Chapter 3

Generalized $\nu$-Spherical Densities

3.1 Introduction

A statistical model is a class of probability distributions that describes random behavior of a character of interest. Statistical models play important role in analyzing the data and to take decisions. In the literature many standard statistical models have been proposed which have been validated by examining for their goodness of fit to various type of data sets. However, due to the existence of many invisible/hidden factors affecting the variables of interest, a well known standard statistical models may not be appropriate to describe the random behavior of the variables. A well-known existing symmetric/asymmetric model may not give a very satisfactory fit for a given data set, may be essentially, due to a typical type of drift exhibited by the data and such a drift which is not necessarily due to a change of origin and scale. As such, there is a need to generate new statistical models and develop related theory for the purpose of inference and data analysis.

Different techniques are available to generate new statistical models by using well known existing standard models. Lai (2004) gives an overview of various methodologies to construct various bivariate distributions. Rattihalli and Basugade (2008) have generated a class of multivariate densities by using contour transformation.
Recently, Rattihalli and Basugade (2009) have introduced basic contour transform leading to new classes of decreasing densities on $R^+$. For some more references on generation of new class of densities, their characteristics and applications one may refer to: Arnold and Lin (2004), Arnold et al. (2008), Azzalini and Capitanio (1999), Azzalini and Capitanio (2003), Azzalini and Dalla Valle (1996).

The main objective of this chapter is to generate new statistical models, essentially derived from $\nu$-spherical densities proposed by Fernandez et al. (1995). We generate a class of generalized $\nu$-spherical densities by introducing an additional parameter into $\nu$-spherical densities (Rattihalli and Patil (2010)). If the initial $\nu$-spherical distribution itself is symmetric, in some sense (for details refer to Liu et al. (1999)) then as a consequence of the introduction of a parameter, it generates a class of asymmetric densities and the degree of asymmetry incorporated depends on the value of the parameter introduced. In the following section, we generalize the class of $\nu$-spherical densities of Fernandez et al. (1995) to $\nu_a$-spherical densities and refer it as the class of generalized $\nu$-spherical densities. We state some properties of generalized $\nu$-spherical densities. Illustrations are given in the Section 3.3 and amongst others, we obtain generalized $p$-variate normal, $t$ and Laplace densities. In the Section 3.4, we develop methods to generate random sample from these distributions. In the Section 3.5, adequacy of $\nu_a$-spherical densities over $\nu$-spherical densities is shown by using real data sets. Concluding remarks are given in Section 3.6.

### 3.2 Generation of generalized $\nu$-spherical densities

Fernandez et al. (1995) have considered a class of $\nu$-spherical densities of the form

$$f(x; \mu, \tau) = \tau^p g(\nu(\tau(x - \mu))), \text{ for all } x \in R^p$$

(3.1)
where $\mu \in \mathbb{R}^p$, $\tau > 0$ and $\nu(.)$ is a scalar function such that (i) $\nu(x) > 0$ (except possibly on a set of Lebesgue measure 0) (ii) $\nu(kx) = k\nu(x)$ for all $k \geq 0$, $x \in \mathbb{R}^p$ and $g(.)$ is a nonnegative function defined on $(0, \infty)$. Furthermore, $g(.)$ and $\nu(.)$ are chosen such that $f(x; \mu, \tau)$ is a probability density function.

If $X$ has the density function (3.1) then the density function of $Y = \tau(X - \mu)$ is $g(\nu(y))$, $y \in \mathbb{R}^p$. Thus, $\mu$ is a location parameter and $\tau^{-1}$ scale in the families of densities defined in (3.1). For notational simplicity, we assume that $\mu = 0$ and $\tau = 1$. Hence, (3.1) reduces to

$$f(x) = g(\nu(x)), \text{ for all } x \in \mathbb{R}^p$$

(3.2)

In the strict sense, the functions $(g(.), \nu(.))$ can not be taken as parameters for the density defined in (3.2), since for $g^*(t) = g(t/k)$ and $\nu^*(x) = k\nu(x)$, $k > 0$, we have for all $x \in \mathbb{R}^p$, $f(x; g, \nu) = g(\nu(x)) = g(k\nu(x)/k) = g(\nu^*(x)/k) = g^*(\nu^*(x)) = f(x; g^*, \nu^*)$. However, for $(g(.), \nu(.))$ to be considered as parameters, we should put an additional condition on $\nu(.)$, say $\nu(1, 0) = 1$. If $(1, 0)$ is not a point in the support of the density, by common scale and/or rotation transformation, it can be included in the support.

