contradiction. Hence no graph exists.

Subcase (ix): If $\langle S \rangle = P_4 \cup K_1$.

Let $v_1, v_2, v_3, v_4$ be the vertices of $P_4$ and $v_5$ be the vertex of $K_1$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to any one of $\{v_1, v_4\}$ and $\{v_5\}$. In this case $\{u_i, v_1, v_3, v_4\}$ forms a $\gamma_{ir}$-set of $G$ so that $\gamma_{ir} = 4$ and $n = 6$, which is a contradiction. Hence no graph exists.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to any one of $\{v_2, v_3\}$ and $v_5$. Then $\gamma_{ir}$-set of $G$ does not exists. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to any one of $\{v_1, v_4\}$ and $u_j$ for $i \neq j$ is adjacent to $v_5$. In this case $\{u_i, u_j, v_3, v_4\}$ forms a $\gamma_{ir}$-set of $G$ so that $\gamma_{ir} = 4$ and $n = 6$, which is a contradiction. Hence no graph exists. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to any one of...
\{v_2, v_3\} and u_j for i \neq j is adjacent to v_5. In this case \{u_i, v_2, v_3, v_5\} forms a \gamma_{ip} -set of G so that \gamma_{ip} = 4 and n = 6, which is a contradiction. Hence no graph exists.

**Subcase (x):** If \langle S \rangle = K_3 \cup K_2.

Let v_1, v_2, v_3 be the vertices of K_3 and v_4, v_5 be the vertices of K_2. Since G is connected, there exists a vertex u_i in K_{n-5} which is adjacent to any one of \{v_1, v_2, v_3\} and v_4, v_5. In this case \{u_i, v_1\} forms a \gamma_{ip} -set of G so that \gamma_{ip} = 2 and n = 4, which is a contradiction. Hence no graph exists. Since G is connected, there exists a vertex u_i in K_{n-5} which is adjacent to any one of \{v_1, v_2, v_3\} and v_4 and u_j for i \neq j is adjacent to v_5. In this case \{u_i, u_j, v_2, v_3\} forms a \gamma_{ip} -set of G, so that \gamma_{ip} = 4 and n = 6, which is a contradiction. Hence no graph exists.

Since G is connected, there exists a vertex u_i in K_{n-5} which is adjacent to any one of \{v_1, v_2, v_3\} and u_j for i \neq j is adjacent to v_4 and v_5. In this case \{u_i, u_j, v_2, v_3, v_5\} forms a \gamma_{ip} -set of G so that \gamma_{ip} = 4 and n = 6, which is a contradiction. Since G is connected, there exists a vertex u_i in K_{n-5} which is adjacent to any one of \{v_1, v_2, v_3\} and u_j for i \neq j is adjacent to v_4 and u_k for i \neq j \neq k is adjacent to v_5. In this case \{u_j, v_2, v_3, u_k\} for i \neq j forms a \gamma_{ip} -set of G so that \gamma_{ip} = 4 and n = 6, which is a contradiction. Hence no graph exists.

**Subcase (xi):** If \langle S \rangle = C_4 \cup K_1.

Let v_1, v_2, v_3, v_4 be the vertices of C_4 and v_5 be the vertex of K_1. Since G is connected, there exists a vertex u_i in K_{n-5} which is adjacent to any one of \{v_1, v_2, v_3, v_4\} and \{v_5\}. In this case \{u_i, v_2, v_3, v_5\} forms a \gamma_{ip} -set of G so that \gamma_{ip} = 4 and n = 6, which is a contradiction. Hence no graph exists.
Since G is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to any one of \{v_1, v_2, v_3, v_4\} and $u_j$ for $i \neq j$ is adjacent to $v_5$. In this case \{u_i, u_j, v_1, v_4\} forms a $\gamma_{ip}$-set of G so that $\gamma_{ip} = 4$ and $n = 6$, which is a contradiction. Hence no graph exists.

**Subcase (xii):** If $<S> = P_2 \cup P_2 \cup K_1$.

Let $v_1, v_2$ be the vertices of $P_2$ and $v_3, v_4$ be the vertices of $P_2$ and $v_5$ be the vertex of $K_1$.

Since G is connected, suppose $u_i$ is adjacent to $v_3$ and $v_5$ and $u_j$ for $i \neq j$ is adjacent to $v_1$. In this case \{u_i, v_3, v_1, v_2\} forms a $\gamma_{ip}$-set of G so that $\gamma_{ip} = 4$ and $n = 6$, which is a contradiction. Hence no graph exists.

Since G is connected, suppose $u_i$ is adjacent to $v_1, u_j$ is adjacent to $v_3, u_k$ is adjacent to $v_5$.

In this case \{v_1, v_2, v_3, v_4, v_5, u_k\} forms a $\gamma_{ip}$-set of G so that $\gamma_{ip} = 6$ and $n = 8$. Now $K = K_3 = <u_1, u_2, u_3>$.

Let $u_1$ be adjacent to $v_3$ and $u_2$ be adjacent to $v_1$, and $u_3$ be adjacent to $v_5$. If $\deg(v_1) = 2$, $\deg(v_3) = 2$, $\deg(v_2) = 1 = \deg(v_4) = \deg(v_5)$, then $G \cong C_3(P_3, P_3, P_2)$.

Since G is connected, suppose $u_i$ is adjacent to $v_1$ and $v_3$ and $u_j$ for $i \neq j$ is adjacent to $v_5, u_k$ for $i \neq j \neq k$. In this case \{v_1, v_2, v_3, v_4, u_j, u_k\} forms a $\gamma_{ip}$-set of G so that $\gamma_{ip} = 6$ and $n = 8$. Now $K = K_3 = <u_1, u_2, u_3>$.

Let $u_1$ be adjacent to $v_1$ and $v_3$ and $u_3$ be adjacent to $v_5$. $\deg(v_1) = \deg(v_3) = 2$ and $\deg(v_2) = \deg(v_4) = \deg(v_5) = 1$. Then $G \cong C_3(2P_3, P_2, 0)$.

