CHAPTER IV

PAIRED TRIPLE CONNECTED DOMINATION NUMBER OF A GRAPH

In this chapter, we introduce the concept of paired triple connected domination number of a graph. We found this number for some standard classes of graphs and obtain some bounds for general graph. Its relationship with other graph theoretical parameters are also investigated.

Many authors have introduced different types of domination parameters by imposing conditions on the dominating set $S$ and/or its complement.

Recently the concept of triple connected graphs was introduced by Paulraj Joseph J. et. al., [25] by considering the existence of a path containing any three vertices of $G$. They have studied the properties of triple connected graph and established many results on them.

A graph $G$ is said to be triple connected if any three vertices lie on a path in $G$. All paths and cycles, complete graphs and wheels are some standard examples of triple connected graphs.

In [9, 10], the authors introduced the concept of triple connected domination number and complementary triple connected domination number of a graph.

A set $S \subseteq V$ is a triple connected dominating set, if $S$ is a dominating set of $G$ and the induced subgraph $<S>$ is triple connected. The triple connected domination number $\gamma_{tc}(G)$ is the minimum cardinality taken over all a triple connected dominating sets of $G$.

A set $S \subseteq V$ is a complementary triple connected dominating set, if $S$ is a dominating set of $G$ and the induced subgraph $<V - S>$ is triple connected. The complementary triple connected domination number $\gamma_{c,tc}(G)$ is the minimum cardinality taken over all a complementary triple connected dominating sets of $G$. 

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In this chapter, we use this idea to develop another new concept called paired triple connected dominating set and paired triple connected domination number of a graph.

We use the following results in the necessary steps in this chapter:

**Theorem I** [25] A tree $T$ is triple connected if and only if $T \cong P_n$, $n \geq 3$.

**Theorem II** [25] A connected graph $G$ is not triple connected if and only if there exists a $H$-cut with $\omega(G - H) \geq 3$ such that $|V(H) \cap N(C_i)| = 1$ for at least three components $C_1$, $C_2$, and $C_3$ of $G - H$.

**Theorem III** [18] $G$ is semi-complete graph with $n \geq 4$ vertices. Then $G$ has a vertex of degree 2 if and only if one of the vertices of $G$ has consequent neighbourhood number $n - 3$.

**Theorem IV** [18] $G$ is semi-complete graph with $n \geq 4$ vertices such that there is a vertex with consequent neighbourhood number $n - 3$. Then $\gamma(G) \leq 4$.

**Definition 4.1** A subset $S$ of $V$ of a nontrivial graph $G$ is said to be paired triple connected dominating set, if $S$ is a triple connected dominating set and the induced sub graph $<S>$ has a perfect matching. The minimum cardinality taken over all paired triple connected dominating sets is called the paired triple connected domination number and is denoted by $\gamma_{ptc}$. Any paired triple connected dominating set with $\gamma_{ptc}$ vertices is called a $\gamma_{ptc}$-set of $G$.

**Example 4.2** For the graph $G_1$ in figure 4.1, $S = \{v_1, v_2, v_3, v_4\}$ forms a paired triple connected dominating set. Hence $\gamma_{ptc}(G_1) = 4$.

![Figure 4.1](image_url)
**Observation 4.3** Paired triple connected dominating set does not exist for all graphs and if exists, then $\gamma_{ptc}(G) \geq 4$.

**Example 4.4** For the graph $G_2$ in figure 4.2, we cannot find any paired triple connected dominating set.

![Figure 4.2](image-url)

**Remark 4.5** Throughout this Chapter, we consider only connected graphs for which paired triple connected dominating set exists.

**Paired Triple connected domination number for some standard graphs are given below**

1) For any path of order $n \geq 4$, $\gamma_{ptc}(P_n) = \begin{cases} 4 & \text{if } n = 4 \\ n-1 & \text{if } n \text{ is odd} \\ n-2 & \text{if } n \text{ is even}. \end{cases}$

2) For any cycle of order $n \geq 4$, $\gamma_{ptc}(C_n) = \begin{cases} 4 & \text{if } n = 4 \\ n-1 & \text{if } n \text{ is odd} \\ n-2 & \text{if } n \text{ is even}. \end{cases}$

3) For the complete bipartite graph of order $n \geq 4$, $\gamma_{ptc}(K_{p,q}) = 4$.

   (where $p, q \geq 2$ and $p + q = n$).

4) For any complete graph of order $n \geq 4$, $\gamma_{ptc}(K_n) = 4$.

5) For any wheel of order $n \geq 4$, $\gamma_{ptc}(W_n) = 4$.

6) For any bistar of order $n \geq 4$, $\gamma_{ptc}(B(p, q)) = 4$.

7) For any helm graph of order $n \geq 9$, $\gamma_{ptc}(H_p) = \begin{cases} \frac{n-1}{2} & \text{if } p \text{ is even} \\ \frac{n-1}{2} + 1 & \text{if } p \text{ is odd}. \end{cases}$
Paired Triple connected domination number for some special graphs are given below

8) The Bidiakis cube is a 3-regular graph with 12 vertices and 18 edges as shown in figure 4.3.

For the Bidiakis cube, $\gamma_{ptc}(G) = 6$.

Here $S = \{v_3, v_6, v_1, v_{11}, v_{12}, v_9\}$ is a minimum paired triple connected dominating set so that $\gamma_{ptc}(G) = 6$.

9) The Chvatal graph is an undirected graph with 12 vertices and 24 edges as shown in figure 4.4

For the Chvatal graph, $\gamma_{ptc}(G) = 4$. 

Figure 4.3 Bidiakis cube

Figure 4.4 Chvatal graph
Here \( S = \{v_1, v_2, v_3, v_4\} \) is a minimum paired triple connected dominating set so that \( \gamma_{ptc}(G) = 4 \).

10) The Durer graph is an undirected graph with 12 vertices and 18 edges as shown in figure 4.5.

For the Durer graph, \( \gamma_{ptc}(G) = 6 \).

![Figure 4.5 Durer graph](image)

Here \( S = \{v_1, v_2, v_3, v_4, v_5, v_6\} \) is a minimum paired triple connected dominating set so that \( \gamma_{ptc}(G) = 6 \).

11) The Franklin graph a 3-regular graph with 12 vertices and 18 edges as shown in figure 4.6.

For the Franklin graph, \( \gamma_{ptc}(G) = 6 \).
Here $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a minimum paired triple connected dominating set so that $\gamma_{ptc}(G) = 6$.

12) The Wagner graph is a 3-regular graph with 8 vertices and 12 edges as shown in figure 4.7

For the Wagner graph, $\gamma_{ptc}(G) = 4$.

Here $S = \{v_1, v_2, v_7, v_8\}$ is a minimum paired triple connected dominating set so that $\gamma_{ptc}(G) = 4$.

13) The Herschel graph is a bipartite undirected graph with 11 vertices and 18 edges as shown in figure 4.8.

For the Herschel graph, $\gamma_{ptc}(G) = 4$. 

Figure 4.6 Franklin graph

Figure 4.7 Wagner graph

Figure 4.8 Herschel graph
Here $S = \{v_1, v_8, v_{11}, v_4\}$ is a minimum paired triple connected dominating set so that
$$\gamma_{ptc}(G) = 4.$$

14) The bull graph is a planar undirected graph with 5 vertices and 5 edges, in the form of a triangle with two disjoint pendant edges as shown in figure 4.9.
For the bull graph, $\gamma_{ptc}(G) = 4$.

Here $S = \{v_1, v_2, v_4, v_5\}$ is a minimum paired triple connected dominating set so that
$$\gamma_{ptc}(G) = 4.$$

15) The Moser spindle is an undirected graph named after mathematicians Leo Moser and his brother William, with seven vertices and eleven edges as shown in figure 4.10.
For the bull graph, $\gamma_{ptc}(G) = 4$. 
Here $S = \{v_1, v_2, v_3, v_4\}$ is a minimum paired triple connected dominating set so that $\gamma_{ptc}(G) = 4$.

16) The Tietze’s graph is an undirected cubic graph with 12 vertices and 18 edges as shown in figure 4.11.

For the Tietze’s graph, $\gamma_{ptc}(G) = 6$.

We use the following notation in the necessary steps in this chapter:

**Notation 4.6** Let $G$ be a connected graph with $m$ vertices $v_1, v_2, \ldots, v_m$. The graph
$G (n_1P_{l_1}, n_2P_{l_2}, n_3P_{l_3}, \ldots, n_mP_{l_m})$ where $n_i \geq 0$ and $1 \leq i \leq m$, is obtained from $G$ by pasting
\( n_1 \) times a pendant vertex of \( P_{l_1} \) on the vertex \( v_1 \), \( n_2 \) times a pendant vertex of \( P_{l_2} \) on the vertex \( v_2 \) and so on.

**Example 4.7** Let \( v_1, v_2, v_3, v_4 \), be the vertices of \( K_4 \), the graph \( K_4(P_2, P_3, 2P_2, P_2) \) is obtained from \( K_4 \) by pasting \( 1 \) times a pendant vertex of \( P_2 \) on \( v_1 \), \( 1 \) times a pendant vertex of \( P_3 \) on \( v_2 \), \( 2 \) times a pendant vertex of \( P_2 \) on \( v_3 \) and \( 1 \) times a pendant vertex of \( P_2 \) on \( v_4 \) and the graph shown below in \( G_3 \) of figure 4.12.

![Figure 4.12](image)

**Observation 4.8** The complement of the paired triple connected dominating set need not be a paired triple connected dominating set.

**Example 4.9** For the graph \( G_4 \) in figure 4.13, \( S = \{v_1, v_2, v_3, v_4\} \) is a paired triple connected dominating set of \( G_4 \). But the complement \( V - S = \{v_5, v_6, v_7, v_8\} \) is not a paired triple connected dominating set.

![Figure 4.13](image)

**Observation 4.10** Every paired triple connected dominating set is a dominating set but not the converse.
Example 4.11 For the graph $G_5$ in figure 4.14, $S_1 = \{v_1, v_2, v_4\}$ is a dominating set. But $S_2 = \{v_1, v_2, v_3, v_4\}$ is a paired triple connected dominating set of $G_5$.

**Figure 4.14**

Observation 4.12 For any connected graph $G$, $\gamma(G) \leq \gamma_c(G) \leq \gamma_{tc}(G) \leq \gamma_{ptc}(G)$ and the inequalities are strict and for a connected graph $G$ with $n \geq 5$ vertices, $\gamma_c(G) \leq \gamma_{tr}(G) \leq \gamma_{tc}(G) \leq \gamma_{ptc}(G)$.