Geometrical structure of the density function (3.1) depends on the function $\nu(.)$ and different types of symmetry can be described by using such $\nu(.)$ functions. In the literature, different types of multivariate symmetry - spherical, elliptical, antipodal and angular symmetry, have been studied. The preceding four notions of symmetry are increasingly less restrictive. That is, spherically symmetric distributions are elliptically symmetric, elliptically symmetric distributions are antipodally symmetric and antipodally symmetric distributions are angularly symmetric. A function $f(x; \mu, \tau)$ is antipodal, spherical or elliptical symmetric if

$$\nu(x - \mu) = \nu(- (x - \mu)),$$

$$\nu(x - \mu) = \{(x - \mu)'(x - \mu)\}^{1/2} \text{ or } \nu(x - \mu) = \{(x - \mu)A'(x - \mu)\}^{1/2},$$

respectively. In angular symmetry, the line integral of the density over the sets $L(x, \mu)$ and $L(-x, \mu)$ are equal, for all $x \in \mathbb{R}^p$, where $L(x, \mu)$ is the line segment starting from $\mu$ and passing through $x$. Liu et al. (1999) have proposed tests for the above...
types of symmetry, by using central convex hulls.

For a given density function \( f(.) \), the set \( \{ \mathbf{x} : f(\mathbf{x}) = k \} \) for \( k > 0 \) is called an isodensity set. The isodensity sets corresponding to (3.2) are \( \{ \mathbf{x} : \nu(\mathbf{x}) = k \} \). The set \( \{ \mathbf{x} : \nu(\mathbf{x} - \mathbf{\theta}) = r \} \) is said to have the centre \( \mathbf{\theta} \) and the radius \( r \) with reference to \( \nu(.) \). For the class of densities (3.2), the isodensity sets are of the form \( kC \), where \( C = \{ \mathbf{x} : \nu(\mathbf{x}) = 1 \ \text{(say)} \} \). Let \( C_a = C + a \). It is to be noted that for any \( \mathbf{x} \in \mathbb{R}^p \), there corresponds a number \( k > 0 \) such that \( \mathbf{x} \in kC_a \). For any \( a \) with \( \nu(a) < 1 \), we define

\[
\nu_a(\mathbf{x}) = k, \text{ if } \mathbf{x} \in kC_a
\]  

(3.3)

We have, \( \nu_0(.) = \nu(.) \).

The set \( kC_a \) has the centre \( ka \) and radius \( k \) with reference to \( \nu_a(.) \). Since by change of location, geometrical structure of a set of points does not change, the shapes of \( C_a \) and \( C \) remain the same but with reference to \( \nu(.) \) their centers are \( a \) and \( 0 \), while with reference to \( \nu_a(.) \) their centers are \( 0 \) and \( -a \), respectively.

A symmetric model may not give a very satisfactory fit for a given data set, this may be due to a typical type of skewness exhibited by the data, which perhaps is not accounted in a well known standard model. In such a situation, to take an account of a typical type of skewness present in the data, a model with isodensity sets \( kC_a \) might be an appropriate. For testing \( H_0: f(.) \) is a \( \nu \)-spherical density, the class of \( \nu_a \)-spherical densities with \( \nu(a) < 1, a \neq 0 \) can be taken as the class of alternative.

In the literature, multivariate extensions of normal, t and elliptical distributions together with their applications have been studied by Azzalini and Capitanio (1999, 2003), Azzalini and Dalla Valle (1996) amongst many others. Arnold and Lin (2004) give a characterization of skew normal distribution by using order statistics. Arnold et al. (2008) have given general method for defining families of distributions with a given class of non-intersecting contours, not necessarily having a common centre.

The property \( \nu(k\mathbf{x}) = k\nu(\mathbf{x}) \), for all \( k \geq 0, \mathbf{x} \in \mathbb{R}^p \) of the function \( \nu(.) \) is not affected by a scalar multiplication, hence without loss of generality we assume that
\( \nu(1, 0) = 1 \). For given \( g(.) \) and \( \nu(.) \), the density function (3.2) reduces to:

\[
f(x; g, \nu) = g(\nu(x)), \quad \text{for all } x \in \mathbb{R}^p
\]  

(3.4)

For \( u \geq 0, A_u = \{x : f(x; g, \nu) = u\} \) is the \( u \)-level isodensity set (also referred as contour). The isodensity sets of (3.4) just differ by scale and hence the entire class of isodensity sets can be specified by any one of them. We shall choose one that passes through the point \((1, 0)\). Let the isodensity set passing through \((1, 0)\) be denoted by \(C\). We note that, the function \( \nu(.) \) can be characterized by the set \( C = \{x : \nu(x) = 1\} \). A condition for isodensity sets \( kC, k \geq 0 \) to be disjoint is that every extended line segment starting from \(0\) intersects \(C\) exactly once. Geometrical structure of the set \(C\) gives fairly better idea of the shape of the density function, hence in the following, we shall use \(C\) instead of \(\nu(.)\). The density function given in (3.4) can be represented as

\[
f(x; g, C) = g(k), \quad \text{if } x \in kC
\]  

(3.5)

It is a \( \nu \)-spherical density function. For given \( g(.) \) and \( C \), by considering points \( a \) with \( \nu(a) < 1 \), we generate a new class of densities \( f(x; g, C, a) \), for simplicity we write it as \( f(x; a) \) and is given by

\[
f(x; a) = g(k), \quad \text{for } x \in kC_a, \ k \geq 0
\]  

(3.6)

Equivalently,

\[
f(x; a) = g(\nu_a(x)), \quad \text{for all } x \in \mathbb{R}^p
\]  

(3.7)

which is a \( \nu_a \)-spherical density. The isodensity sets for density function \( f(x; a) \) are \( \{kC_a : k > 0\} \). In particular, for \( p = 2 \), isodensity sets for the densities (3.4) and (3.7) namely, \( kC \) and \( kC_a \) for \( \nu(x) = (x'x)^{1/2} \) are given below in Figure-3.1.