**Subcase (xiii):** If $<S> = K_{1,3} \cup K_1$.

Let $v_1$ be the vertex of $K_1$ and let $v_2, v_3, v_4, v_5$ be the vertices of $K_{1,3}$ where $v_2$ be the root vertex. Since G is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to $v_1$ and $u_j$ for
i \neq j \text{ which is adjacent to } v_3. \text{ In this case } \{u_i, v_2, v_5, u_j\} \text{ for } i \neq j \text{ forms a } \gamma_{ip} \text{-set of } G \text{ so that } \gamma_{ip} = 4 \text{ and } n = 6, \text{ which is a contradiction. Hence no graph exists.}

Since } G \text{ is connected, there exists a vertex } u_i \text{ in } K_{n-5} \text{ which is adjacent to } v_1 \text{ and } u_j \text{ for } i \neq j \text{ which is adjacent to } v_2. \text{ In this case } \{u_i, u_j\} \text{ forms a } \gamma_{ip} \text{-set of } G \text{ so that } \gamma_{ip} = 2 \text{ and } n = 4, \text{ which is a contradiction. Hence no graph exists.}

**Subcase (xiv):** If } <S> = C_3(P_2) \cup K_1 = \[
\begin{array}{c}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5
\end{array}
\]

Since } G \text{ is connected, there exists a vertex } u_i \text{ in } K_{n-5} \text{ which is adjacent to } v_1 \text{ and } v_5. \text{ In this case } \{u_i, v_5, v_1, v_4\} \text{ forms a } \gamma_{ip} \text{-set of } G \text{ so that } \gamma_{ip} = 4 \text{ and } n = 6, \text{ which is a contradiction. Hence no graph exists.}

Since } G \text{ is connected, there exists a vertex } u_i \text{ in } K_{n-5} \text{ which is adjacent to } v_1 \text{ and } v_5. \text{ In this case } \{u_i, v_1\} \text{ forms a } \gamma_{ip} \text{-set of } G \text{ so that } \gamma_{ip} = 2 \text{ and } n = 4, \text{ which is a contradiction. Hence no graph exists.}

**Subcase (xv):** If } <S> = W_5

Let } v_1, v_2, v_3, v_4, v_5 \text{ be the vertices of } W_5 \text{ and } v_5 \text{ be the root vertex.}

Since } G \text{ is connected, there exists a vertex } u_i \text{ in } K_{n-5} \text{ which is adjacent to } v_3 \text{ and } u_j \text{ for } i \neq j \text{ and } u_k \text{ for } i \neq j \neq k. \text{ In this case } \{v_1, v_3, u_j, u_k\} \text{ forms a } \gamma_{ip} \text{-set of } G \text{ so that } \gamma_{ip} = 4 \text{ and } n = 6 \text{ which is a contradiction. Hence no graph exists.}

Since } G \text{ is connected, there exists a vertex } u_i \text{ in } K_{n-5} \text{ which is adjacent to } v_5. \text{ In this case } \{u_i, v_5\} \text{ forms a } \gamma_{ip} \text{-set of } G \text{ so that } \gamma_{ip} = 2 \text{ and } n = 4 \text{ which is a contradiction. Hence no graph exists.}
Subcase (xvi): \(< S > = K_{1,4}\)

Let \(v_1, v_2, v_3, v_4, v_5\) be the vertices of \(K_{1,4}\), where \(v_1\) be the root vertex. Since \(G\) is connected, there exists a vertex \(u_i\) in \(K_{n-5}\) which is adjacent to \(v_1\). In this case \(\{u_i, v_1\}\) forms a \(\gamma_{ip}\)-set of \(G\) so that \(\gamma_{ip} = 2\) and \(n = 4\) which is a contradiction. Hence no graph exists.

Since \(G\) is connected, there exists a vertex \(u_i\) in \(K_{n-5}\) which is adjacent to \(v_2\) and \(u_j\) for \(i \neq j\). In this case \(\{u_i, u_j, v_1, v_5\}\) forms a \(\gamma_{ip}\)-set of \(G\) so that \(\gamma_{ip} = 4\) and \(n = 6\), which is a contradiction. Hence no graph exists.

Subcase (xvii): If \(< S > = C_4 (P_2) = \)

Since \(G\) is connected, there exists a vertex \(u_i\) in \(K_{n-5}\) which is adjacent to \(v_5\). In this case \(\{v_1, v_3, v_5, u_i\}\) forms a \(\gamma_{ip}\)-set of \(G\) so that \(\gamma_{ip} = 4\) and \(n = 6\) which is a contradiction. Hence no graph exists.

Since \(G\) is connected, there exists a vertex \(u_i\) in \(K_{n-5}\) which is adjacent to \(v_3\) and \(u_j\) for \(i \neq j\) which is adjacent to \(v_5\). In this case \(\{v_1, v_3, v_5, u_j\}\) forms a \(\gamma_{ip}\)-set of \(G\) so that \(\gamma_{ip} = 4\) and \(n = 6\), which is a contradiction. Hence no graph exists.

Subcase (xviii): If \(< S > = K_5-\{e\} = \)

Since \(G\) is connected, there exists a vertex \(u_i\) in \(K_{n-5}\) which is adjacent to \(v_3\). In this case \(\{v_3, u_i\}\) forms a \(\gamma_{ip}\)-set of \(G\) so that \(\gamma_{ip} = 2\) and \(n = 4\) which is a contradiction. Hence no graph exists.
Since G is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to $v_1$. In this case \{u_i, u_j, v_2, v_3\} $i \neq j$ forms a $P_{ip}$-set of G so that $P_{ip} = 4$ and $n = 6$, which is a contradiction. Hence no graph exists.

**Subcase (xix):** If $<S> = C_3(P_2, P_2, 0)$ =

Since G is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to $v_5$. In this case \{v_5, u_i, v_2, v_3\} forms a $P_{ip}$-set of G so that $P_{ip} = 4$ and $n = 6$ which is a contradiction. Hence no graph exists.

Since G is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to $v_5$ and $u_j$ for $i \neq j$ which is adjacent to $v_3$. In this case \{u_i, v_5, v_2, v_3\} forms a $P_{ip}$-set of G so that $P_{ip} = 4$ and $n = 6$, which is a contradiction. Hence no graph exists.

Since G is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to $v_1$ and $u_j$ for $i \neq j$ and $u_k$ for $i \neq j \neq k$. In this case \{v_1, v_2, u_j, u_k\} forms a $P_{ip}$-set of G so that $P_{ip} = 4$ and $n = 6$ which is a contradiction. Hence no graph exists.

**Subcase (xx):** If $<S> = K_4(P_2)$ =

Since G is connected, there exists a vertex $u_i$ in $K_{n-5}$ which is adjacent to $v_5$ and $u_j$ for $i \neq j$ and $u_k$ for $i \neq j \neq k$. In this case \{u_j, u_k, v_4, v_5\} forms a $P_{ip}$-set of G so that $P_{ip} = 4$ and $n = 6$ which is a contradiction. Hence no graph exists.
Subcase (xxi): If \( <S> = F_2 = \)

Since G is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_3 \). In this case \( \{v_3, u_i\} \) forms a \( \gamma_{ip} \)-set of G so that \( \gamma_{ip} = 2 \) and \( n = 4 \) which is a contradiction. Hence no graph exists.

Since G is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_5 \). In this case \( \{u_i, v_5, v_1, v_2\} \) forms a \( \gamma_{ip} \)-set of G so that \( \gamma_{ip} = 4 \) and \( n = 6 \) which is a contradiction. Hence no graph exists.

Subcase (xxii): If \( <S> = C_3(2P_2) = \)

Since G is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_2 \). In this case \( \{u_i, v_2\} \) forms a \( \gamma_{ip} \)-set of G so that \( \gamma_{ip} = 2 \) and \( n = 4 \) which is a contradiction. Hence no graph exists.