Example 4.13 For the graph $G_6$ in figure 4.15, $\gamma(G_6) = \gamma_c(G_6) = \gamma_{tc}(G_6) = \gamma_{ptc}(G_6) = 4$ and for $C_6$, $\gamma_c(C_6) = \gamma_{tr}(C_6) = \gamma_{tc}(C_6) = \gamma_{ptc}(C_6) = 4$.

**Figure 4.15**

Theorem 4.14 If the induced subgraph of all connected dominating set of $G$ has more than two pendant vertices, then $G$ does not contains a paired triple connected dominating set.

Proof This theorem follows from theorem II.

Example 4.15 For the graph $G_7$ in figure 4.16, $S = \{v_4, v_5, v_6, v_7, v_8\}$ is a minimum connected dominating set so that $\gamma_c(G_7) = 6$. Here we notice that the induced subgraph of $S$
has three pendant vertices and hence \( G \) does not contains a paired triple connected dominating set.

\[ \begin{array}{c}
\text{Figure 4.16} \\
\end{array} \]

**Observation 4.16** If a spanning sub graph \( H \) of a graph \( G \) has a paired triple connected dominating set then \( G \) also has a paired triple connected dominating set.

**Example 4.17** Consider \( C_5 = v_1v_2v_3v_4v_5 \) and its spanning subgraph \( P_5 = v_1v_2v_3v_4v_5 \), \( S = \{v_1, v_2, v_3, v_4\} \) is both a paired triple connected dominating set of \( P_5 \) and \( C_5 \).

**Observation 4.18** Let \( G \) be a connected graph and \( H \) be a spanning sub graph of \( G \). If \( H \) has a paired triple connected dominating set, then \( \gamma_{ptc}(G) \leq \gamma_{ptc}(H) \) and the bound is sharp.

**Example 4.19** Consider \( C_6 \) and its spanning subgraphs \( P_6, \gamma_{ptc}(C_6) = \gamma_{ptc}(P_6) = 4 \).

**Observation 4.20** For any connected graph \( G \) with \( n \) vertices, \( \gamma_{ptc}(G) = n \) if and only if \( G \cong P_4, C_4, K_4, C_3(P_2), K_4 - \{e\} \).

**Theorem 4.21** For any connected graph \( G \) with \( n \geq 5 \), we have \( 4 \leq \gamma_{ptc}(G) \leq n-1 \) and the bounds are sharp.

**Proof** The lower and upper bounds follows from definition 4.1 and observation 4.3 and observation 4.20. For \( P_5 \), the lower bound is attained and for \( C_7 \) the upper bound is attained.
Theorem 4.22 For a connected graph $G$ with 5 vertices, $\gamma_{ptc}(G) = n - 1$ if and only if $G$ is isomorphic to $P_5$, $C_5$, $W_5$, $K_5$, $K_{2,3}$, $F_2$, $K_5 - \{e\}$, $K_4(P_2)$, $C_4(P_2)$, $C_3(P_3)$, $C_3(2P_2)$, $C_3(P_2, P_2, 0)$, $P_4(0, P_2, 0, 0)$ or any one of the graphs shown in figure 4.17.

Proof Suppose $G$ is isomorphic to $P_5$, $C_5$, $W_5$, $K_5$, $K_{2,3}$, $F_2$, $K_5 - \{e\}$, $K_4(P_2)$, $C_4(P_2)$, $C_3(P_3)$, $C_3(2P_2)$, $C_3(P_2, P_2, 0)$, $P_4(0, P_2, 0, 0)$ or any one of the graphs $G_1$ to $G_7$ given in figure 4.17., then clearly $\gamma_{ptc}(G) = n - 1$.

Conversely, Let $G$ be a connected graph with 5 vertices and $\gamma_{ptc}(G) = n - 1$. Let $S = \{w, x, y, z\}$ be a paired triple connected dominating set of $G$. Let $V - S = V(G) - V(S) = \{v\}$.

Case (i) $<S>$ is not a tree.

Then $<S>$ contains a cycle $C$. Let $C = wxyz$ and let $z$ be adjacent to $w$. Since $S$ is a paired triple connected dominating set, there exists a vertex say $w$ or $x$ (or $y$) or $z$ is adjacent to $v$.

Let $w$ be adjacent to $v$. If $d(w) = 4$, $d(x) = d(y) = 2$, $d(z) = 1$, then $G \cong C_3(2P_2)$. 

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Let $x$ be adjacent to $v$. If $d(w) = d(x) = 3$, $d(y) = 2$, $d(z) = 1$, then $G \cong C_3(P_2, P_2, 0)$.

Let $z$ be adjacent to $v$. If $d(w) = 3$, $d(x) = d(y) = d(z) = 2$, then $G \cong C_3(P_3)$.

Now by adding edges to $C_3(2P_2)$, $C_3(P_2, P_2, 0)$, and $C_3(P_3)$, we have $G \cong W_5$, $K_5$, $K_{2,3}$, $F_2$, $K_5 - \{e\}$, $K_d(P_2)$ or any one of the graphs $G$, to $G_7$ given in Figure 4.16.

**Case (ii) $\langle S\rangle$ is a tree.**

Since $S$ is a paired triple connected dominating set. Therefore by *Theorem I*, we have $\langle S\rangle \cong P_{n-1}$. Since $S$ paired triple connected dominating set, there exists a vertex say $w$ (or $z$) or $x$ (or $y$) is adjacent to $v$.

Let $w$ be adjacent to $v$. If $d(w) = d(x) = d(y) = 2$, $d(z) = 1$, then $G \cong P_5$.

Let $w$ be adjacent to $v$ and let $z$ be adjacent to $v$. If $d(w) = d(x) = d(y) = d(z) = 2$, then $G \cong C_5$.

Let $w$ be adjacent to $v$ and let $y$ be adjacent to $v$. If $d(w) = d(x) = 2$, $d(y) = 3$, $d(z) = 1$, then $G \cong C_d(P_2)$.

Let $x$ be adjacent to $v$. If $d(w) = d(z) = 1$, $d(x) = 3$, $d(y) = 2$, then $G \cong P_d(0, P_2, 0, 0)$.

In all the other cases, no new graph exists.

**Theorem 4.23** If $G$ is a graph such that $G$ and $\tilde{G}$ have no isolates of order $n \geq 5$, then $\gamma_{ptc}(G) + \gamma_{ptc}(\tilde{G}) \leq 2(n - 1)$ and the bound is sharp.

**Proof** The bound directly follows from the theorem 4.21. For the cycle $C_5$, $\gamma_{ptc}(G) + \gamma_{ptc}(\tilde{G}) = 2(n - 1)$. 

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**Theorem 4.24** If $G$ is a semi-complete graph with $n \geq 4$ vertices such that there is a vertex with consequent neighbourhood number $n - 3$, then $\gamma_{ptc}(G) = 4$.

**Proof** By theorem III, it follows that there is a vertex $v$ of degree 2 in $G$. Let $N(v) = \{v_1, v_2\}$ (say). Let $u \in V(G) - N[v]$.

Since $G$ is semi-complete and $u, v \in V(G)$, there is a $w \in V(G)$ such that $\{u, w, v\}$ is a path in $G$. Clearly $w \in N(v)$.

Therefore the vertices not in $N[v]$ is dominated by either $v_1$ or $v_2$. Here $S = \{u, v_1, v, v_2\}$ is a paired triple connected dominating set of $G$. Hence $\gamma_{ptc}(G) = 4$.

**Paired Triple Connected Domination Number and Other Graph Theoretical Parameters.**

Let us now discuss the relationship of paired triple connected domination number with other graph theoretical parameters.

**Theorem 4.25** For any connected graph $G$ with $n \geq 5$ vertices, $\gamma_{ptc}(G) + \kappa(G) \leq 2n - 2$ and the bound is sharp if and only if $G \cong K_5$.

**Proof** Let $G$ be a connected graph with $n \geq 5$ vertices. We know that $\kappa(G) \leq n - 1$ and by theorem 4.21, $\gamma_{ptc}(G) \leq n - 1$. Hence $\gamma_{ptc}(G) + \kappa(G) \leq 2n - 2$. Suppose $G$ is isomorphic to $K_5$. Then clearly $\gamma_{ptc}(G) + \kappa(G) = 2n - 2$.

Conversely, Let $\gamma_{ptc}(G) + \kappa(G) = 2n - 2$. This is possible only if $\gamma_{ptc}(G) = n - 1$ and $\kappa(G) = n - 1$. But $\kappa(G) = n - 1$, and so $G \cong K_n$ for which $\gamma_{ptc}(G) = 4 = n - 1$ so that $n = 5$. Hence $G \cong K_5$.

**Theorem 4.26** For any connected graph $G$ with $n \geq 5$ vertices, $\gamma_{ptc}(G) + \kappa(G) = 2n - 3$ if and only if $G \cong K_6, W_5, K_5 - \{e\}$.
Proof} Let $G$ be a connected graph with $n \geq 5$ vertices. If $G$ is isomorphic to $K_6$, $W_5$, $K_5 - \{e\}$, then clearly $\gamma_{ptc}(G) + \kappa(G) = 2n - 3$.

Conversely, let $\gamma_{ptc}(G) + \kappa(G) = 2n - 3$. This is possible only if

(i) $\gamma_{ptc}(G) = n - 1$ and $\kappa(G) = n - 2$

(ii) $\gamma_{ptc}(G) = n - 2$ and $\kappa(G) = n - 1$.

Case (i) $\gamma_{ptc}(G) = n - 1$ and $\kappa(G) = n - 2$.

Then $n - 2 \leq \delta(G)$. If $\delta(G) = n - 2$, then $G$ is isomorphic to $K_n - Y$ where $Y$ is a matching in $K_n$. Hence $\gamma_{ptc}(G) = 4$ so that $n = 5$. Since $\kappa(G) = n - 2$, we have $G \cong W_5$, $K_5 - \{e\}$.

Case (ii) $\gamma_{ptc}(G) = n - 2$ and $\kappa(G) = n - 1$.

But $\kappa(G) = n - 1$, and so $G \cong K_n$ for which $\gamma_{ptc}(G) = 4 = n - 2$ so that $n = 6$. Hence $G \cong K_6$.

Theorem 4.27 For any connected graph $G$ with $n \geq 5$ vertices, $\gamma_{ptc}(G) + \Delta(G) \leq 2n - 2$ and the bound is sharp.