If the set \(C\) is symmetric about \(0\) then \( a \) in (3.7) can be viewed as skewness parameter introduced in \( f(x; 0) \), \((\nu\)-spherical density). As a nomenclature, similar to \textbf{epsilon-skew-normal density}, \( f(x; a) \) can be referred as \textbf{\( a \)-skew-\( \nu \)-density}. If \( g(.) \) is a strictly decreasing function, then \( f(x; a) \) is a unimodal density with modal value \(0\). Furthermore, one may introduce the parameters \( \mu \) and \( \tau \) in (3.7).
Figure 3.1: Isodensity Sets

Following are some properties of the density function \( f(x; a) \).

**Lemma 3.1:**

(i) \( f(x; a) \) is a proper density function with mode at \( 0 \).

(ii) If \( X \) and \( X_a \) have probability densities given by (3.5) and (3.6), respectively then \( \nu(X) \) and \( \nu_a(X_a) \) have identical distribution.

**Proof:**

(i) We have \( C(u) = \{ x : f(x) \geq u \} \), \( C_a(u) = \{ x : f(x; a) \geq u \} \) for all \( u \geq 0 \) and \( \Lambda(C_a(u)) = \Lambda(C(u)) \). Hence, \( \int_{R^p} f(x; a)dx = \int_0^{f(0; a)} \Lambda(C_a(u))du = \int_0^{f(0)} \Lambda(C(u))du = 1. \)

(ii) We have \( \bar{C}_a = \{ x : \nu_a(x) \leq 1 \} \)

Consider, \( P_r\{\nu_a(X_a) \leq k\} = P_r\{X_a \in k\bar{C}_a\} = \int_{\nu_a(x) \leq k} g(\nu_a(x))dx \)

\( = \int_{\nu_a(x) \leq k} \{g(k) + (g(\nu_a(x)) - g(k))\}dx = kg(k)\Lambda(C_a) + \int_{\nu_a(x) \leq k} \{g(\nu(x)) du \}dx \)
\[ = kg(k)\Lambda(C_0) + \int_{g(k)}^{0} \Lambda[\tilde{C}_0(u)]du, \text{ since } \Lambda(\tilde{C}_a) = \Lambda(\tilde{C}_2) \]

\[ = P_r\{X \in k\tilde{C}\} = P_r\{\nu(X) \leq k\}. \]

It shows that \(\nu(X)\) and \(\nu_a(X_a)\) have identical distribution and hence distribution of \(\nu_a(X_a)\) is independent of \(a\).

**Lemma 3.2:** If \(X\) has a \(C\)-isodensity density function then

\[ P_r\{X_1 > 0\} = \Lambda(\tilde{C}^+)/\Lambda(\tilde{C}). \quad (3.8) \]

**Proof:** If isodensity sets corresponding to a density \(f(.)\) are \(kA\) for some \(k > 0\), then \(f(.)\) is said to be an \(A\)-isodensity function. The function defined in (3.7) is \(C_a\) - isodensity function.

Let \(X\) has a \(C\)-isodensity density function \(f(.)\). Then for any \(u, \ 0 < u \leq f(0)\), the set \(C(u) = \{x : f(x) > u\} = k\tilde{C}\) for some \(k > 0\) and \(\tilde{C} = \{x : \nu(x) \leq 1\}\).

Let \(\tilde{C}^+ = \{x : x \in \tilde{C} \text{ and } x_1 > 0\}\) then \(\Lambda(C^+(u))/\Lambda(C(u)) = \Lambda(k\tilde{C}^+)/\Lambda(k\tilde{C}) = \Lambda(\tilde{C}^+)/\Lambda(\tilde{C})\), which is independent of \(u\), for \(0 < u \leq f(0)\).

Let \(C^+(u) = \{x : f(x) > u \text{ and } x_1 > 0\}\) and \(\bar{x}(2) = (x_2, x_3, \ldots, x_p)\).

Consider,

\[ P_r\{X_1 > 0\} = \int_0^\infty \int_{R^{p-1}} f(x_1, \bar{x}(2))dx_1d\bar{x}(2) = \int_0^\infty \int_{R^{p-1}} (\int_0^{f(x_1, \bar{x}(2))} du)dx_1d\bar{x}(2) \]

\[ = \int_0^{f(0)} (\int_{C^+(u)} du)dx_1d\bar{x}(2) = \int_0^{f(0)} \Lambda(C^+(u))du \]

\[ = \int_0^{f(0)} \{\Lambda(C^+(u))/\Lambda(C(u))\}\Lambda(C(u))du \]

\[ = \Lambda(\tilde{C}^+)/\Lambda(\tilde{C}). \]

Hence the lemma.