Since G is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_1 \) and \( u_j \) for \( i \neq j \) and \( u_k \) for \( i \neq j \neq k \). In this case \( \{u_i, u_k, v_1, v_2\} \) forms a \( \gamma_{ip} \)-set of G so that \( \gamma_{ip} = 4 \) and \( n = 6 \) which is a contradiction. Hence no graph exists. If G does not contain a clique K on \( n - 5 \) vertices, then it can be verified that no new graph exists.
Subcase (xxiii): If \( <S> = C_3(P_3) = \)

Since G is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_4 \). In this case \( \{ u_i , v_4 , v_1 , v_2 \} \) forms a \( y_{ip} \)-set of G so that \( y_{ip} = 4 \) and \( n = 6 \) which is a contradiction. Hence no graph exists.

Since G is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_2 \) and \( u_j \) for \( i \neq j \) and \( u_k \) for \( i \neq j \neq k \). In this case \( \{ u_i , u_k , v_2 , v_3 \} \) forms a \( y_{ip} \)-set of G so that \( y_{ip} = 4 \) and \( n = 6 \) which is a contradiction. Hence no graph exists.

Since G is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_1 \). In this case \( \{ u_i , v_1 , v_3 , v_4 , v_5 \} \) forms a \( y_{ip} \)-set of G so that \( y_{ip} = 4 \) and \( n = 6 \) which is a contradiction. Hence no graph exists.

Subcase (xxiv): If \( <S> = P_2 \cup R_3 \)

Let \( v_1 , v_2 \) be the vertices of \( P_2 \) and let \( v_3 , v_4 , v_5 \) be the vertices of \( R_3 \).

Since G is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_1 \) and \( v_3 \) and \( v_4 \) and \( v_5 \). In this case \( \{ u_i , v_1 \} \) forms a \( y_{ip} \)-set of G so that \( y_{ip} = 2 \) and \( n = 4 \) which is a contradiction. Hence no graph exists.

Since G is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_1 \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_3 \) and \( v_4 \) and \( v_5 \) and \( u_k \) for \( i \neq j \neq k \). In this case \( \{ v_1 , v_2 , u_j , u_k \} \) forms a \( y_{ip} \)-set of G so that \( y_{ip} = 4 \) and \( n = 6 \) which is a contradiction. Hence no graph exists.

Since G is connected, there exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( v_1 \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_3 \) and \( v_4 \) and \( u_k \) for \( i \neq j \neq k \) is adjacent to \( v_5 \). In this case \( \{ v_1 , v_2 , u_i , u_k \} \) forms a \( y_{ip} \)-set of G so that \( y_{ip} = 4 \) and \( n = 6 \) which is a contradiction. Hence no graph exists.
Subcase (xxv): If $S =$

Subcase (xxvi): $S =$

Subcase (xxvii): If $S =$

Subcase (xxviii): If $S =$

Subcase (xxix): If $S =$
Subcase (xxx): If $S =$

Subcase (xxxi): If $S =$

Subcase (xxxii): If $S =$

Subcase (xxxiii): If $S =$

Subcase (xxxiv): If $S =$
In all the above sub cases, from xxv to subcase xxxiv, by various arguments it can be verified that no graph exists.

If \( G \) does not contain clique \( K \) on \( n - 5 \) vertices, then it can be verified that no new graph exists.

**Case (iii):** \( p_{\mathrm{ip}} = n - 3 \) and \( \chi = n - 4 \).

Since \( \chi = n - 4 \), \( G \) contains a clique \( K \) on \( n - 4 \) vertices or does not contain a clique \( K \) on \( n - 4 \) vertices.

Let \( G \) contain a clique \( K \) on \( n - 4 \) vertices.

**Figure 3.3 (c)**

Let \( S = V(G) - V(K) = \{v_1, v_2, v_3, v_4\} \). Then the induced subgraph \( <S> \) has the following possible cases: \( <S> = K_4, P_4, P_4, C_4, K_{1,3}, P_3 \cup K_1, K_2 \cup K_2, K_3 \cup K_1, K_2 \cup K_2, K_4 - \{e\}, C_3(1, 0, 0) \).
Subcase (i): If \( <S> = K_4 \).

Let \( \{v_1, v_2, v_3, v_4\} \) be the vertices of \( K_4 \). Since G is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to any one of \( \{v_1, v_2, v_3, v_4\} \). Let \( u_i \) be adjacent to \( v_1 \) for some \( i \) in \( K_{n-4} \). Then \( \{u_i, v_1\} \) forms a \( \gamma_{ip} \)-set of G so that \( \gamma_{ip} = 2 \) and \( n = 5 \), which is a contradiction. Hence no graph exists.

Subcase (ii): \( <S> = R_4 \).

Let \( \{v_1, v_2, v_3, v_4\} \) be the vertices of \( R_4 \). Since G is connected, two vertices of \( R_4 \) are adjacent to one vertex say \( u_i \) and the remaining two vertex of \( R_4 \) are adjacent to one vertex say \( u_j \) for \( i \neq j \). In this case \( \{u_i, u_j\} \) for \( i \neq j \) forms a \( \gamma_{ip} \)-set of G so that \( \gamma_{ip} = 2 \) and \( n = 5 \), which is a contradiction. Hence no graph exists. Since G is connected, one vertex of \( R_4 \) is adjacent to one vertex say \( u_i \) and the remaining three vertex of \( R_4 \) are adjacent to one vertex say \( u_j \) for \( i \neq j \). In this case \( \{u_i, u_j\} \) for \( i \neq j \) forms a \( \gamma_{ip} \)-set of G so that \( \gamma_{ip} = 2 \) and \( n = 5 \), which is a contradiction. Hence no graph exists.

Since G is connected, two vertices of \( R_4 \) is adjacent to one vertex say \( u_i \) and the remaining one vertex of \( R_4 \) is adjacent to one vertex say \( u_j \) for \( i \neq j \) and the remaining one vertex is adjacent to a vertex say \( u_k \) for \( i \neq j \neq k \). Then \( \gamma_{ip} \)-set of G does not exists. Since G is connected, let \( u_i \) be adjacent to \( v_1 \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_2 \) and \( u_k \) for \( i \neq j \neq k \) is adjacent to \( v_3 \) and \( u_l \) for \( i \neq j \neq k \neq l \) is adjacent to \( v_4 \). Then \( \gamma_{ip} \)-set of G does not exists.

Subcase (iii): If \( <S> = P_4 \).