Proof} Let $G$ be a connected graph with $n \geq 5$ vertices. We know that $\Delta(G) \leq n - 1$ and by theorem 4.21, $\gamma_{ptc}(G) \leq n - 1$. Hence $\gamma_{ptc}(G) + \Delta(G) \leq 2n - 2$. For $K_5$ the bound is sharp.

Theorem 4.28 For any connected graph $G$ with $n \geq 5$ vertices, $\gamma_{pr}(G) + \gamma_{ptc}(G) \leq 2n - 2$ and the bound is sharp.

Proof} Let $G$ be a connected graph with $n \geq 5$ vertices. We know that $\gamma_{pr}(G) \leq n - 1$ and by theorem 4.21, $\gamma_{ptc}(G) \leq n - 1$. Hence $\gamma_{pr}(G) + \gamma_{ptc}(G) \leq 2n - 2$. 

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For $P_5$ the bound is sharp. Since $P_5 = v_1v_2v_3v_4v_5$, take $S = \{v_1, v_2, v_3, v_4\}$ is both paired dominating set as well as paired triple connected dominating set. Hence
\[\gamma_{pr}(P_5) + \gamma_{ptc}(P_5) = 2n - 2.\]

**Theorem 4.29** For any connected graph $G$ with $n \geq 5$ vertices, $\gamma_{tc}(G) + \gamma_{ptc}(G) \leq 2n - 3$ and the bound is sharp.

**Proof** Let $G$ be a connected graph with $n \geq 5$ vertices. We know that $\gamma_{tc}(G) \leq n - 2$ and by theorem 4.21, $\gamma_{ptc}(G) \leq n - 1$. Hence $\gamma_{tc}(G) + \gamma_{ptc}(G) \leq 2n - 3$.

For $P_7$ the bound is sharp. Since $P_7 = v_1v_2v_3v_4v_5v_6v_7$, we have $S_1 = \{v_2, v_3, v_4, v_5, v_6\}$ is a triple connected dominating set and $S_2 = \{v_2, v_3, v_4, v_5, v_6, v_7\}$ is a paired triple connected dominating set so that $\gamma_{tc}(P_7) + \gamma_{ptc}(P_7) = 2n - 3$.

**Theorem 4.30** For any connected graph $G$ with $n \geq 5$ vertices, $\gamma_{ptc}(G) + \chi(G) \leq 2n - 1$ and the bound is sharp if and only if $G \cong K_5$.

**Proof** Let $G$ be a connected graph with $n \geq 5$ vertices. We know that $\chi(G) \leq n$ and by theorem 4.21, $\gamma_{ptc}(G) \leq n - 1$. Hence $\gamma_{ptc}(G) + \chi(G) \leq 2n - 1$.

Suppose $G$ is isomorphic to $K_5$. Then clearly $\gamma_{ptc}(G) + \chi(G) = 2n - 1$.

Conversely, Let $\gamma_{ptc}(G) + \chi(G) = 2n - 1$. This is possible only if $\gamma_{ptc}(G) = n - 1$ and $\chi(G) = n$. But $\chi(G) = n$, and so $G$ is isomorphic to $K_n$ for which $\gamma_{ptc}(G) = 4 = n - 1$ so that $n = 5$. Hence $G \cong K_5$.

In [5, 6, 7], already characterized the corresponding extremal graphs whose sum of chromatic number and various difference types of domination parameters of order $2n - 1$, $2n - 2, 2n - 3, 2n - 4$ and $2n - 5$ etc.
Motivated by the above we extend the characterization of extremal graphs whose
sum of paired triple connected domination number and chromatic number of order 2n – 2,
2n – 3, 2n – 4 and 2n – 5.

Let us now characterize the extremal graphs in detail.

**Theorem 4.31** For any connected graph $G$ with $n \geq 5$ vertices, $\gamma_{ptc}(G) + \chi(G) = 2n – 2$ if and only if $G \cong K_4(P_2), K_6$ or any one of the graphs shown in figure 4.18.

![Figure 4.18](image)

**Proof** Suppose $G$ is a connected graph with $n \geq 5$ vertices. Let $G$ be isomorphic to $K_4(P_2), K_6$, or the graphs given in figure 4.18, then clearly $\gamma_{ptc}(G) + \chi(G) = 2n – 2$.

Conversely, Let $\gamma_{ptc}(G) + \chi(G) = 2n – 2$. The various possible cases are

(i) $\gamma_{ptc}(G) = n – 1$ and $\chi(G) = n – 1$

(ii) $\gamma_{ptc}(G) = n – 2$ and $\chi(G) = n$.

**Case(i)** $\gamma_{ptc}(G) = n – 1$ and $\chi(G) = n – 1$.

Since $\chi(G) = n – 1$, $G$ contains a clique $K$ on $n – 1$ vertices or does not contains a clique $K$ on $n – 1$ vertices.

Let $G$ contains a clique $K$ on $n – 1$ vertices.
Let \( u \) be the vertex other than the vertices of \( K_{n-1} \).

Since \( G \) is connected, \( u \) must be adjacent to a vertex \( v_i \) in \( K_{n-1} \). Now \( S = \{ u, v_i, v_j, v_k \} \) for \( i \neq j \neq k \) is a paired triple connected dominating set of \( G \). Since \( \gamma_{pct}(G) = n - 1 \), so that \( n = 5 \). Hence \( K_{n-1} = K_4 = \langle v_1, v_2, v_3, v_4 \rangle \).

![Figure 4.18 (a)](image)

From the figure 4.18 (a), Let \( u \) be adjacent to \( v_i \) in \( K_4 \). If \( d(u) = 1 \), then \( G \cong K_4(P_2) \).

If \( u \) is adjacent to \( v_i, v_2 \) in \( K_4 \). If \( d(u) = 2 \), then \( G \cong G_1 \).

If \( u \) is adjacent to \( v_i, v_2 \) and \( v_3 \) in \( K_4 \). If \( d(u) = 3 \), then \( G \cong G_2 \).

In all the other cases, no graph exists.

If \( G \) does not contain the clique \( K \) on \( n - 1 \) vertices, then it can be verified that no graph exists.

**Case(ii)** \( \gamma_{pct}(G) = n - 2 \) and \( \chi(G) = n \).

But \( \chi(G) = n \), we have \( G \cong K_n \) and for \( K_n \), \( \gamma_{pct}(K_n) = 4 \) so that \( n = 6 \). Hence \( G \cong K_6 \).

**Theorem 4.32** For any connected graph \( G \) with \( n \geq 5 \) vertices, \( \gamma_{pct}(G) + \chi(G) = 2n - 3 \) if and only if \( G \cong W_5, F_2, K_7, K_3(P_3), K_3(2P_2), K_3(2P_2), K_3(P_2, 0) \) or any one of the graphs shown in figure 4.19.
**Proof** Let $G$ be a connected graph with $n \geq 5$ vertices. Suppose $G \cong W_5, F_2, K_7, K_3 (P_3), K_3 (2P_2), K_5 (P_2), K_3 (P_2, P_2, 0)$ or any one of the graphs given in figure 4.19, then clearly

$$\gamma_{ptc}(G) + \chi(G) = 2n - 3.$$ 

Conversely, Let $\gamma_{ptc}(G) + \chi(G) = 2n - 3$. The various possible cases are

(i) $\gamma_{ptc}(G) = n - 1$ and $\chi(G) = n - 2$

(ii) $\gamma_{ptc}(G) = n - 2$ and $\chi(G) = n - 1$

(iii) $\gamma_{ptc}(G) = n - 3$ and $\chi(G) = n$.

**Case(i)** $\gamma_{ptc}(G) = n - 1$ and $\chi(G) = n - 2$. 

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**Figure 4.19**

$G_1$ $G_2$ $G_3$ $G_4$ $G_5$ $G_6$ $G_7$ $G_8$ $G_9$
Since \( \chi(G) = n - 2 \), \( G \) contains a clique \( K \) on \( n - 2 \) vertices or does not contains a clique \( K \) on \( n - 2 \) vertices.

Let \( G \) contains a clique \( K \) on \( n - 2 \) vertices.

Let \( T = \{x, y\} \) be the vertices other than the \( n - 2 \) vertices in \( K_{n-2} \). Then the induced subgraph of \( T \) may be \( K_2 \) or \( \overline{K}_2 \).

**Subcase (i)** \( <T> = K_2 \).

Since \( G \) is connected, there exists a vertex \( x \) in \( T \) which is adjacent to a vertex \( u_i \) in \( K_{n-2} \). Now \( S = \{x, y, u_i, u_j\} \) for \( i \neq j \) forms a paired triple connected dominating set of \( G \).

Since \( \gamma_{ptc}(G) = n - 1 \), so that \( n = 5 \). Hence \( K_{n-2} = K_3 = uvwu. \)

![Figure 4.19 (a)](image)

From the figure 4.19 (a), Let \( x \) be adjacent to \( u \) in \( K_3 \). If \( d(x) = 2 \), \( d(y) = 1 \), then \( G \cong K_3(P_2) \).

Let \( x \) be adjacent to \( u \) and \( v \) in \( K_3 \). If \( d(x) = 3 \), \( d(y) = 1 \), then \( G \cong G_1 \).

Let \( x \) be adjacent to \( u \) and let \( y \) be adjacent to \( u \) in \( K_3 \). If \( d(x) = 2 \), \( d(y) = 2 \), then \( G \cong F_2 \).
Let $x$ be adjacent to $u$ in $K_3$ and $y$ be adjacent to $v$ in $K_3$. If $d(x) = d(y) = 2$, then $G \equiv G_2$.

Let $x$ be adjacent to $u$ in $K_3$ and $y$ be adjacent to $u$ and $v$ in $K_3$. If $d(x) = 2$, $d(y) = 3$, then $G \equiv G_3$.

Let $x$ be adjacent to $u$ and $v$ in $K_3$ and $y$ be adjacent to $u$ and $w$ in $K_3$. If $d(x) = 2$, $d(y) = 3$, then $G \equiv G_4$.

In all the other cases, no new graph exists.

**Subcase (ii)** $<T> = \bar{K}_2$.

Since $G$ is connected, $x$ and $y$ in $T$ are adjacent to a vertex $u_i$ in $K_{n-2}$. Then $S = \{x, u_i, u_j, u_k\}$ for $i \neq j \neq k$ forms a paired triple connected dominating set of $G$. Since $\gamma_{ptc}(G) = n - 1$, so that $n = 5$. Hence $K_{n-2} = K_3 = uvwu$.

![Figure 4.19 (b)](image-url)
From the figure 4.19 (b), Let $x$ and $y$ be adjacent to $u$ in $K_3$. If $d(x) = d(y) = 1$, then $G \cong K_3(2P_2)$.