Thus if a random vector \(X_a\) has density function (3.7), \(a = (a, 0)\), \(0 < \nu(a) < 1\), \(\nu(.)\) is known and satisfies the condition \(\nu(x) = \nu(-x)\) and \(g(.)\) is not necessarily known then \(P_r\{X_1a > 0\} = (\Psi(a), \text{ say })\) is given by

\[ \Psi(a) = \Lambda(\tilde{C}_a^+)/\Lambda(\tilde{C}_a) \quad (3.9) \]
The function $\Psi(a)$ is independent of $g(.)$ and continuous, differentiable, strictly increasing function with $\Psi(a) \to 0$ as $a \to -1$ and $\Psi(a) \to 1$ as $a \to 1$.

In the following section, we give explicit expressions of $f(x; a)$ for different choices of $g(.)$ and $C$.

### 3.3 Illustrations

As an illustration, in this section, we generalize three well known p-variate classes of densities namely, normal, t and Laplace, together with some other examples (for details on p-variate densities, one may refer to Johnson et al. (1972), Kotz et al. (2000)). In Subsection 3.3.1, by using Euclidian norm we define generalized p-variate normal, t densities and a generalized bivariate conical density. In Subsection 3.3.2, we define generalized p-variate Laplace and generalized bivariate pyramid shaped density by using taxicab norm.

#### 3.3.1 Generating density by using Euclidian norm

Consider the Euclidian norm, $\nu(x) = (x'x)^{1/2} = ||x||$ (say). The corresponding set $C$ is given by $C = \{x : x'x = 1\}$. Let $||a|| < 1$, $\bar{C} = \{x : x'x \leq 1\}$ and $\bar{C}_a = \bar{C} + a = \{x + a : x'x \leq 1\} = \{x : (x - a)'(x - a) \leq 1\}$. We note that, $0 \in \bar{C}_a$ and for any $x(\neq 0) \in \mathbb{R}^p$, there exists unique $k > 0$ such that $x \in k\bar{C}_a$ and hence, $\nu_a(x) = k > 0$ for $x \in k\bar{C}_a$. Furthermore, $x \in k\bar{C}_a$ if and only if $(x - ka)'(x - ka) = k^2$. If $\nu_a(x) = k$ then $(x-ka)'(x-ka) = k^2$ and hence for any $m > 0$, $(mx - mk_a)'(mx - mk_a) = m^2k^2$, which in turn implies that $\nu_a(mx) = m\nu_a(x)$. Thus, $\nu_a(x)$ satisfies conditions (a) and (b) in (1) of Fernandez et al. (1995). As for any $x(\neq 0) \in \mathbb{R}^p$, there exists unique $k > 0$ such that $x \in k\bar{C}_a$ and we have $\nu_a(x) = k$ if $x \in k\bar{C}_a$, that is $\nu_a(x) = k$ if $x$ satisfies the condition

\[(x - ka)'(x - ka) = k^2\] (3.10)
The roots of the equation (3.10) for $k$ are given by

$$k = \begin{cases} 
-\mathbf{x}'a + [(\mathbf{x}'a)^2 + (1 - a'a)x'x]^{1/2} / (1 - a'a) > 0 \\
-\mathbf{x}'a - [(\mathbf{x}'a)^2 + (1 - a'a)x'x]^{1/2} / (1 - a'a) < 0
\end{cases}$$

(3.11)

Hence, for $\nu(\mathbf{x}) = ||\mathbf{x}||$ and $||\mathbf{a}|| < 1$,

$$\nu_a(\mathbf{x}) = \{-\mathbf{x}'a + [(\mathbf{x}'a)^2 + (1 - a'a)x'x]^{1/2} / (1 - a'a)\}$$

(3.12)

Thus, for $\nu(\mathbf{x}) = ||\mathbf{x}||$, from equation (3.7)

$$f(\mathbf{x}; a) = g\{(\mathbf{x}'a + [(\mathbf{x}'a)^2 + (1 - a'a)x'x]^{1/2}) / (1 - a'a)\}$$

for all $\mathbf{x} \in \mathbb{R}^p$, $||\mathbf{a}|| < 1$

(3.13)

It is easy to observe that:

1. $\nu(\mathbf{x}) = (\mathbf{x}'A\mathbf{x})^{1/2}$, for some positive definite and symmetric matrix $A$, then for $\mathbf{a}$ such that $\mathbf{a}'A\mathbf{a} < 1$, we have

$$\nu_a(\mathbf{x}) = \{-\mathbf{x}'A\mathbf{a} + [(\mathbf{x}'A\mathbf{a})^2 + (1 - a'A\mathbf{a})x'\mathbf{a}x]^{1/2} / (1 - a'A\mathbf{a})\}$$

2. Let $X_a = (X_{1,a}, X_{2,a}, \ldots, X_{p,a})$ have density (3.13). Then

(i) $X_a$ and $-X_{-a}$ are identically distributed.