Let \( \{v_1, v_2, v_3, v_4\} \) be the vertices of \( P_4 \). Since G is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to any one of \( \{v_1, v_4\} \) or any one of \( \{v_2, v_3\} \). Let \( u_i \) be adjacent to \( v_1 \) for some \( i \) in \( K_{n-4} \). Then \( \{u_i, v_1, v_3, v_4\} \) forms a \( \gamma_{ip} \)-set of G so that \( \gamma_{ip} = 4 \) and \( n = 7 \) and hence
Let \( K = K_3 = \langle u_1, u_2, u_3 \rangle \). If \( u_1 \) is adjacent to \( v_1 \), then \( G \cong C_3 \) \((P_3)\). Let \( u_i \) be adjacent to \( v_2 \) for some \( i \) in \( K_{n-4} \). Then \( \{u_j, u_k, v_2, v_3\} \) for \( i \neq j \neq k \) forms a \( \gamma_{ip} \)-set of \( G \) so that \( \gamma_{ip} = 4 \) and \( n = 7 \) and hence \( K = K_3 = \langle u_1, u_2, u_3 \rangle \). If \( u_1 \) is adjacent to \( v_2 \). If \( \deg (v_1) = 1 = \deg (v_4) \), \( \deg (v_2) = 3 \), \( \deg (v_3) = 2 \), then \( G \cong G_3 \).

**Subcase (iv):** \( \{S\} = C_4 \).

Let \( \{v_1, v_2, v_3, v_4\} \) be the vertices of \( C_4 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_3 \).

Let \( u_i \) for some \( i \) in \( K_{n-4} \) be adjacent to \( v_3 \), then \( \{u_i, u_j, v_2, v_4\} \) for \( i \neq j \) forms a \( \gamma_{ip} \)-set of \( G \) so that \( \gamma_{ip} = 4 \) and \( n = 7 \). Hence \( K = K_3 = \langle u_1, u_2, u_3 \rangle \). If \( u_1 \) is adjacent to \( v_1 \), then

\[
\deg (v_1) = 3, \quad \deg (v_2) = \deg (v_3) = \deg (v_4) = 2 \quad \text{and so} \quad G \cong G_4.
\]

Let \( u_1 \) be adjacent to \( v_1 \) and \( v_2 \). If \( \deg (v_1) = \deg (v_2) = 3 \), \( \deg (v_3) = \deg (v_4) = 2 \), then \( G \cong G_5 \). Let \( u_1 \) be adjacent to \( v_1 \) and \( v_2 \) and \( u_3 \) be adjacent to \( v_2 \). If \( \deg (v_1) = 3 \), \( \deg (v_2) = 4 \), then \( G \cong G_6 \).

**Subcase (v):** \( \{S\} = K_{1,3} \).

Let \( v_1 \) be the root vertex and \( v_2, v_3, v_4 \) are adjacent to \( v_1 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_1 \) (or) any one of \( \{v_2, v_3, v_4\} \) and \( v_4 \). Let \( u_i \) for some \( i \) in \( K_{n-4} \) be the vertex adjacent to \( v_1 \), then \( \{u_i, v_1\} \) is a \( \gamma_{ip} \)-set of \( G \) so that \( \gamma_{ip} = 2 \) and \( n = 5 \), which is a contradiction. Hence no such graph exists.

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to any one of \( \{v_2, v_3, v_4\} \). Then \( u_i \) for some \( i \), is adjacent to \( v_2 \). In this case, \( \{u_i, u_j, v_1, v_4\} \) for \( i \neq j \) is an \( \gamma_{ip} \)-set of \( G \) so that \( \gamma_{ip} = 4 \) and \( n = 7 \), and hence \( K = K_3 = \langle u_1, u_2, u_3 \rangle \). Let \( u_1 \) be adjacent to \( v_2 \). If \( \deg (v_1) = 3 \), \( \deg (v_3) = \deg (v_4) = 1 \), \( \deg (v_2) = 2 \), then \( G \cong G_7 \). Let \( u_1 \) be adjacent to \( v_2 \) and \( v_3 \). If
deg \((v_1) = 3, \deg (v_2) = \deg (v_3) = 2, \deg (v_4) = 1\), then \(G \cong G_8\). Let \(u_1\) be adjacent to \(v_2\) and \(v_4\). If 
\(deg (v_1) = 3, \deg (v_2) = \deg (v_4) = 2, \deg (v_3) = 1\), then \(G \cong G_9\). Let \(u_1\) be adjacent to \(v_2\) and \(u_2\) be adjacent to \(v_4\). If \(deg (v_1) = 3, \deg (v_2) = \deg (v_4) = 2, \deg (v_3) = 1\), then \(G \cong G_{10}\).

**Subcase (vi):** \(<S> = K_3 \cup K_1\).

Let \(v_1, v_2\) and \(v_3\) be the vertices of \(K_3\) and \(v_4\) be the vertex of \(K_1\). Since \(G\) is connected, there exists a vertex \(u_i\) in \(K_{n-4}\) which is adjacent to any one of \(\{v_1, v_2, v_3\}\) and \(\{v_4\}\). In this case \(\{u_i, v_1\}\) is a \(P_{ip}\)-set of \(G\) so that \(p_{ip} = 2\) and \(n = 5\), which is a contradiction. Hence no graph exists.

Since \(G\) is connected, there exists a vertex \(u_i\) in \(K_{n-4}\) which is adjacent to \(v_2\) and \(u_j\) for \(i \neq j\), is adjacent to \(v_4\). In this case, \(\{u_i, u_j, v_1, v_3\}\) is a \(P_{ip}\)-set of \(G\) so that \(p_{ip} = 4\) and \(n = 7\). Hence \(K = \langle K_3 = \langle u_1, u_2, u_3 \rangle \rangle\). Let \(u_1\) be adjacent to \(v_2\) and \(u_3\) be adjacent to \(v_4\). If \(deg (v_1) = 2, \deg (v_3) = 2, \text{ and } \deg (v_4) = 1\), then \(G \cong G_{11}\).

**Subcase (vii):** \(<S> = P_3 \cup K_1\).

Let \(v_1, v_2, v_3\) be the vertices of \(P_3\) and \(v_4\) be the vertex of \(K_1\). Since \(G\) is connected, there exists a vertex \(u_i\) in \(K_{n-4}\) which is adjacent to any one of \(\{v_1, v_2, v_3\}\) and \(\{v_4\}\). Let \(u_i\) be adjacent to \(v_1\) and \(v_4\). In this case \(\{u_i, v_2, v_3, v_4\}\) is a \(P_{ip}\)-set of \(G\) so that \(p_{ip} = 4\) and \(n = 7\). Hence \(K = \langle K_3 = \langle u_1, u_2, u_3 \rangle \rangle\). Let \(u_1\) be adjacent to \(v_1\) and \(v_4\). If \(deg (v_1) = \deg (v_2) = 2, \deg (v_3) = 1, \deg (v_4) = 1\), then \(G \cong C_{3}(u(P_4, P_2))\). Let \(u_1\) be adjacent to \(v_1\) and \(v_4\), and let \(u_3\) be adjacent to \(v_2\). If \(deg (v_1) = 2, \deg (v_2) = 3, \deg (v_3) = \deg (v_4) = 1\), then \(G \cong G_{12}\).