Let $x$ be adjacent to $u$ and $v$ in $K_3$ and $y$ be adjacent to $u$ in $K_3$. If $d(x) = 2$, $d(y) = 1$, then $G \cong G_5$.

Let $x$ be adjacent to $u$ and $v$ in $K_3$ and $y$ be adjacent to $u$ and $v$ in $K_3$. If $d(x) = 2$, $d(y) = 2$, then $G \cong G_6$.

In all the other cases, no graph exists.

Since $G$ is connected, $x$ in $T$ is adjacent to $v_i$ in $K_{n-2}$ and $y$ in $T$ is adjacent to $u_j$ for $i \neq j$ in $K_{n-2}$. Then $S = \{x, u_i, u_j, y\}$ for $i \neq j$ forms a paired triple connected dominating set of $G$.

Since $\gamma_{ptc}(G) = n - 1$, so that $n = 5$. Hence $K_{n-2} = K_3 = uvwu$.

Let $x$ be adjacent to $u$ in $K_3$ and $y$ be adjacent to $v$ in $K_3$. If $d(x) = d(y) = 1$, then $G \cong K_3(P_2, P_2, 0)$.

In all the other cases, no new graph exists.

If $G$ does not contains the clique $K$ on $n - 2$ vertices, then it can be verified that no graph exists.

**Case (ii)** $\gamma_{ptc}(G) = n - 2$ and $\chi(G) = n - 1$.

Since $\chi(G) = n - 1$, $G$ contains a clique $K$ on $n - 1$ vertices or does not contains a clique $K$ on $n - 1$ vertices.

Let $G$ contains a clique $K$ on $n - 1$ vertices.

Let $x$ be the vertex other than the $n - 1$ vertices in $K_{n-1}$.
Since $G$ is connected $x$ is adjacent to a vertex $v_i$ in $K_{n-1}$. Then $S = \{x, v_i, v_j, v_k\}$ for $i \neq j \neq k$ forms a paired triple connected dominating set of $G$. Since $\gamma_{ptc}(G) = n - 2$, so that $n = 6$. Hence $K_{n-1} = K_5 = \langle v_1, v_2, v_3, v_4, v_5 \rangle$.

From the figure 4.19 (c), Let $x$ be adjacent to $v_1$ in $K_5$. If $d(x) = 1$, then $G \cong K_5(P_2)$.

Let $x$ be adjacent to $v_1$ and $v_2$ in $K_5$. If $d(x) = 2$, then $G \cong G_7$.

Let $x$ be adjacent to $v_1$, $v_2$ and $v_3$ in $K_5$. If $d(x) = 3$, then $G \cong G_8$.

Let $x$ be adjacent to $v_1$, $v_2$, $v_3$ and $v_4$ in $K_5$. If $d(x) = 4$, then $G \cong G_9$.

In all the other cases, no graph exists.

If $G$ does not contains the clique $K$ on $n - 1$ vertices, then it can be verified that no graph exists.

**Case(iii)** $\gamma_{ptc}(G) = n - 3$ and $\chi(G) = n$.

Since $\chi(G) = n$, we have $G \cong K_n$. But For $K_n$, $\gamma_{ptc}(K_n) = 4$ so that $n = 7$. Hence $G \cong K_7.$
Theorem 4.33 For any connected graph $G$ with $n \geq 5$ vertices $\gamma_{ptc}(G) + \chi(G) = 2n - 4$ if and only if $G \in S^*(K_{1,3})$, $P_5$, $C_4(P_2)$, $K_4(P_3)$, $K_4(2P_2)$, $K_4(P_2, P_2, 0, 0)$, $K_6(2)$, $K_6(3)$, $K_6(4)$, $K_6(5)$, $K_8$ or any one of the following graphs in figure 4.20.
Figure 4.20

**Proof:** Let $G$ be a connected graph with $n \geq 5$ vertices. Suppose $G$ is isomorphic to $P_5$, $S^*(K_{1,3})$, $C_4(P_2)$, $K_4(P_3)$, $K_4(2P_2)$, $K_4(P_2, P_2, 0, 0)$, $K_6(P_2)$, $K_6(2)$, $K_6(3)$, $K_6(4)$, $K_6(5)$, $K_8$ or any one of the graphs given in figure 4.20, then clearly $\gamma_{ptc}(G) + \chi(G) = 2n - 4$.

Conversely, let $\gamma_{ptc}(G) + \chi(G) = 2n - 4$. This is possible if

(i) $\gamma_{ptc}(G) = n - 1$ and $\chi(G) = n - 3$

(ii) $\gamma_{ptc}(G) = n - 2$ and $\chi(G) = n - 2$

(iii) $\gamma_{ptc}(G) = n - 3$ and $\chi(G) = n - 1$

(iv) $\gamma_{ptc}(G) = n - 4$ and $\chi(G) = n$.

**Case i.** $\gamma_{ptc}(G) = n - 1$ and $\chi(G) = n - 3$

Since $\chi(G) = n - 3$, $G$ contains a clique $K$ on $n - 3$ vertices or does not contain a clique $K$ on $n - 3$ vertices.

Let $G$ contains a clique $K$ on $n - 3$ vertices.

Let $S = V(G) - V(K) = \{v_1, v_2, v_3\}$. Then the induced subgraph $<S>$ has the following possible cases. $<S> = K_3, \overline{K}_3, P_3, K_2 \cup K_1$. 

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Subcase i. \(<S> = K_3\).

Let \(v_1, v_2, v_3\) be the vertices of \(K_3\). Since \(G\) is connected, there exists a vertex \(u_i\) in \(K_{n-3}\) which is adjacent to any one of \(\{v_1, v_2, v_3\}\). Let \(u_i\) be adjacent to \(v_2\), then \(\{u_i, v_2, v_3, v_1\}\) is a \(\gamma_{ptc}\) -set of \(G\), so that \(\gamma_{ptc} = 4\) and \(n = 5\), which is a contradiction. Hence no graph exists.

Subcase ii. \(<S> = \overline{K}_3\).

Let \(v_1, v_2, v_3\) be the vertices of \(\overline{K}_3\). Since \(G\) is connected, there exists a vertex \(u_i\) be adjacent to \(v_1, v_2, v_3\) and \(u_j\) for \(i \neq j\) and \(u_k\) for \(i \neq j \neq k\). In this case \(\{v_1, u_i, u_j, u_k\}\) is a \(\gamma_{ptc}\) -set of \(G\), so that \(\gamma_{ptc} = 4\) and \(n = 5\). Hence \(K = K_2\), which is a contradiction. Hence no graph exists.

If \(u_i\) is adjacent to \(v_1\) and \(u_j\) for \(i \neq j\) is adjacent to \(v_2\) and \(v_3\), and \(u_k\) for \(i \neq j \neq k\), then \(\{v_1, u_i, u_j, u_k\}\) is an \(\gamma_{ptc}\) -set of \(G\), so that \(\gamma_{ptc} = 4\) and \(n = 5\). Hence \(K = K_2 = u_1u_2\).

\[\text{Figure 4.20 (a)}\]

From the figure 4.20 (a) If \(u_1\) is adjacent to \(v_1\) and \(u_2\) is adjacent to \(v_2\) and \(v_3\).

If \(\deg(v_1) = 1 = \deg(v_2) = \deg(v_3)\), then \(G \cong S^*(K_{1,3})\).

Since \(G\) is connected, there exists a vertex \(u_i\) in \(K_{n-3}\) which is adjacent to \(v_1\) and \(u_j\) for \(i \neq j\) in \(K_{n-3}\) is adjacent to \(v_2\) and \(u_k\) for \(i \neq j \neq k\) in \(K_{n-3}\), which is adjacent to \(v_3\). In this case \(\{u_i, u_j, u_k, v\}\) for some \(v\) in \(K_{n-3}\) is a \(\gamma_{ptc}\)-set of \(G\), so that \(\gamma_{ptc} = 4\) and \(n = 5\), which is a contradiction. Hence no graph exists.
Subcase iii. $<P_3> = v_1v_2v_3$.

Let $v_1, v_2, v_3$ be the vertices of $P_3$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ which is adjacent to $v_1$ (or equivalently $v_3$) or $v_2$. If $u_i$ is adjacent to $v_2$ and $u_j$ for $i \neq j$ then \{v_1, v_2, u_i, u_j\} is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$. Hence $K = K_2 = u_1u_2$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{clique.png}
\caption{Clique $K_2$}
\end{figure}

From the figure 4.20 (b), Let $u_1$ be adjacent to $v_2$. If deg ($v_1$) = 1 = deg ($v_3$), deg ($v_2$) = 3, then $G \cong S^*(K_{1,3})$.

If $u_1$ is adjacent to $v_2$ and $u_2$ is adjacent to $v_1$. If deg ($v_1$) = 2, deg ($v_2$) = 3, deg ($v_3$) = 1, then $G \cong C_4(P_2)$.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ which is adjacent to $v_1$, then \{u_i, v_1, v_2, v_3\} for some i, is a $\gamma_{ptc}$ -set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$ and hence $K = K_2 = \langle u_1, u_2 \rangle$.

Let $u_1$ be adjacent to $v_1$, then $G \cong P_5$.

Let $u_1$ be adjacent to $v_1$ and $v_3$. If deg ($v_1$) = 2 = deg ($v_2$), deg ($v_3$) = 2, then $G \cong C_4(P_2)$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If deg($v_1$) = 2, deg ($v_2$) = 3, deg ($v_3$) = 1, then $G \cong C_4(P_2)$.  

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Subcase iv. $<S> = K_2 \cup K_1$.

Let $v_1$, $v_2$ be the vertices of $K_2$ and $v_3$ be the isolated vertex. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ which is adjacent to $v_1$ and $u_j$ for $i \neq j$ is adjacent to $v_3$. In this case $\{v_2, v_1, u_i, u_j\}$ is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$ and hence $K = K_2 = \langle u_1, u_2 \rangle$.

![Figure 4.20 (c)](image)

From the figure 4.20 (c), Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_3$. If $\text{deg} (v_1) = 2$, $\text{deg} (v_2) = 1 = \text{deg} (v_3)$, then $G \cong P_5$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_3$ and $v_2$. If $\text{deg} (v_1) = 2 = \text{deg} (v_2)$, $\text{deg} (v_3) = 1$, then $G \cong C_4(P_2)$.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ which is adjacent to $v_1$ and $v_3$. In this case $\{v_2, v_1, u_i, v_3\}$ is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$ and hence $K = K_2 = \langle u_1, u_2 \rangle$.