(ii) $X_{1,a}, X_{2,a}, \ldots, X_{p,a}$ are exchangeable if and only if $\mathbf{a} = (a, a, \ldots, a)$.

In the following, we illustrate how appropriate choices of function $g(.)$ in (3.13) gives us:

(A) Generalized p-variate normal density, (B) Generalized p-variate t density and

(C) Generalized bivariate conical density.

(A) **Generalized p-variate normal density**: Let $g(t) = (2\pi)^{-p/2}exp(-t^2/2)$, for all $t > 0$. Now from (3.4), we have

$$f(\mathbf{x}; g, \nu) = (2\pi)^{-p/2}exp(-\mathbf{x}'\mathbf{x}/2)$$

for all $\mathbf{x} \in \mathbb{R}^p$, 

the p-variate density function of independent standard normal variates. From (3.7) and (3.11), the \textit{generalized p-variate normal density} is

\[
 f(x; a) = (2\pi)^{-p/2} \exp\left(-\frac{1}{2}\left(-\mathbf{x}'a + [(\mathbf{x}'a)^2 + (1 - a'a)x'x]^{1/2}\right)^2/(1 - a'a)^2\right)
\]

for all \( x \in \mathbb{R}^p, ||a|| < 1 \). Plots of the density function (3.14) for \( a = (0.35, 0) \) and \( (0.75, 0) \) for bivariate case are given in Figure-3.2 and Figure-3.4, respectively. Its corresponding isodensity sets (contours) are given below in Figure-3.3 and Figure-3.5.

\textbf{Figure 3.2: Density Plot of (3.14)}

\[
p = 2 \text{ and } a = (0.35, 0).
\]
Figure 3.3: Contour Plot of (3.14)

\[ p = 2 \text{ and } \bar{a} = (0.35, 0). \]

Figure 3.4: Density Plot of (3.14)

\[ p = 2 \text{ and } \bar{a} = (0.75, 0). \]
Figure 3.5: Contour Plot of (3.14)

$\mathbf{p} = 2$ and $\mathbf{a} = (0.75, 0)$.

(B) **Generalized p-variate t density:** Let $g(t) = \frac{\Gamma((n+p)/2)}{(n\pi)^{p/2}\Gamma(n/2)} \times (1 + t^2/n)^{-(n+p)/2}$, for all $t > 0$. Now from (3.4), we have

$$f(x; g, \nu) = \frac{\Gamma((n+p)/2)}{(n\pi)^{p/2}\Gamma(n/2)} \times (1 + x'x/n)^{-(n+p)/2}$$

for all $x \in \mathbb{R}^p$, the p-variate t density function. From (3.7) and (3.11), the generalized p-variate t density is

$$f(x; a) = \frac{\Gamma((n+p)/2)}{(n\pi)^{p/2}\Gamma(n/2)} \times \left(1 + \frac{-x'a + [(x'a)^2 + (1 - a'a)x'x]^{1/2}}{n(1 - a'a)^2}\right)^{-(n+p)/2}$$

for all $x \in \mathbb{R}^p$ and $||a|| < 1$.

(C) **Generalized bivariate conical density:** Let $p = 2$, $g(t) = (3/\pi)(1 - t)$, for all $0 \leq t \leq 1$. Now from (3.4), we have

$$f(x; g, \nu) = \frac{3}{\pi} \left(1 - \sqrt{x'x} \right),$$

for all $0 \leq ||x|| \leq 1$, which is the bivariate conical density on the unit circular disc with centre at the origin. From (3.7) and (3.11), the generalized bi-variate
conical density is
\[ f(x; a) = \frac{3}{\pi} \left[ 1 - \{ -x^T a + [(x^T a)^2 + (1 - a^T a)x^T x]^{1/2} \} / (1 - a^T a) \right], \] (3.16)
if $||x - a|| \leq 1$ and $||a|| < 1$.

### 3.3.2 Generating density by using taxicab norm

Consider the taxicab norm, $\nu(x) = \sum_{i=1}^{p} |x_i|$. Here, $C = \{ x : \sum_{i=1}^{p} |x_i| = 1 \}$. For $\sum_{i=1}^{p} |a_i| < 1$, $C_a = \{ x : \sum_{i=1}^{p} |x_i - a_i| = 1 \}$. For any $x \in \mathbb{R}^p$, $\nu_a(x) = k$ if and only if $x \in kC_a$. Further, $x \in kC_a$ if and only if $\sum_{i=1}^{p} |x_i - ka_i| = k$. For any $x \in \mathbb{R}^p$, there exists unique $k \geq 0$ such that
\[ \sum_{i=1}^{p} |x_i - ka_i| = k \] (3.17)

If $a = (a, 0)$ and $||a|| < 1$, then solving (3.17) for $k$ we have,
\[ \nu_a(x) = \begin{cases} \sum_{i=2}^{p} |x_i - x_1| / (1 - a) & \text{if } x_1 \leq a \sum_{i=2}^{p} |x_i| \\ \sum_{i=2}^{p} |x_i + x_1| / (1 + a) & \text{if } x_1 > a \sum_{i=2}^{p} |x_i| \end{cases} \] (3.18)