Since \(G\) is connected, there exists a vertex \(u_i\) in \(K_{n-4}\) which is adjacent to \(v_1\) and there exists \(u_j\) for \(i \neq j\), is adjacent to \(v_4\). In this case, \(\{u_i, u_j, v_2, v_3\}\) is a \(P_{ip}\)-set of \(G\) so that \(p_{ip} = 4\) and \(n = 7\), and hence \(K = \langle K_3 = \langle u_1, u_2, u_3 \rangle \rangle\). Let \(u_1\) be adjacent to \(v_1\) and let \(u_3\) be adjacent to \(v_4\). If
\[ \text{deg} (v_1) = \text{deg} (v_2) = 2, \text{deg} (v_3) = \text{deg} (v_4) = 1, \text{then } G \cong G_3. \]

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_2 \) and \( v_4 \). In this case, \( \{u_i, v_2\} \) is a \( P \)-set of \( G \) so that \( \text{deg} (v_1) = 2 \) and \( n = 5 \), which is a contradiction. Hence no graph exists.

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_2 \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_4 \). In this case, \( \{u_i, u_j, v_2, v_3\} \) is a \( P \)-set of \( G \) so that \( \text{deg} (v_1) = 4 \) and \( n = 7 \), and hence \( K = K_3 = \langle u_1, u_2, u_3 \rangle \). Let \( u_1 \) be adjacent to \( v_2 \) and let \( u_3 \) be adjacent to \( v_4 \). If \( \text{deg} (v_1) = \text{deg} (v_3) = \text{deg} (v_4) = 1 \), then \( G \cong G_{13} \).

**Subcase (viii):** \( S = K_2 \cup K_2. \)

Let \( v_1 \) and \( v_2 \) be the vertices of \( K_2 \) and \( v_3, v_4 \) be the vertices of \( K_2 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to any one of \( \{v_1, v_2\} \) and any one of \( \{v_3, v_4\} \). Let \( u_i \) be adjacent to \( v_1 \) and \( v_3 \). In this case \( \{u_i, u_k, v_1, v_2, v_3, v_4\} \) forms a \( P \)-set of \( G \) so that \( \text{deg} (v_1) = 6 \) and \( n = 9 \). Hence \( K = K_5 = \langle u_1, u_2, u_3, u_4, u_5 \rangle \). Let \( u_1 \) be adjacent to \( v_1 \) and \( v_3 \). If \( \text{deg} (v_1) = 2, \text{deg} (v_3) = 2, \text{deg} (v_2) = 1 = \text{deg} (v_4) \), then \( G \cong K_5(2P_3) \).

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_1 \) and there exists \( u_j \) for \( i \neq j \) is adjacent to \( v_3 \). In this case \( \{u_i, v_1, v_2, v_3\} \) for \( i \neq j \) forms a \( P \)-set of \( G \) so that \( \text{deg} (v_1) = 4 \) and \( n = 7 \), and hence \( K = K_3 = \langle u_1, u_2, u_3 \rangle \).

Let \( u_1 \) be adjacent to \( v_3 \) and let \( u_2 \) be adjacent to \( v_1 \). If \( \text{deg} (v_1) = \text{deg} (v_3) = 2, \text{deg} (v_2) = \text{deg} (v_4) = 1 \), then \( G \cong G_3. \)

Let \( u_1 \) be adjacent to \( v_3 \) and \( u_2 \) be adjacent to \( v_1 \). If \( \text{deg} (v_1) = \text{deg} (v_3) = \text{deg} (v_4) = 2, \text{deg} (v_2) = 1 \), then \( G \cong G_{14}. \)

Let \( u_1 \) be adjacent to \( v_3 \) and \( u_2 \) be adjacent to \( v_1 \) and let \( u_3 \) be adjacent to \( v_1 \). If \( \text{deg} (v_1) = 3, \text{deg} (v_3) = 2, \text{deg} (v_4) = 2, \text{deg} (v_2) = 1 \), then \( G \cong G_{15}. \)

Let \( u_1 \) be adjacent to \( v_3 \) and let \( u_2 \) be adjacent to \( v_1 \) and \( v_3 \). If \( \text{deg} (v_1) = 2, \text{deg} (v_2) = \text{deg} (v_4) = 1, \text{deg} (v_3) = 3 \), then \( G \cong G_{16}. \)
be adjacent to $v_1$ and $v_3$, and let $u_3$ be adjacent to $v_4$. If $\deg (v_1) = 2$, $\deg (v_2) = 1$, $\deg (v_4) = 2$, $\deg (v_3) = 3$, then $G \cong G_{17}$.

Subcase (ix): $\langle S \rangle = K_2 \cup K_2$.

Let $v_1$ and $v_2$ be the vertices of $K_2$ and $v_3$, $v_4$ be the vertices of $K_2$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$ and $v_2$ and any one of $\{v_3, v_4\}$. Let $u_i$ be adjacent to $v_1, v_2, v_3$. In this case $\{u_i, v_j\}$ forms a $\gamma_{ip}$-set of $G$, so that $\gamma_{ip} = 2$ and $n = 5$, which is a contradiction. Hence no graph exists.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$ and there exists $u_j$ for $i \neq j$, is adjacent to $v_2$ and $v_3$. In this case, $\gamma_{ip}$-set of $G$ does not exists.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$ and there exists $u_j$ for $i \neq j$, is adjacent to $v_2$ and $u_k$ for $i \neq j \neq k$ is adjacent to $v_3$. In this case, $\{u_i, u_j, v_3, v_4\}$ for $i \neq j$ forms a $\gamma_{ip}$-set of $G$ so that $\gamma_{ip} = 4$ and $n = 7$. Hence $K = K_3 = \langle u_1, u_2, u_3 \rangle$. Let $u_1$ be adjacent to $v_2$ and let $u_2$ be adjacent to $v_1$ and $u_3$ be adjacent to $v_3$. If $\deg (v_1) = \deg (v_2) = \deg (v_4) = 1$, $\deg (v_3) = 2$, then $G \cong C_3(P_3, P_2, P_2)$. Let $u_1$ be adjacent to $v_2$, $u_2$ be adjacent to $v_1$ and $v_2$ and let $u_3$ is adjacent to $v_3$. If $\deg (v_1) = \deg (v_4) = 1$, $\deg (v_2) = 2$, $\deg (v_3) = 2$, then $G \cong G_{18}$. Let $u_1$ be adjacent to $v_2$, $u_2$ be adjacent to $v_1$ and $v_2$ and let $u_3$ be adjacent to $v_1$ and $v_3$. If $\deg (v_1) = 2$, $\deg (v_4) = 1$, $\deg (v_2) = 2$, $\deg (v_3) = 2$, then $G \cong G_{19}$. Let $u_1$ be adjacent to $v_2$, $u_2$ be adjacent to $v_1$ and let $u_3$ be adjacent to $v_3$ and $v_1$. If $\deg (v_1) = 2$, $\deg (v_3) = 2$, $\deg (v_2) = \deg (v_4) = 1$, then $G \cong G_{20}$.