Let $u_1$ be adjacent to $v_1$ and $v_3$. If $\text{deg} (v_1) = 2$, $\text{deg} (v_2) = 1 = \text{deg} (v_3)$, then $G \cong S^*(K_{1,3})$.

Let $u_1$ be adjacent to $v_1$ and $v_3$ and $u_2$ be adjacent $v_2$. If $\text{deg} (v_1) = 2 = \text{deg} (v_2)$, $\text{deg} (v_3) = 1$, then $G \cong C_4(P_2)$.

If $G$ does not contains the clique $K$ on $n - 3$ vertices, then it can be verified that no graph exists.
Case ii. $\gamma_{ptc} = n - 2$ and $\chi = n - 2$.

Since $\chi = n - 2$, $G$ contains a clique $K$ on $n - 2$ vertices or does not contain a clique $K$ on $n - 2$ vertices.

Let $G$ contains a clique $K$ on $n - 2$ vertices.

Let $S = V(G) - V(K) = \{v_1, v_2\}$. Then $<S> = K_2, \overline{K}_2$.

Subcase i. $<S> = K_2$.

Let $v_1, v_2$ be the vertices of $K_2$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-2}$ is adjacent to $v_1$ and $u_j$ for $i \neq j$ then $\{v_2, v_1, u_i, u_j\}$ is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 6$ and hence $K = K_4 = <u_1, u_2, u_3, u_4>$.

![Figure 4.20 (d)](image)

From the figure 4.20 (d), Let $u_1$ be adjacent to $v_1$. If $\deg (v_1) = 2$, $\deg (v_2) = 1$, then $G \cong K_4(P_3)$.

Let $u_1$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$. If $\deg (v_1) = 3$, $\deg (v_2) = 1$, then $G \cong G_1$.

Let $u_1$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$ and $u_3$ be adjacent to $v_1$. If $\deg (v_1) = 4$, $\deg (v_2) = 1$ then $G \cong G_2$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_1$. If $\deg (v_1) = 3$, $\deg (v_2) = 1$, then $G \cong G_1$. 

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Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 4$, $\deg(v_2) = 1$, then $G \cong G_2$.

Let $u_1$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$ and $u_3$ be adjacent to $v_1$. If $\deg(v_1) = 4$, $\deg(v_2) = 1$, then $G \cong G_2$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 2 = \deg(v_2)$, then $G \cong G_3$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_2$. If $\deg(v_1) = 2$, $\deg(v_2) = 3$, then $G \cong G_4$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_2$ and $u_4$ be adjacent to $v_2$. If $\deg(v_1) = 2$, $\deg(v_2) = 4$, then $G \cong G_5$.

Let $u_1$ be adjacent to $v_1$ and $v_2$. If $\deg(v_1) = 2 = \deg(v_2)$ then $G \cong G_6$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_3$ be adjacent to $v_2$ if $\deg(v_1) = 2$, $\deg(v_2) = 3$, then $G \cong G_7$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_3$ be adjacent to $v_2$ and $u_4$ be adjacent to $v_2$. If $\deg(v_1) = 2$, $\deg(v_2) = 4$, then $G \cong G_8$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_1$ and $v_2$ and $u_3$ be adjacent to $v_2$ and $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 4$, $\deg(v_2) = 4$ then $G \cong G_9$.

**Subcase ii.** Let $<S> = K_2$.

Let $v_1$, $v_2$ be the vertices of $K_2$. Since $G$ is connected, $v_1$ and $v_2$ are adjacent to a common vertex say $u_i$ of $K_{n-2}$ (or) $v_1$ is adjacent to $u_i$ for some $i$ and $v_2$ is adjacent to $u_j$ for some $i \neq j$ in $K_{n-2}$. In both cases $\{v_1, u_i, u_j, u_k\}$ is a $\gamma_{pc}$-set of $G$, so that $\gamma_{pc} = 4$ and $n = 6$ and hence $K = K_4 = <u_1, u_2, u_3, u_4>$.

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From the figure 4.20 (e), Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 1$, $\deg(v_2) = 1$, then $G \cong K_4(P_2, P_2, 0, 0)$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 2$, $\deg(v_2) = 1$, then $G \cong G_{10}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 3$, $\deg(v_2) = 1$, then $G \cong G_{11}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_1$ and $v_2$. If $\deg(v_1) = 2$, $\deg(v_2) = 1$, then $G \cong G_{12}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 3$, $\deg(v_2) = 1$, then $G \cong G_{13}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 1$, $\deg(v_2) = 2$, then $G \cong G_{12}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_3$ be adjacent to $v_2$. If $\deg(v_1) = 1$, $\deg(v_2) = 3$, then $G \cong G_{13}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_1$ and $v_2$. If $\deg(v_1) = 2$, $\deg(v_2) = 2$, then $G \cong G_{14}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_1$ and $v_2$ and $u_3$ be adjacent to $v_2$ and $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 3$, $\deg(v_2) = 3$, then $G \cong G_{15}$.
Let $u_1$ be adjacent to $v_1$ and $v_2$. If $\deg(v_1) = 1 = \deg(v_2)$, then $G \cong K_4(2P_2)$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 2$, $\deg(v_2) = 1$, then $G \cong G_{12}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_3$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 3$, $\deg(v_2) = 1$, then $G \cong G_{13}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_3$ be adjacent to $v_2$. If $\deg(v_1) = 1$, $\deg(v_2) = 2$, then $G \cong G_{12}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_3$ be adjacent to $v_2$ and $u_4$ be adjacent to $v_2$. If $\deg(v_1) = 1$, $\deg(v_2) = 3$, then $G \cong G_{13}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 1$, $\deg(v_2) = 2$, then $G \cong G_{12}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_2$. If $\deg(v_1) = 1$, $\deg(v_2) = 3$, then $G \cong G_{13}$.

If $G$ does not contain the clique $K$ on $n - 2$ vertices, then it can be verified that no graph exists.

**Case iii.** $\gamma_{ptc} = n - 3$ and $\chi = n - 1$.

Since $\chi = n - 1$, $G$ contains a clique $K$ on $n - 1$ vertices or does not contains a clique $K$ on $n - 1$ vertices.

Let $G$ contains a clique $K$ on $n - 1$ vertices.

Let $v_1$ be the vertex not on $K_{n-1}$. Since $G$ is connected, there exists a vertex $v_1$ adjacent to one vertex $u_i$ of $K_{n-1}$ and $u_j$ for $i \neq j$ and $u_k$ for $i \neq j \neq k$. In this case $\{v_1, u_i, u_j, u_k\}$ is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 7$ and hence $K = K_6 = \langle u_1, u_2, u_3, u_4, u_5, u_6 \rangle$. 

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From the figure 4.20 (f), Let $u_1$ be adjacent to $v_1$. If $\text{deg} (v_1) = 1$, then $G \cong K_6(P_2)$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_1$. If $\text{deg} (v_1) = 2$, then $G \cong K_6(2)$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_1$ and $u_3$ be adjacent to $v_1$. If $\text{deg} (v_1) = 3$, then $G \cong K_6(3)$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_1$ and $u_3$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$. If $\text{deg} (v_1) = 4$, then $G \cong K_6(4)$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_1$ and $u_3$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$ and $u_5$ be adjacent to $v_1$. If $\text{deg} (v_1) = 5$, then $G \cong K_6(5)$.

If $G$ does not contain the clique $K$ on $n - 1$ vertices, then it can be verified that no graph exists.

**Case iv.** $\gamma_{\text{ptc}} = n - 4$ and $\chi = n$.

Since $\chi = n$, we have $G \cong K_n$, But for $K_n$, $\gamma_{\text{ptc}}(K_n) = 4$, so that $n = 8$. Hence $G \cong K_8$.

**Theorem 4.34** For any connected graph $G$ with $n \geq 5$ vertices $\gamma_{\text{ptc}}(G) + \chi(G) = 2n - 5$ if and only if $G \cong C_3(3P_2)$, $C_3(P_3, P_2, 0)$, $C_3(2P_2, P_2, 0)$, $C_3(P_2, P_2, P_2)$, $C_3(P_4)$, $C_3(u(P_3, P_2)$,
$K_5(P_3), K_5(2P_2), K_5(P_2, P_2, 0, 0, 0), K_7(P_2), K_7(2), K_7(3), K_7(4), K_7(5), K_7(6), K_9$ or any one of the graphs in the following Figure 4.21.
**Proof:** Let $G$ be a connected graph with $n \geq 5$ vertices. Suppose $G$ is isomorphic to $C_3(3P_2)$, $C_3(P_3, P_2, 0)$, $C_3(2P_2, P_2, 0)$, $C_3(P_2, P_2, P_2)$, $C_3(P_4)$, $C_3(u(P_3, P_2)$, $K_5(P_3)$, $K_5(2P_2)$, $K_5(P_2, P_2, 0, 0, 0)$, $K_7(P_2)$, $K_7(2)$, $K_7(3)$, $K_7(4)$, $K_7(5)$, $K_7(6)$, $K_9$ or any one of the graphs in figure 4.21, then clearly $\gamma_{ptc}(G) + \chi(G) = 2n - 5$.

Conversely, let $\gamma_{ptc}(G) + \chi(G) = 2n - 5$. This is possible if

(i) $\gamma_{ptc}(G) = n - 1$ and $\chi(G) = n - 4$

(ii) $\gamma_{ptc}(G) = n - 2$ and $\chi(G) = n - 3$

(iii) $\gamma_{ptc}(G) = n - 3$ and $\chi(G) = n - 2$

(iv) $\gamma_{ptc}(G) = n - 4$ and $\chi(G) = n - 1$

(v) $\gamma_{ptc}(G) = n - 5$ and $\chi(G) = n$.

**Case i.** $\gamma_{ptc}(G) = n - 1$ and $\chi(G) = n - 4$.

Since $\chi(G) = n - 4$, $G$ contains a clique on $n - 4$ vertices or does not contain clique on $n - 4$ vertices.

Let $G$ contains a clique on $n - 4$ vertices. Let $S = \{v_1, v_2, v_3, v_4\}$.
Then the induced subgraph $<S>$ has the following possible cases.