Hence, from (3.7) and (3.18) we have,
\[ f(x; a) = \begin{cases} g \left( \sum_{i=2}^{p} |x_i - x_1| / (1 - a) \right) & \text{if } x_1 \leq a \sum_{i=2}^{p} |x_i| \\ g \left( \sum_{i=2}^{p} |x_i + x_1| / (1 + a) \right) & \text{if } x_1 > a \sum_{i=2}^{p} |x_i| \end{cases} \] (3.19)

In the following we consider two $g(.)$ functions, which gives us: (A) Generalized p-variate Laplace density and (B) Generalized bivariate pyramid shaped density.
(A) Generalized p-variate Laplace density: Let \( g(t) = 2^{-p} \exp\{-|t|\} \), for all \( t \in \mathbb{R} \). Now from (3.4), we have \( f(\mathbf{x}; g, \nu) = 2^{-p} \exp\{-\sum_{i=1}^{p} |x_i|\} \) for all \( \mathbf{x} \in \mathbb{R}^p \), the p-variate density function of independent Laplace variates. From (3.7) and (3.18) the generalized p-variate Laplace density is

\[
f(\mathbf{x}; a) = \begin{cases} 
2^{-p} \exp\left(-\sum_{i=2}^{p} |x_i - x_1|/(1 - a)\right) & \text{if } x_1 \leq a \sum_{i=2}^{p} |x_i| \\
2^{-p} \exp\left(-\sum_{i=2}^{p} |x_i + x_1|/(1 + a)\right) & \text{if } x_1 > a \sum_{i=2}^{p} |x_i|,
\end{cases}
\]

where \(|a| < 1\).

(B) Generalized bivariate pyramid shaped density: Let \( g(t) = (3/2)(1 - t) \), if \( 0 \leq t \leq 1 \). From (3.4), \( f(\mathbf{x}; g, \nu) = (3/2)(1 - (|x_1| + |x_2|)) \), if \( 0 \leq |x_1| + |x_2| \leq 1 \), the bivariate pyramid shaped density function defined on the support \( \{(x_1, x_2) : 0 \leq |x_1| + |x_2| \leq 1\} \). From (3.7) and (3.18), the generalized bi-variate pyramid shaped density is

\[
f(\mathbf{x}; a) = \begin{cases} 
\frac{3}{2} \left(1 - (|x_2| - x_1)/(1 - a)\right) & \text{if } x_1 \leq a|x_2| \\
\frac{3}{2} \left(1 - (|x_2| + x_1)/(1 + a)\right) & \text{if } x_1 > a|x_2|,
\end{cases}
\]

where \(|a| < 1\).

When \( p = 2 \), plots of the density function (3.20) for \( a = (0.35, 0) \) and \( (0.75, 0) \) are given in Figure- 3.6 and Figure- 3.8, respectively. Its corresponding isodensity sets are given below in Figure-3.7 and Figure-3.9.
Figure 3.6: Density Plot of (3.20)

\[ p = 2 \text{ and } \underline{a} = (0.35, 0). \]

Figure 3.7: Contour Plot of (3.20)

\[ p = 2 \text{ and } \underline{a} = (0.35, 0). \]
Figure 3.8: Density Plot of (3.20)

\[ p = 2 \text{ and } \mathbf{a} = (0.75, 0). \]

Figure 3.9: Contour Plot of (3.20)

\[ p = 2 \text{ and } \mathbf{a} = (0.75, 0). \]
3.4 Sampling from generalized $\nu$-spherical densities

Whenever analytical study of the performance of an estimator or of a test is not possible, performance study can be performed based on extensive simulation from the members of the class of distributions considered. To study the performance of estimator and/or test corresponding to parameter $a$, random sampling from $f(x; a)$ seems inevitable. Model sampling can be done by inverting the marginal and/or conditional distributions, provided the concerned distributions have closed and invertible forms. Whenever such closed invertible forms for the distributions are not available one can use a method called the acceptance-rejection method (Devroye (1986)). In the following, we describe acceptance-rejection method for sampling from $f(x; a)$. This method involves of finding $f^*(x)$, a density function and $K$, a constant such that

(i) $f(x; a) \leq K f^*(x)$, for all $x \in \mathbb{R}^p$ and

(ii) An observation can easily be generated from $f^*(x)$.

By using this method the proportion of observations we reject is $(K-1)/K$, hence it is desirable to use an $f^*(x)$ with least value of $K$ ($\geq 1$). In acceptance-rejection method, steps to generate random observation from $f(x; a)$ are:

**Step-1** Generate $X$ from $f^*(x)$.

**Step-2** Generate $U$ from $U(0, K f^*(x)) = U(0, f(x/(1+||a||); 0))$: uniform distribution.

**Step-3** If $U \leq f(x; a)$ then accept $X$ as a random observation from $f(x; a)$ else reject it and go to Step-1.

Repeat Step-1 to 3 for desired sample size.