Subcase (x): $\langle S \rangle = K_4 - \{e\}$.

Let $v_1, v_2, v_3, v_4$ be the vertices of $K_4$.

Let $e$ be any one of the edges inside the cycle $C_4$. 

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Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_1 \) of degree 3. In this case \( \{u_i, v_1\} \) is a \( \gamma_i \)-set of \( G \), so that \( \gamma_i = 2 \) and \( n = 5 \), which is a contradiction. Hence no graph exists.

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_2 \) of degree 2. In this case \( \{u_i, u_j, v_1, v_3\} \) is a \( \gamma_i \)-set of \( G \) so that \( \gamma_i = 4 \) and \( n = 7 \). Hence \( K = K_3 = \langle u_1, u_2, u_3 \rangle \).

Let \( u_1 \) be adjacent to \( v_2 \). If \( \text{deg} (v_1) = \text{deg} (v_2) = \text{deg} (v_3) = 3, \text{deg} (v_4) = 2 \), then \( G \cong G_{21} \). Let \( u_1 \) be adjacent to \( v_2 \) and let \( u_3 \) adjacent to \( v_2 \). If \( \text{deg} (v_1) = 3, \text{deg} (v_2) = 4, \text{deg} (v_3) = 3, \text{deg} (v_4) = 2 \), then \( G \cong G_{22} \).

**Subcase (xi):** \( \langle S \rangle = C_3(1, 0, 0) \).

Let \( v_1, v_2, v_3 \) be the vertices of \( C_3 \) and let \( v_4 \) be adjacent to \( v_1 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_2 \). In this case \( \{u_i, u_k, v_1, v_2\} \) for \( i \neq j \) is a \( \gamma_i \)-set of \( G \), so that \( \gamma_i = 4 \) and \( n = 7 \). Hence \( K = K_3 = \langle u_1, u_2, u_3 \rangle \). If \( u_1 \) is adjacent to \( v_2 \), \( \text{deg} (v_1) = \text{deg} (v_2) = 3, \text{deg} (v_3) = 2, \text{deg} (v_4) = 1 \), then \( G \cong G_{23} \).

Let \( u_1 \) be adjacent to \( v_2 \) and let \( u_2 \) be adjacent to \( v_2 \). If \( \text{deg} (v_1) = 3, \text{deg} (v_2) = 4, \text{deg} (v_3) = 2, \text{deg} (v_4) = 1 \), then \( G \cong G_{24} \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_1 \). In this case \( \{u_i, v_1\} \) is a \( \gamma_i \)-set of \( G \), so that \( \gamma_i = 2 \) and \( n = 5 \), which is a contradiction. Hence no graph exists. Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_4 \). In this case \( \{u_i, u_j, v_2, v_4\} \) for \( i \neq j \) forms a \( \gamma_i \)-set of \( G \), so that \( \gamma_i = 4 \) and \( n = 7 \).

Hence \( K = K_3 = \langle u_1, u_2, u_3 \rangle \). If \( u_1 \) is adjacent to \( v_4 \), \( \text{deg} (v_1) = 3, \text{deg} (v_2) = \text{deg} (v_3) = 2, \text{deg} (v_4) = 2 \), then \( G \cong G_{25} \). Let \( u_1 \) be adjacent to \( v_4 \) and let \( u_3 \) be adjacent to \( v_4 \). If \( \text{deg} (v_1) = 3, \text{deg} (v_2) = \text{deg} (v_3) = 2, \text{deg} (v_4) = 3 \), then \( G \cong G_{26} \). Let \( u_1 \) be adjacent to \( v_4 \), \( u_2 \) be adjacent to \( v_2 \) and let \( u_3 \) be adjacent to \( v_2 \). If \( \text{deg} (v_1) = 3, \text{deg} (v_3) = 2, \text{deg} (v_4) = 2, \text{deg} (v_2) = 4 \), then \( G \cong G_{27} \).
If G does not contain clique K on n - 4 vertices, then it can be verified that no new graph exists.

**Case (iv):** \( \gamma_{ip} = n - 4 \) and \( \chi = n - 3 \).

Since \( \chi = n - 3 \), G contains a clique K on n - 3 vertices or does not contain a clique K on n - 3 vertices.

![Figure 3.3 (d)](image)

Let \( S = V(G) - V(K) = \{v_1, v_2, v_3\} \). Then the induced sub graph \( <S> \) has the following possible cases. \( <S> = K_3, \overline{K}_3, P_3, K_2 \cup K_1 \).

**Subcase (i):** \( <S> = K_3 \).

Let \( v_1, v_2, v_3 \) be the vertices of K3. Since G is connected, there exists a vertex \( u_i \) in \( K_{n-3} \) which is adjacent to any one of \( \{v_1, v_2, v_3\} \). Let \( u_i \) be adjacent to \( v_1 \), then \( \{u_i, v_1\} \) is a \( \gamma_{ip} \)-set of G, so that \( \gamma_{ip} = 2 \) and \( n = 6 \). Hence \( K = K_3 = <u_1, u_2, u_3> \). Let \( u_1 \) be adjacent to \( v_2 \). If \( \text{deg} (v_2) = 3 \), \( \text{deg}(v_3) = 2 = \text{deg} (v_1) \), then \( G \cong G_{28} \). Let \( u_1 \) be adjacent to \( v_1 \) and \( v_3 \). If \( \text{deg} (v_1) = \text{deg} (v_3) = 3 \), \( \text{deg} (v_2) = 2 \), then \( G \cong G_{29} \). Let \( u_1 \) be adjacent to \( v_1 \), \( u_2 \) be adjacent to \( v_3 \) and let \( u_3 \) be adjacent to
If $\deg(v_1) = \deg(v_2) = \deg(v_3) = 3$, then $G \cong G_{30}$. Let $u_1$ be adjacent to both the vertices $v_1$ and $v_3$, $u_2$ be adjacent to $v_3$, and let $u_3$ be adjacent to $v_2$. If $\deg(v_1) = 3$, $\deg(v_2) = 3$, $\deg(v_3) = 4$, then $G \cong G_{31}$.