$<S> = K_4, \overline{K_4}, P_4, C_4, K_{1,3}, K_2 \cup K_2, K_3 \cup K_1, K_4 - \{e\}, C_3(1, 0, 0), P_3 \cup K_1, K_2 \cup \overline{K_2}$

**Subcase i.** Let $<S> = K_4$.

Let $\{v_1, v_2, v_3, v_4\}$ be the vertices of $K_4$. Since $G$ is connected, there exists a vertex $u_i$ of $K_{n-4}$ which is adjacent to any one of $\{v_1, v_2, v_3, v_4\}$. Let $u_i$ be adjacent to $v_4$ for some $i$ in $K_{n-4}$. Then $\{u_i, v_4, v_1, v_2\}$ is an $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

**Subcase ii.** Let $<S> = \overline{K_4}$.

Let $\{v_1, v_2, v_3, v_4\}$ be the vertices of $\overline{K_4}$. Since $G$ is connected, two vertices of the $\overline{K_4}$ are adjacent to one vertex say $u_i$ and the remaining two vertices of $\overline{K_4}$ are adjacent to one vertex say $u_j$ for $i \neq j \neq k$ and $u_k$ for $i \neq j \neq k \neq s$. In this case $\{u_i, u_j, u_k, u_s\}$ is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

Since $G$ is connected, one vertex of $\overline{K_4}$ is adjacent to $u_i$ and the remaining three vertices of $\overline{K_4}$ are adjacent to vertex say $u_j$ for $i \neq j \neq k$ and $u_k$ for $i \neq j \neq k \neq s$. In this case $\{u_i, u_j, u_k, u_s\}$ forms a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

Since $G$ is connected, there exists a vertex $u_i$ in $\overline{K_4}$ adjacent to $v_1$ and $v_2$ and $u_j$ for $i \neq j$ is adjacent to $v_3$ and $u_k$ for $i \neq j \neq k$ is adjacent to $v_4$ and $u_s$ for $i \neq j \neq k \neq s$. In this case $\{u_i, u_j, u_k, u_s\}$ forms a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

Since $G$ is connected, there exists a vertex $u_i$ of $\overline{K_4}$ adjacent to $v_1$ and $u_j$ for $i \neq j$ is adjacent to $v_2$ and $u_k$ for $i \neq j \neq k$ is adjacent to $v_3$ and $u_s$ for $i \neq j \neq k \neq s$ is adjacent to $v_4$. In this case $\{u_i, u_j, u_k, u_s\}$ for $i \neq j \neq k \neq s$ is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.
Subcase iii. Let $<S> = P_4 = v_1v_2v_3v_4$.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$ or $v_4$ and $v_2$ or $v_3$. If $u_i$ is adjacent to $v_1$, then $\{u_i, v_1, v_2, v_3\}$ forms a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

If $u_i$ is adjacent to $v_2$, then $\{u_i, v_2, v_3, v_4\}$ forms a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

Subcase iv. Let $<S> = K_2 \cup K_2$.

Let $v_1, v_2$ be the vertices of $K_2$ and $v_3, v_4$ be the vertices of $K_2$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$ and $v_3$. In this case $\{v_1, u_i, v_3, v_4\}$ forms an $\gamma_{ptc}$-set of $G$ so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$ and $u_j$ for $i \neq j$ is adjacent to $v_3$. In this case $\{v_1, u_i, u_j, v_3\}$ forms an $\gamma_{ptc}$-set of $G$ so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

Subcase v. Let $<S> = K_2 \cup \overline{K}_2$.

Let $v_1, v_2$ be the vertices of $\overline{K}_2$ and $v_3, v_4$ be the vertices of $K_2$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$ and $v_2$ and any one of $\{v_3, v_4\}$. Let $u_i$ be adjacent to $v_1, v_2, v_3$ and $u_j$ for $i \neq j$. In this case $\{u_j, u_i, v_3, v_4\}$ is a $\gamma_{ptc}$-set of $G$ so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$ and there exists a vertex $u_j$ for $i \neq j$ in $K_{n-4}$ is adjacent to $v_2$ and $v_3$. In this case $\{u_i, u_j, v_3, v_4\}$ forms an $\gamma_{ptc}$-set of $G$ so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_1$ and $u_j$ for $i \neq j$ is adjacent to $v_2$ and $u_k$ for $i \neq j \neq k$ is adjacent to $v_3$. In this case $\{u_i, u_j, u_k, v_3\}$ forms a $\gamma_{ptc}$-set of $G$. So that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.
Subcase vi. \( <S> = P_3 \cup K_1 \).

Let \( v_1, v_2, v_3 \) be the vertices of \( P_3 \) and \( v_4 \) be the vertex of \( K_1 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to any one of \( \{v_1, v_2, v_3\} \) and \( v_4 \). Then \( u_i \) be adjacent to \( v_1 \) and \( v_4 \). In this case \( \{v_4, u_i, v_1, v_2\} \) is a \( \gamma_{ptc} \)-set of \( G \) so that \( \gamma_{ptc} = 4 \) and \( n = 5 \), which is a contradiction. Hence no graph exists.

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_1 \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_4 \). In this case \( \{u_j, u_i, v_1, v_2\} \) is a \( \gamma_{ptc} \)-set of \( G \), so that \( \gamma_{ptc} = 4 \) and \( n = 5 \), which is a contradiction. Hence no graph exists.

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_2 \) and \( v_4 \). In this case \( \{v_4, u_i, v_1, v_2\} \) is a \( \gamma_{ptc} \)-set of \( G \), so that \( \gamma_{ptc} = 4 \) and \( n = 5 \), which is a contradiction. Hence no graph exists.

Subcase vii. \( <S> = K_3 \cup K_1 \).

Let \( v_1, v_2, v_3 \) be the vertices of \( K_3 \) and \( v_4 \) be the vertices of \( K_1 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) is adjacent to \( v_1 \). In this case \( \{v_4, u_i, v_1, v_2\} \) is a \( \gamma_{ptc} \)-set of \( G \), so that \( \gamma_{ptc} = 4 \) and \( n = 5 \), which is a contradiction. Hence no graph exists.

Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_2 \) and \( u_j \) for \( i \neq j \) is adjacent to \( v_4 \). In this case \( \{u_j, u_i, v_2, v_1\} \) for \( i \neq j \neq k \) is a \( \gamma_{ptc} \)-set of \( G \), so that \( \gamma_{ptc} = 4 \) and \( n = 5 \), which is a contradiction. Hence no graph exists.

Subcase viii. \( <S> = K_4 - \{e\} \)

Let \( v_1, v_2, v_3, v_4 \) be the vertices of \( K_4 \). Let \( \{e\} \) be any one the edge inside the cycle \( C_4 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( v_1 \). In this case
\{u, v_1, v_2, v_3\} is a \(\gamma_{ptc}\)-set of \(G\), so that \(\gamma_{ptc} = 4\) and \(n = 5\), which is a contradiction. Hence no graph exists.

Since \(G\) is connected, there exists a vertex \(u\) in \(K_{n-4}\) which is adjacent to \(v_2\). In this case \(\{u, v_2, v_1, v_4\}\) is a \(\gamma_{ptc}\)-set of \(G\), so that \(\gamma_{ptc} = 4\) and \(n = 5\), which is a contradiction. Hence no graph exists.

Subcase ix. \(\langle S\rangle = C_3(1, 0, 0)\).

Let \(v_1, v_2, v_3\) be the vertices of \(C_3\) and let \(v_4\) be adjacent to \(v_1\).

Since \(G\) is connected, there exists a vertex \(u\) in \(K_{n-4}\) which is adjacent to \(v_2\) and \(u\) for \(i \neq j\). In this case \(\{u_j, u, v_2, v_1\}\) is a \(\gamma_{ptc}\)-set of \(G\), so that \(\gamma_{ptc} = 4\) and \(n = 5\), which is a contradiction. Hence no graph exists.

Since \(G\) is connected, there exists a vertex \(u\) in \(K_{n-4}\) which is adjacent to \(v_1\) and \(u\) for \(i \neq j\). In this case \(\{u_j, u, v_1, v_4\}\) is a \(\gamma_{ptc}\)-set of \(G\), so that \(\gamma_{ptc} = 4\) and \(n = 5\), which is a contradiction. Hence no graph exists.

Subcase x. \(\langle S\rangle = K_{1,3}\).

Let \(v_1\) be the root vertex and \(v_2, v_3, v_4\) are adjacent to \(v_1\). Since \(G\) is connected, there exists a vertex \(u\) in \(K_{n-4}\) which is adjacent to \(v_1\) and \(u\) for \(i \neq j\). In this case \(\{u_j, u, v_1, v_2\}\) is a \(\gamma_{ptc}\)-set of \(G\), so that \(\gamma_{ptc} = 4\) and \(n = 5\), which is a contradiction. Hence no graph exists.

Subcase xi. \(\langle S\rangle = C_4\).

Let \(v_1, v_2, v_3, v_4\) be the vertices of \(C_4\).
Since G is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $v_4$. In this case, \( \{u_i, v_4, v_3, v_2\} \) is a $\gamma_{ptc}$-set of G, so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

If G does not contain the clique $K$ on $n - 4$ vertices, then it can be verified that no graph exists.

**Case ii.** $\gamma_{ptc} = n - 2$ and $\chi = n - 3$.

Since $\chi = n - 3$, G contains a clique $K$ on $n - 3$ vertices or does not contain a clique $K$ on $n - 3$ vertices.

Let $G$ contains a clique $K$ on $n - 3$ vertices.

Let $S = V(G) - V(K) = \{v_1, v_2, v_3\}$. Then the induced subgraph $<S>$ has the following possible cases. $<S> = K_3, \overline{K}_3, P_3, K_2 \cup K_1$.

**Subcase i.** $<S> = K_3$.

Let $v_1, v_2, v_3$ be the vertices of $K_3$.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ which is adjacent to $v_2$ and $u_j$ for $i \neq j$. Then $\{u_j, u_i, v_2, v_1\}$ is a $\gamma_{ptc}$-set of G, so that $\gamma_{ptc} = 4$ and $n = 6$ and hence $K = K_3 = \langle u_1, u_2, u_3 \rangle$.

![Diagram](Figure 4.21 (a))
From the figure 4.21 (a), Let \( u_1 \) be adjacent to \( v_2 \). If \( \deg (v_1) = 2 = \deg (v_3), \deg (v_2) = 3 \), then \( G \cong G_1 \).

Let \( u_1 \) be adjacent to \( v_2 \) and \( u_2 \) be adjacent to \( v_2 \). If \( \deg (v_1) = 2 = \deg (v_3), \deg (v_2) = 4 \), then \( G \cong G_2 \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( v_2 \). If \( \deg (v_1) = 3, \deg (v_3) = 2, \deg (v_2) = 3 \), then \( G \cong G_2 \).

Let \( u_1 \) be adjacent to \( v_2 \) and \( u_3 \) be adjacent to \( v_3 \). If \( \deg (v_1) = 2, \deg (v_3) = 3, \deg (v_2) = 3 \), then \( G \cong G_3 \).