As a particular case, we illustrate how to find a suitable density function $f^*(x)$ and
a constant $K$ in order to generate random observation from density $f(x; a)$ given by (3.13). Here, $\nu(x) = ||x||$ and $g(t) = (2\pi)^{-p/2}e^{t^2/2}$, for all $t > 0$.

$$K f^*(\underline{x}) = f\left(\frac{x}{1 + ||a||}; 0\right) = \frac{1}{(2\pi)^{p/2}} e^{x\cdot \frac{1}{2(1 + ||a||)^2} \sum_{i=1}^{p} x_i^2}$$

is an upper bound for $f(x; a)$. A normalizing constant $K^*$ is obtained such that:

$$K^* \int_{\underline{x} \in \mathbb{R}^p} f\left(\frac{\underline{x}}{1 + ||a||}; 0\right) d\underline{x} = 1.$$

That is,

$$K^* \int_{\underline{x} \in \mathbb{R}^p} \frac{1}{(2\pi)^{p/2}} e^{x\cdot \frac{1}{2(1 + ||a||)^2} \sum_{i=1}^{p} x_i^2} d\underline{x} = 1,$$

or

$$K^* \int_{\underline{x} \in \mathbb{R}^p} \frac{1}{(2\pi)^{p/2}} e^{x\cdot \frac{1}{2} (1 + ||a||)^2 I_p}^{-1} d\underline{x} = 1,$$

gives

$$K^* = 1/(1 + ||a||)^p.$$

Hence, $K = 1/K^* = (1 + ||a||)^p$.

Here, the proportion of observations we reject is equal to \{$(1 + ||a||)^p - 1)/(1 + ||a||)^p$.\}

$$f^*(\underline{x}) = K^* f\left(\frac{\underline{x}}{1 + ||a||}; 0\right) = \frac{1}{(1 + ||a||)^p(2\pi)^{p/2}} e^{x\cdot \frac{1}{2} (1 + ||a||)^2 I_p}^{-1} \underline{x}$$

$$= N_p(\underline{x}; 0, (1 + ||a||)^2 I_p): p\text{-variate normal density at } \underline{x} \text{ with mean vector zero and co-variance matrix } (1 + ||a||)^2 I_p \text{ where } ||a|| < 1 \text{ and } I_p \text{ is an identity matrix of order } p.$$

Therefore, steps to generate random observation from a density (3.14) are:

**Step-1** Generate $\underline{X}$ from $N_p(0, (1 + ||a||)^2 I_p): p\text{-variate normal density with mean } 0 \text{ and co-variance matrix } (1 + ||a||)^2 I_p$.

**Step-2** Generate $U$ from $U(0, f(\underline{X}/(1 + ||a||); 0))$ distribution. That is from

$$U\left(0, \frac{1}{(2\pi)^{p/2}} e^{x\cdot \frac{1}{2} (1 + ||a||)^2 I_p}^{-1} \underline{X}\right) \text{ distribution}.$$

**Step-3** If $U \leq f(\underline{X}; a)$ then accept $\underline{X}$ as a random observation from $f(\underline{x}; a)$ given by (3.14) and stop else reject $\underline{X}$ as an observation from $f(\underline{x}; a)$ and go to Step-1.

We have used this algorithm of generating random sample in Section 4.6 to study the performance of proposed estimators and tests for the skewness parameter $a$. 
3.5 Goodness of fit by using generalized $\nu$-spherical densities

(1) Univariate case (epsilon-skew-normal density): Epsilon-skew-normal density considered by Mudholkar and Hutson (2000) is

$$f(x; \epsilon) = \begin{cases} 
(2\pi)^{-1/2} \exp \left\{-x^2/2(1+\epsilon)^2\right\} & \text{if } x < 0 \\
(2\pi)^{-1/2} \exp \left\{-x^2/2(1-\epsilon)^2\right\} & \text{if } x \geq 0, \text{ where } |\epsilon| < 1.
\end{cases} \tag{3.22}$$

In the following we shall show that this density is same as the density (3.7) with $a = -\epsilon$. Let $p = 1$, $\nu(x) = |x|$. Hence, from (3.12)

$$\nu_a(x) = \{ -ax + [(ax)^2 + (1-a^2)x^2]^{1/2}/(1-a^2) \}, \text{ for all } x \in \mathbb{R}$$

$$= \{ -ax + [x^2]^{1/2}/(1-a^2) \} = \{ -ax + |x|/(1-a^2) \}, \text{ for all } x \in \mathbb{R}$$

$$\nu_a(x) = \begin{cases} 
-x/(1-a) & \text{if } x < 0 \\
x/(1+a) & \text{if } x \geq 0, \text{ where } |a| < 1.
\end{cases}$$

In addition to this, $g(t) = (2\pi)^{-1/2} \exp \{-t^2/2\}$, for all $t > 0$. From (3.7), we have

$$f(x; a) = g(\nu_a(x)) = \begin{cases} 
(2\pi)^{-1/2} \exp \{-(x/(1-a))^2/2\} & \text{if } x < 0 \\
(2\pi)^{-1/2} \exp \{-(x/(1+a))^2/2\} & \text{if } x \geq 0, \text{ where } |a| < 1.
\end{cases}$$

Hence, the density in (3.22) is of the form (3.7) with $p = 1$, $g(t) = (2\pi)^{-1/2} \exp \{-t^2/2\}$, for all $t > 0$ and $a = -\epsilon$.