**Subcase (ii):** $<S> = R_3$.

Let $v_1$, $v_2$, $v_3$ be the vertices of $R_3$. Since $G$ is connected, one of the vertices of $K_{n-3}$ say $u_i$ is adjacent to all the vertices of $S$ (or) $u_i$ be adjacent to $v_1$, $v_2$ and $u_j$ be adjacent to $v_3$ for $i \neq j$ (or) $u_i$ be adjacent to $v_1$ and $u_j$ be adjacent to $v_2$ and $u_k$ be adjacent to $v_3$ for $i \neq j \neq k$. If $u_i$ for some $i$, is adjacent to all the vertices of $S$, then $\{u_i, u_j\}$ for $i \neq j$, is a $\gamma_{lp}$-set of $G$, so that $\gamma_{lp} = 2$ and $n = 6$. Hence $K = K_3 = <u_1, u_2, u_3>$. If $u_1$ is adjacent to all the vertices $v_1$, $v_2$, $v_3$, then $G \cong C_3(K_{1,3})$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ is adjacent to $v_1$ and $u_j$ for $i \neq j$ is adjacent to $v_2$ and $v_3$, then $\{u_i, u_j\}$ is a $\gamma_{lp}$-set of $G$, so that $\gamma_{lp} = 2$ and $n = 6$. Hence $K = K_3 = <u_1, u_2, u_3>$. Let $u_1$ be adjacent to $v_1$ and $v_2$ and let $u_3$ be adjacent to $v_3$. If $\deg(v_1) = \deg(v_2) = \deg(v_3) = 1$ then $G \cong C_3(2P_2, P_2, 0)$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ is adjacent to $v_1$ and $u_j$ for $i \neq j$ is adjacent to $v_2$ and $u_k$ for $i \neq j \neq k$ in $K_{n-3}$ is adjacent to $v_3$, then $\gamma_{lp}$-set of $G$ does not exists.

**Subcase (iii):** $<S> = P_3$.

Let $v_1$, $v_2$, $v_3$ be the vertices of $P_3$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ which is adjacent to $v_1$ (or equivalently $v_3$) or $v_2$. If $u_i$ is adjacent to $v_2$, then $\{u_i, v_2\}$ is a $\gamma_{lp}$-set of $G$, so that $\gamma_{lp} = 2$ and $n = 6$. Hence $K = K_3 = <u_1, u_2, u_3>$. Let $u_1$ be adjacent to $v_2$. If $\deg(v_2) = 3$, $\deg(v_1) = \deg(v_3) = 1$, then $G \cong G_{32}$.

Let $u_1$ be adjacent to $v_2$ and let $u_2$ be adjacent to $v_2$. If $\deg(v_2) = 4$, $\deg(v_1) = \deg(v_3) = 1$, then $G \cong G_{33}$. Let $u_1$ be adjacent to $v_2$ and let $u_3$ be adjacent to $v_3$. If $\deg(v_1) = 1$, $\deg(v_2) = 3$,
deg (v_3) = 2, then \( G \cong G_{34} \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-3} \) is adjacent to \( v_1 \) then \( \{u_i, u_j, v_2, v_3\} \) for \( i \neq j \) is a \( \varphi_{ip} \) -set of \( G \), so that \( \varphi_{ip} = 4 \) and \( n = 8 \). Hence \( K = K_5 = \langle u_1, u_2, u_3, u_4, u_5 \rangle \).

Let \( u_1 \) be adjacent to \( v_1 \). If \( \deg (v_1) = \deg (v_2) = 2, \deg (v_3) = 1 \), then \( G \cong K_5(P_3) \). Let \( u_1 \) be adjacent to \( v_1 \) and let \( u_5 \) be adjacent to \( v_1 \). If \( \deg (v_1) = 3, \deg (v_2) = 2, \deg (v_3) = 1 \), then \( G \cong G_{35} \). If \( u_1 \) is adjacent to \( v_1 \), \( u_4 \) be adjacent to \( v_1 \) and let \( u_5 \) be adjacent to \( v_1 \). If \( \deg (v_1) = 4, \deg (v_2) = 2, \deg (v_3) = 1 \), then \( G \cong G_{36} \).

**Subcase (iv):** \( \langle S \rangle = K_2 \cup K_4 \).

Let \( v_1, v_2 \) be the vertices of \( K_2 \) and \( v_3 \) be the vertex of \( K_1 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-3} \) which is adjacent to any one of \( \{v_1, v_2\} \) and \( \{v_3\} \) (or) \( u_i \) is adjacent to any one of \( \{v_1, v_2\} \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_3 \). In this case, \( \{u_i, v_1, v_2, v_3\} \) forms a \( \varphi_{ip} \) -set of \( G \), so that \( \varphi_{ip} = 4 \) and \( n = 8 \). Hence \( K = K_5 = \langle u_1, u_2, u_3, u_4, u_5 \rangle \).

Let \( u_1 \) be adjacent to \( v_1 \) and let \( u_2 \) be adjacent to \( v_3 \). If \( \deg (v_1) = 2, \deg (v_2) = \deg (v_3) = 1 \), then \( G \cong K_5(P_3, P_2, 0, 0, 0) \). Let \( u_1 \) be adjacent to \( v_1 \), \( u_2 \) be adjacent to \( v_3 \) and let \( u_5 \) be adjacent to \( v_1 \). If \( \deg (v_1) = 3, \deg (v_2) = 1, \deg (v_3) = 1 \), then \( G \cong G_{37} \). Let \( u_1 \) be adjacent to \( v_1 \), \( u_2 \) be adjacent to \( v_3 \) and let \( u_4 \) be adjacent to \( v_1 \) and \( u_5 \) be adjacent to \( v_1 \). If \( \deg (v_1) = 4, \deg (v_2) = 1, \deg (v_3) = 1 \), then \( G \cong G_{38} \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-3} \) which is adjacent to \( v_1, v_3 \), so that \( \{u_i, v_1\} \) is a \( \varphi_{ip} \) -set of \( G \). Hence \( \varphi_{ip} = 2 \) and \( n = 6 \), so that \( K = K_3 = \langle u_1, u_2, u_3 \rangle \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( v_3 \). If \( \deg (v_1) = 2, \deg (v_2) = \deg (v_3) = 1 \), then \( G \cong C_3(u(P_3, P_2)) \). Let \( u_1 \) be adjacent to \( v_1 \) and \( v_3 \) and let \( u_2 \) be adjacent to \( v_1 \). If \( \deg (v_1) = 3, \deg (v_2) = \deg (v_3) = 1, \) then \( G \cong G_{39} \).
Let \( u_1 \) be adjacent to \( v_1 \) and \( v_3 \), \( u_2 \) be adjacent to \( v_1 \) and let \( u_3 \) be adjacent to \( v_3 \). If \( \deg(v_1) = 3, \deg(v_2) = 1, \deg(v_3) = 2 \), then \( G \cong G_{40} \).

If \( G \) does not contain a clique \( K \) on \( n - 3 \) vertices, then it can be verified that no new graph exists.

**Case (v):** \( \gamma_{ip} = n - 5 \) and \( \chi = n - 2 \).

Since \( \chi = n - 2 \), \( G \) contains a clique \( K \) on \( n - 2 \) vertices or does not contain a clique \( K \) on \( n - 2 \) vertices.

Let \( G \) contains a clique \( K \) on \( n - 2 \) vertices.