Let \( u_1 \) be adjacent to \( v_2 \) and \( u_2 \) be adjacent to \( v_3 \) and \( u_3 \) be adjacent to \( v_3 \). If \( \deg (v_1) = 2, \deg (v_2) = 3, \deg (v_3) = 4 \), then \( G \cong G_4 \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( v_2 \) and \( u_3 \) be adjacent to \( v_3 \). If \( \deg (v_1) = 3, \deg (v_2) = 3, \deg (v_3) = 3 \), then \( G \cong G_4 \).

Let \( u_1 \) be adjacent to \( v_2 \) and \( u_2 \) be adjacent to \( v_1 \) and \( u_3 \) be adjacent to \( v_3 \). If \( \deg (v_1) = 3, \deg (v_2) = 3 = \deg (v_3) \), then \( G \cong G_5 \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( v_2 \) and \( u_2 \) be adjacent to \( v_1 \) and \( u_3 \) be adjacent to \( v_3 \). If \( \deg (v_1) = 4, \deg (v_2) = 3 = \deg (v_3) \), then \( G \cong G_6 \).

**Subcase ii.** \( \langle S \rangle = \overline{K}_3 \).

Let \( v_1, v_2, v_3 \) be the vertices of \( \overline{K}_3 \).

Since \( G \) is connected, one of the vertices of \( K_{n-3} \) say \( u_i \) is adjacent to all the vertices of \( S \) and \( u_j \) for \( i \neq j \) and \( u_k \) for \( i \neq j \neq k \). Then \( \{v_1, u_i, u_j, u_k\} \) is a \( \gamma_{ptc} \)-et of \( G \), so that \( \gamma_{ptc} = 4 \) and \( n = 6 \) and hence \( K = K_3 = \langle u_1, u_2, u_3 \rangle \).
From the figure 4.21 (b), If $u_1$ is adjacent to $v_1$, $v_2$ and $v_3$, then $G \cong C_3(3P_2)$.

If $u_1$ is adjacent to $v_1$, $v_2$ and $v_3$ and $u_2$ be adjacent to $v_1$. If $\deg(v_1) = 2$, $\deg(v_2) = 1 = \deg(v_3)$, then $G \cong G_7$.

If $u_1$ is adjacent to $v_1$, $v_2$ and $v_3$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 1 = \deg(v_3)$, $\deg(v_2) = 2$, then $G \cong G_7$.

If $u_1$ is adjacent to $v_1$, $v_2$ and $v_3$ and $u_3$ be adjacent to $v_3$. If $\deg(v_1) = 1 = \deg(v_2)$, $\deg(v_3) = 2$, then $G \cong G_7$.

If $u_i$ is adjacent to $v_1$ and $u_j$ for $i \neq j$ is adjacent to $v_2$ and $v_3$ and $u_k$ for $i \neq j \neq k$, then \{v_1, u_i, u_j, u_k\} is an $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 6$. Hence $K = K_3 = \langle u_1, u_2, u_3 \rangle$. If $u_1$ is adjacent to $v_1$ and $v_2$ and $u_3$ is adjacent to $v_3$, then $G \cong C_3(2P_2, P_2, 0)$.

If $u_1$ is adjacent to $v_1$ and $v_2$ and $u_2$ is adjacent to $v_3$ and $u_3$ is adjacent to $v_3$. If $\deg(v_1) = 1$, $\deg(v_2) = 1$, $\deg(v_3) = 2$, then $G \cong G_8$.

If $u_1$ is adjacent to $v_1$ and $v_2$ and $u_3$ is adjacent to $v_3$ and $u_2$ be adjacent to $v_1$. If $\deg(v_1) = 2$, $\deg(v_2) = 1 = \deg(v_3)$, then $G \cong G_9$.

If $u_1$ is adjacent to $v_1$ and $v_2$ and $u_3$ is adjacent to $v_3$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 1 = \deg(v_3)$, $\deg(v_2) = 2$, then $G \cong G_9$. 

\[ \begin{figure} 
\centering
\includegraphics[width=0.5\textwidth]{figure421b.png}
\caption{Figure 4.21 (b)}
\end{figure} \]
Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ which is adjacent to $v_1$ and $u_j$ for $i \neq j$ in $K_{n-3}$ is adjacent to $v_2$ and $u_k$ for $i \neq j \neq k$ in $K_{n-3}$, which is adjacent to $v_3$ and $u_s$ for $i \neq j \neq k \neq s$. In this case $\{u_i, u_j, u_k, u_s\}$ is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 6$, and hence

$K = K_3 = \langle u_1, u_2, u_3 \rangle$.

Let $u_1$ be adjacent to $v_2$ and $u_2$ be adjacent to $v_1$ and $u_3$ be adjacent to $v_3$. If $\deg (v_1) = \deg (v_2) = \deg (v_3) = 1$, then $G \cong C_3(P_2,P_2,P_2)$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_1$ and $u_3$ be adjacent to $v_3$. If $\deg (v_1) = 2$, $\deg (v_2) = \deg (v_3) = 1$, then $G \cong G_9$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_1$ and $v_3$ and $u_3$ be adjacent to $v_3$. If $\deg (v_1) = 2 = \deg (v_3)$, $\deg (v_2) = 1$, then $G \cong G_{10}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_1$ and $v_2$ and $u_3$ be adjacent to $v_3$. If $\deg (v_1) = 2 = \deg (v_2)$, $\deg (v_3) = 1$, then $G \cong G_{11}$.

**Subcase iii.** $\langle P_3 \rangle = v_1v_2v_3$.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ which is adjacent to $v_2$ and $u_j$ for $i \neq j$. Then $\{v_1, v_2, u_i, u_j\}$ is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 6$. Hence

$K = K_3 = \langle u_1, u_2, u_3 \rangle$.

![Figure 4.21 (c)](image-url)
From the figure 4.21 (c), If $u_1$ is adjacent to $v_2$. If $\deg (v_1) = 1 = \deg (v_3)$, $\deg (v_2) = 3$ then $G \cong G_{12}$.

If $u_1$ is adjacent to $v_2$ and $u_2$ is adjacent to $v_2$. If $\deg (v_1) = 1 = \deg (v_3)$, $\deg (v_2) = 4$ then $G \cong G_8$.

If $u_1$ is adjacent to $v_2$ and $u_2$ is adjacent to $v_1$. If $\deg (v_1) = 2$, $\deg (v_2) = 3$, $\deg (v_2) = 1$, then $G \cong G_{13}$.

If $u_1$ is adjacent to $v_2$ and $u_2$ is adjacent to $v_1$ and $u_3$ be adjacent to $v_1$. If $\deg (v_1) = 3$, $\deg (v_2) = 3$, $\deg (v_3) = 1$, then $G \cong G_{14}$.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ which is adjacent to $v_1$ and $u_j$ for $i \neq j$, then $\{u_i, u_j, v_1, v_2\}$ for some $i \neq j$ is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 6$ and hence $K = K_3 = < u_1, u_2, u_3 >$.

Let $u_1$ be adjacent to $v_1$, then $G \cong C_3(P_4)$

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_1$. If $\deg (v_1) = 3$, $\deg (v_2) = 2$, $\deg (v_3) = 1$, then $G \cong G_{15}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$. If $\deg (v_1) = 2$, $\deg (v_2) = 3$, $\deg (v_3) = 1$, then $G \cong G_{16}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_3$ be adjacent to $v_2$. If $\deg (v_1) = 2$, $\deg (v_2) = 4$, $\deg (v_3) = 1$, then $G \cong G_{17}$.

Let $u_1$ be adjacent to $v_1$ and $u_3$ be adjacent to $v_3$. If $\deg (v_1) = 2 = \deg (v_2)$, $\deg (v_3) = 2$, then $G \cong G_{18}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_3$ and $u_3$ be adjacent to $v_3$. If $\deg (v_1) = 2$, $\deg (v_2) = 2$, $\deg (v_3) = 3$, then $G \cong G_{19}$.

**Subcase iv.** $<S> = K_2 \cup K_1$.

Let $v_1$, $v_2$ be the vertices of $K_2$ and $v_3$ be the isolated vertex. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-3}$ which is adjacent to any one of $\{v_1, v_2\}$ and $\{v_3\}$ (or) $u_i$ is
adjacent to any one of \{v_1, v_2\} and u_j for \(i \neq j\) is adjacent to \(v_3\). In this case \{u_j, u_i, v_1, v_2\} is a \(\gamma_{ptc}\)-set of G, so that \(\gamma_{ptc} = 4\) and \(n = 6\) and hence \(K = K_3 = \langle u_1, u_2, u_3 \rangle\).

From the figure 4.21 (d), Let \(u_1\) be adjacent to \(v_1\) and \(v_3\). If \(\deg(v_1) = 2\), \(\deg(v_2) = 1\), \(\deg(v_3) = 1\), then \(G \cong C_3(u(P_3, P_2))\).

Let \(u_1\) be adjacent to \(v_1\) and \(v_3\) and \(u_2\) be adjacent to \(v_1\). If \(\deg(v_1) = 3\), \(\deg(v_2) = 1\), \(\deg(v_3) = 1\), then \(G \cong G_9\).

Let \(u_1\) be adjacent to \(v_1\) and \(v_2\) and \(v_3\). If \(\deg(v_1) = 2 = \deg(v_2), \deg(v_3) = 1\), then \(G \cong G_{20}\).

Let \(u_1\) be adjacent to \(v_1\) and \(v_2\) and \(v_3\) and \(u_2\) be adjacent to \(v_2\). If \(\deg(v_1) = 2\), \(\deg(v_2) = 3\), \(\deg(v_3) = 1\), then \(G \cong G_{21}\).

Let \(u_1\) be adjacent to \(v_1\) and \(v_3\). If \(\deg(v_1) = 2\), \(\deg(v_2) = 1 = \deg(v_3)\), then \(G \cong C_3(P_3, P_2, 0)\).

Let \(u_1\) be adjacent to \(v_1\) and \(u_2\) be adjacent to \(v_1\) and \(u_3\) be adjacent to \(v_3\). If \(\deg(v_1) = 3\), \(\deg(v_2) = 1 = \deg(v_3)\), then \(G \cong G_{22}\).

Let \(u_1\) be adjacent to \(v_1\) and \(u_2\) be adjacent to \(v_2\) and \(u_3\) be adjacent to \(v_3\). If \(\deg(v_1) = 2\), \(\deg(v_2) = 2\), \(\deg(v_3) = 1\), then \(G \cong G_{23}\).
Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_2$ and $v_3$. If $\deg(v_1) = 2$, $\deg(v_2) = 3$, $\deg(v_3) = 1$, then $G \cong G_{24}$.