Also, Mudholkar and Hutson (2000) have shown that epsilon-skew-normal density gives better fit than normal density for the data on heights of world’s volcanoes.
(2) **Bivariate case** (generalized bivariate standard normal density): From (3.12), with \( p = 2 \) and \( \mathbf{a} = (a, 0) \), \( |a| < 1 \), we get

\[
f(x; \mathbf{a}) = (2\pi)^{-1} \exp \left( -\frac{1}{2} \left\{ -ax_1 + [(ax_1)^2 + (1 - a^2)(x'_x)]^{1/2} \right\}^2 / (1 - a^2) \right),
\]

(3.23)

for all \( x \in \mathbb{R}^2 \).

The usefulness of the proposed model (3.23) is shown by fitting it to the standardized Fisher’s Iris data set of 50 observations on the versicolor species (Fisher (1936)). It gives better fit than the model with \( \mathbf{a} = 0 \) \( f(x; 0) \) of Fernandez et al. (1995) with \( \nu(x) = ||x|| \). In Fisher’s Iris data, \( X_1, X_2, X_3 \) and \( X_4 \) are sepal length, sepal width, petal length and petal width respectively.

We consider all the six bivariate data sets of standardized variables. For fitting of the models, we consider four quadrants as the cells and \( N_i \) be the number of pairs in the \( i^{th} \) quadrant, for \( i = 1, 2, 3, 4 \). For the model (3.23), the probabilities of the four quadrants are respectively \( \Psi(a)/2, (1 - \Psi(a))/2, (1 - \Psi(a))/2 \) and \( \Psi(a)/2 \), where \( \Psi(a) \) is defined in (3.9). The parameter in the model (3.23) is estimated by using (3.9) (it is not a minimum chi-square estimate). Table -3.1 gives observed value of \( N_i \)’s, estimated values of \( a \), \( \chi^2 \) and p-values.

As expected, chi-square values under \( \nu_\Sigma \)-spherical model are smaller than those under \( \nu \)-spherical model. It is observed that, except for the pair of variables \( (X_3, X_4) \), the p-values of \( \chi^2 \)-statistic with 2 degrees of freedom (for \( \nu_\Sigma \)-spherical model) are larger than those of \( \chi^2 \)-statistic with 3 degrees of freedom (for that of \( \nu \)-spherical model). That is, \( \nu_\Sigma \)-spherical model gives better fit to these five bivariate data sets than that of \( \nu \)-spherical model. For the pair of variables \( (X_3, X_4) \), \( \hat{a} \) is very close to zero, which indicate that \( \nu \)-spherical model itself seems to be an adequate model.
Table 3.1: Observed Values of $N_i$’s, $\hat{a}$, $\chi^2$ and p-Values for bivariate data extracted from Fisher Iris Data

<table>
<thead>
<tr>
<th>Variables</th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$N_3$</th>
<th>$N_4$</th>
<th>$\nu$-Spherical Model</th>
<th>$\nu_{\hat{a}}$-Spherical Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\chi^2$-Stat</td>
<td>p-Val.</td>
</tr>
<tr>
<td>(X1, X2)</td>
<td>13</td>
<td>17</td>
<td>12</td>
<td>8</td>
<td>3.2800</td>
<td>0.3504</td>
</tr>
<tr>
<td>(X1, X3)</td>
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<td>15</td>
<td>14</td>
<td>10</td>
<td>1.3600</td>
<td>0.7149</td>
</tr>
<tr>
<td>(X1, X4)</td>
<td>11</td>
<td>13</td>
<td>16</td>
<td>10</td>
<td>1.6800</td>
<td>0.6414</td>
</tr>
<tr>
<td>(X2, X3)</td>
<td>17</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>2.8000</td>
<td>0.4235</td>
</tr>
<tr>
<td>(X2, X4)</td>
<td>15</td>
<td>9</td>
<td>11</td>
<td>15</td>
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<td>0.5399</td>
</tr>
<tr>
<td>(X3, X4)</td>
<td>13</td>
<td>11</td>
<td>13</td>
<td>13</td>
<td>0.2400</td>
<td>0.9709</td>
</tr>
</tbody>
</table>

3.6 Conclusion

A new class of densities - generalized $\nu$-spherical densities ($\nu_{\hat{a}}$-spherical) is proposed by introducing an additional parameter $\hat{a}$ to the class of $\nu$-spherical densities of Fernandez et al.(1995). As illustrations of generalized of $\nu$-spherical densities, amongst others, we have introduced classes of generalized p-variate normal, t and Laplace densities. A method to generate random sample from $\nu_{\hat{a}}$-spherical density function is described. An adequacy of $\nu_{\hat{a}}$-spherical densities over $\nu$-spherical densities is shown by using real data sets.