![Figure 3.3 (e)](image_url)

Let \( S = V(G) - V(K) = \{v_1, v_2\} \). Then the induced subgraph \( <S> \) has the following possible cases. \( <S> = K_2, K_3 \).

**Subcase (i):** \( <S> = K_2 \).

Let \( v_1, v_2 \) be the vertices of \( K_2 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-2} \) which is adjacent to any one of \( \{v_1, v_2\} \), then \( \{u_i, v_1\} \) is a \( \gamma_{ip} \)-set of \( G \), so that \( \gamma_{ip} = 2 \) and \( n = 7 \). Hence \( K = K_5 = <u_1, u_2, u_3, u_4, u_5> \).
Let $u_1$ be adjacent to $v_1$. If $\deg (v_1) = 2$, $\deg (v_2) = 1$, then $G \cong K_5(P_3)$. Let $u_1$ be adjacent to $v_1$ and let $u_5$ be adjacent to $v_1$. If $\deg (v_1) = 3$, $\deg (v_2) = 1$, then $G \cong G_{41}$. Let $u_1$ be adjacent to $v_1$, $u_4$ be adjacent to $v_1$ and let $u_5$ be adjacent to $v_1$. If $\deg (v_1) = 4$, $\deg (v_2) = 1$, then $G \cong G_{42}$. Let $u_1$ be adjacent to $v_1$, $u_3$ be adjacent to $v_1$ and let $u_5$ be adjacent to $v_1$. If $\deg (v_1) = 5$, $\deg (v_2) = 1$, then $G \cong G_{43}$. Let $u_1$ be adjacent to $v_1$ and let $u_2$ be adjacent to $v_2$. If $\deg (v_1) = \deg (v_2) = 2$, then $G \cong G_{44}$. Let $u_1$ be adjacent to $v_1$, $u_2$ be adjacent to $v_2$ and let $u_3$ be adjacent to $v_2$. If $\deg (v_2) = 3$, $\deg (v_1) = 2$, then $G \cong G_{45}$.

Subcase (ii): $<S> = \overrightarrow{K}_2$.

Let $v_1$, $v_2$ be the vertices of $\overrightarrow{K}_2$. Since $G$ is connected, $v_1$ and $v_2$ are adjacent to a common vertex say $u_i$ in $K_{n-2}$ (or) $v_1$ is adjacent to $u_i$ and $v_2$ is adjacent to $u_j$ for some $i \neq j$ in $K_{n-2}$.

In both cases, $\{u_i, u_j\}$ is a $\mathcal{T}_\mathcal{P}$ -set of $G$, so that $\mathcal{T}_\mathcal{P} = 2$ and $n = 7$. Hence $K = K_5 = <u_1, u_2, u_3, u_4, u_5>$. Let $u_1$ be adjacent to $v_1$ and $v_2$. If $\deg (v_1) = 1 = \deg (v_2)$, then $G \cong K_5(2P_2)$. Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg (v_1) = \deg (v_2) = 1$, then $G \cong K_5(P_2, P_2, 0, 0, 0)$. Let $u_1$ be adjacent to $v_1$ and $v_2$ and let $u_2$ be adjacent to $v_2$. If $\deg (v_1) = 1$, $\deg (v_2) = 2$, then $G \cong G_{46}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$, $u_2$ be adjacent to $v_2$ and let $u_5$ be adjacent to $v_1$. If $\deg (v_1) = \deg (v_2) = 2$, then $G \cong G_{47}$. Let $u_1$ be adjacent to $v_1$, $u_2$ be adjacent to $v_2$ and let $u_3$ be adjacent to $v_2$, $u_5$ be adjacent to $v_1$. If $\deg (v_1) = \deg (v_2) = 2$, then $G \cong G_{48}$. Let $u_1$ be adjacent to $v_1$, $u_2$ be adjacent to $v_2$ and let $u_3$ be adjacent to $v_2$, $u_4$ be adjacent to $v_1$, and let $u_5$ be adjacent to $v_1$. If $\deg (v_1) = 3$, $\deg (v_2) = 2$, then $G \cong G_{49}$. Let $u_1$ be adjacent to $v_1$ and $v_2$, $u_2$ be adjacent to $v_2$ and let $u_3$ be adjacent to $v_2$. If $\deg (v_1) = 1$, $\deg (v_2) = 3$, then $G \cong G_{50}$.

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If $G$ does not contain a clique $K$ on $n - 2$ vertices, then it can be verified that no new graph exists.

**Case (vi):** $\gamma_{ip} = n - 6$ and $\chi = n - 1$.

Since $\chi = n - 1$, $G$ contains a clique $K$ on $n - 1$ vertices.

![Figure 3.3 (f)](image)

Let $\{v_1\}$ be a vertex not on $K_{n-1}$. Since $G$ is connected, there exists a vertex $v_1$ is adjacent to one vertex $u_i$ of $K_{n-1}$. In this case $\{u_i, v_1\}$ is a $\gamma_{ip}$-set of $G$ so that $\gamma_{ip} = 2$ and $n = 8$. Hence $K = K_7 = \langle u_1, u_2, u_3, u_4, u_5, u_6, u_7 \rangle$.

If $u_1$ is adjacent to $v_1$, $\deg(v_1) = 1$, then $G \cong K_7(P_2)$. Let $u_1$ be adjacent to $v_1$ and let $u_2$ be adjacent to $v_1$. If $\deg(v_1) = 2$, then $G \cong K_7(2)$. Let $u_1$ be adjacent to $v_1$, $u_2$ be adjacent to $v_1$ and let $u_3$ be adjacent to $v_1$. If $\deg(v_1) = 3$, then $G \cong K_7(3)$.

Let $u_1$ be adjacent to $v_1$, $u_2$ be adjacent to $v_1$ and let $u_3$ be adjacent to $v_1$, $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 4$, then $G \cong K_7(4)$. Let $u_1$ be adjacent to $v_1$, $u_2$ be adjacent to $v_1$ and let $u_3$ be
adjacent to \(v_1\), \(u_4\) be adjacent to \(v_1\) and let \(u_5\) be adjacent to \(v_1\). If \(\deg(v_1) = 5\), then \(G \cong K_7(5)\).

Let \(u_1\) be adjacent to \(v_1\), \(u_2\) be adjacent to \(v_1\) and let \(u_3\) be adjacent to \(v_1\), \(u_4\) be adjacent to \(v_1\) and let \(u_5\) be adjacent to \(v_1\), \(u_6\) be adjacent to \(v_1\). If \(\deg(v_1) = 6\), then \(G \cong K_7(6)\).

If \(G\) does not contain a clique \(K\) on \(n - 1\) vertices, then it can be verified that no new graph exists.

**Case (vii):** \(\pi_{ip} = n - 7\) and \(\chi = n\)

Since \(\chi = n\), \(G \cong K_n\). But for \(K_n\), \(\pi_{ip} = 2\) so that \(n = 9\). Hence \(G \cong K_9\).