If a vertex $u_i$ in $K_{n-3}$ is adjacent to $v_1$ and $v_3$ then $\{u_j, u_i, v_1, v_2\}$ is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 6$ and hence $K = K_3 = \langle u_1, u_2, u_3 \rangle$.

Let $u_1$ be adjacent to $v_1$ and $v_3$. If $\deg(v_1) = 2$, $\deg(v_2) = 1 = \deg(v_3)$ then $G \cong C_3(u(P_3, P_2))$.

Let $u_1$ be adjacent to $v_1$ and $v_3$ and $u_2$ be adjacent to $v_1$. If $\deg(v_1) = 3$, $\deg(v_2) = 1$, $\deg(v_3) = 1$ then $G \cong G_9$.

Let $u_1$ be adjacent to $v_1$ and $v_3$ and $u_3$ be adjacent to $v_3$. If $\deg(v_1) = 2 = \deg(v_3)$, $\deg(v_2) = 1$, then $G \cong G_{25}$.

Let $u_1$ be adjacent to $v_1$ and $v_3$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 2 = \deg(v_2)$, $\deg(v_3) = 1$, then $G \cong G_{26}$.

If $G$ does not contains the clique $K$ on $n - 3$ vertices, then it can be verified that no graph exists.

**Case iii.** $\gamma_{ptc} = n - 3$ and $\chi = n - 2$.

Since $\chi = n - 2$, $G$ contains a clique $K$ on $n - 2$ vertices or does not contain a clique $K$ on $n - 2$ vertices.

Let $G$ contains a clique $K$ on $n - 2$ vertices.

Let $S = V(G) - V(K) = \{v_1, v_2\}$. Then $\langle S \rangle = K_2, R_2$.

**Subcase i.** $\langle S \rangle = K_2$.

Let $v_1, v_2$ be the vertices of $K_2$.

Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-2}$ is adjacent to any one of $\{v_1, v_2\}$ then $\{u_j, u_i, v_1, v_2\}$ is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 7$ and hence $K = K_5 = \langle u_1, u_2, u_3, u_4, u_5 \rangle$.
From the figure 4.21 (e), Let $u_1$ be adjacent to $v_1$. If $\deg(v_1) = 2$, $\deg(v_2) = 1$, then $G \cong K_5(P_3)$.

Let $u_1$ be adjacent to $v_1$ and $u_5$ be adjacent to $v_1$. If $\deg(v_1) = 3$, $\deg(v_2) = 1$, then $G \cong G_{27}$.

Let $u_1$ be adjacent to $v_1$ and $u_5$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 4$, $\deg(v_2) = 1$ then $G \cong G_{28}$.

Let $u_1$ be adjacent to $v_1$ and $u_5$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 5$, $\deg(v_2) = 1$, then $G \cong G_{29}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 2 = \deg(v_2)$, then $G \cong G_{30}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_2$. If $\deg(v_1) = 2$, $\deg(v_2) = 3$, then $G \cong G_{31}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_2$ and $u_4$ be adjacent to $v_2$. If $\deg(v_1) = 2$, $\deg(v_2) = 4$, $G \cong G_{32}$.

**Subcase ii.** Let $\langle S \rangle = \overline{K}_2$.

Let $v_1, v_2$ be the vertices of $\overline{K}_2$. 
Since $G$ is connected, $v_1$ and $v_2$ are adjacent to a common vertex say $u_i$ of $K_{n-2}$ for some $i$ and $u_j$ for some $i \neq j$ in $K_{n-2}$ and $u_k$ for some $i \neq j \neq k$. In this case $\{v_1, u_i, u_j, u_k\}$ is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 7$ and hence $K = K_5 = \langle u_1, u_2, u_3, u_4, u_5 \rangle$.

From the figure 4.21 (f), Let $u_1$ be adjacent to $v_1$ and $v_2$. If $\deg(v_1) = 1 = \deg(v_2)$, then $G \cong K_5(2P_2)$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 1$, $\deg(v_2) = 2$, then $G \cong G_{33}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_2$. If $\deg(v_1) = 1$, $\deg(v_2) = 3$, then $G \cong G_{34}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to and $v_2$ and $u_3$ be adjacent to $v_2$ and $u_4$ be adjacent to $v_2$. If $\deg(v_1) = 1$, $\deg(v_2) = 4$, then $G \cong G_{35}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_2$ and $u_5$ be adjacent to $v_1$. If $\deg(v_1) = 2 = \deg(v_2)$, then $G \cong G_{36}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_2$ and $u_5$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 3$, $\deg(v_2) = 2$, then $G \cong G_{37}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$. If $\deg(v_1) = 4$, $\deg(v_2) = 2$, then $G \cong G_{38}$. 

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Let $u_i$ be adjacent to $v_1$ and $u_j$ for $i \neq j$ is adjacent to $v_2$ and $u_k$ for $i \neq j \neq k$. In this case 
\{v_1, u_i, u_j, u_k\} is a $\gamma_{ptc}$-set of $G$, so that $\gamma_{ptc} = 4$ and $n = 7$ and hence $K = K_5 = <u_1, u_2, u_3, u_4, u_5>$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 1 = \deg(v_2)$, then $G \cong K_5(P_2, P_2, 0, 0, 0)$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_5$ be adjacent to $v_1$. If $\deg(v_1) = 2, \deg(v_2) = 1$, then $G \cong G_{39}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_4$ be adjacent to $v_1$ and $u_5$ be adjacent to $v_1$. If $\deg(v_1) = 3, \deg(v_2) = 1$, then $G \cong G_{40}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_1$ and $u_4$ be adjacent to $v_1$ and $u_5$ be adjacent to $v_1$. If $\deg(v_1) = 4, \deg(v_2) = 1$, then $G \cong G_{41}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_2$. If $\deg(v_1) = 1, \deg(v_2) = 2$, then $G \cong G_{39}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_2$ and $u_4$ be adjacent to $v_2$. If $\deg(v_1) = 1, \deg(v_2) = 3$, then $G \cong G_{40}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 1, \deg(v_2) = 2$, then $G \cong G_{33}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_2$. If $\deg(v_1) = 1, \deg(v_2) = 3$, then $G \cong G_{34}$.

Let $u_1$ be adjacent to $v_1$ and $v_2$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_2$ and $u_4$ be adjacent to $v_2$. If $\deg(v_1) = 1, \deg(v_2) = 4$, then $G \cong G_{35}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_2$ and $u_5$ be adjacent to $v_1$. If $\deg(v_1) = 2 = \deg(v_2)$, then $G \cong G_{42}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$ and $u_3$ be adjacent to $v_2$ and $u_4$ be adjacent to $v_1$ and $u_5$ be adjacent to $v_1$. If $\deg(v_1) = 3, \deg(v_2) = 1$, then $G \cong G_{43}$. 

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If G does not contain the clique K on n – 2 vertices, then it can be verified that no graph exists.

**Case iv.** \( \gamma_{ptc} = n - 4 \) and \( \chi = n - 1 \).

Since \( \chi = n - 1 \), G contains a clique K on n - 1 vertices or does not contain a clique K on n - 1 vertices.

Let G contain a clique K on n - 1 vertices.

Let \( v_1 \) be the vertex not on \( K_{n-1} \). Since G is connected, there exists a vertex \( v_1 \) is adjacent to one vertex \( u_i \) of \( K_{n-1} \) and \( u_j \) for \( i \neq j \) and \( u_k \) for \( i \neq j \neq k \). In this case \( \{v_1, u_i, u_j, u_k\} \) is a \( \gamma_{ptc} \)-set of G, so that \( \gamma_{ptc} = 4 \) and \( n = 8 \) and hence \( K = K_7 = \langle u_1, u_2, u_3, u_4, u_5, u_6, u_7 \rangle \).

![Clique K_7](image)

**Figure 4.21 (g)**

From the figure 4.21 (g), let \( u_1 \) be adjacent to \( v_1 \). If \( \text{deg}(v_1) = 1 \), then \( G \cong K_7(P_2) \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( u_2 \) be adjacent to \( v_1 \). If \( \text{deg}(v_1) = 2 \), then \( G \cong K_7(2) \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( u_2 \) be adjacent to \( v_1 \) and \( u_3 \) be adjacent to \( v_1 \). If \( \text{deg}(v_1) = 3 \), then \( G \cong K_7(3) \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( u_2 \) be adjacent to \( v_1 \) and \( u_3 \) be adjacent to \( v_1 \) and \( u_4 \) be adjacent to \( v_1 \). If \( \text{deg}(v_1) = 4 \), then \( G \cong K_7(4) \).

Let \( u_1 \) be adjacent to \( v_1 \) and \( u_2 \) be adjacent to \( v_1 \) and \( u_3 \) be adjacent to \( v_1 \) and \( u_4 \) be adjacent to \( v_1 \) and \( u_5 \) be adjacent to \( v_1 \). If \( \text{deg}(v_1) = 5 \), then \( G \cong K_7(5) \).
Let \( u_1 \) be adjacent to \( v_1 \) and \( u_2 \) be adjacent to \( v_1 \) and \( u_3 \) be adjacent to \( v_1 \) and \( u_4 \) be adjacent to \( v_1 \) and \( u_5 \) be adjacent to \( v_1 \) and \( u_6 \) be adjacent to \( v_1 \). If \( \text{deg}(v_1) = 6 \), then \( G \cong K_7(6) \).

If \( G \) does not contain the clique \( K \) on \( n - 1 \) vertices, then it can be verified that no graph exists.

**Case v.** \( \gamma_{ptc} = n - 5 \) and \( \chi = n \).

Since \( \chi = n \), we have \( G \) is isomorphic to \( K_n \). But for \( K_n \), \( \gamma_{ptc}(K_n) = 4 \), so that \( n = 9 \). Hence \( G \cong K_9 \).

**Conclusion and Scope:**

In this chapter a new domination parameter paired triple connected domination number of a graph have been introduced. The lower and upper bounds of the paired triple connected domination number are obtained. The paired triple connected domination number for standard and various classes of the graphs are found and investigated many new results. Its relationship with other graph theoretical parameters have also been discussed. Like [9, 10], the authors introduced some more new domination parameters also. Relationship between the new domination parameter and other graph theoretical parameters are still open for further investigation.

The following problems are given for further investigation.

1. Characterize the graphs for which \( \gamma_{ptc}(G) = \gamma(G) \).
2. Characterize the graphs for which \( \gamma_{ptc}(G) = \chi(G) \).