CHAPTER 2

ATTEMPTS TO PROVE EUCLID V

2.1 INTRODUCTION

Geometers doubted the validity of the fifth postulate. Comparing with the first four postulates, it is neither brief nor simple or self evident or easy to carry out. The complex statement of Euclid V reveals that it is not an assumption but a theorem to be proved. Even Euclid was aware of this fact and did not content himself with this. Euclid himself tried his best to obtain a proof for the fifth postulate. Since he was not able to get a consistent result he was forced to abandon his efforts. Finally in order to avoid delay in the process and completion of his ‘Elements’ he had to apply this postulate from proposition 29 of The Elements I. After Euclid that is from 300 B.C. to 900 A.D. geometers have applied their great minds to establish the fifth Euclidean postulate from the first four postulates. But unfortunately even the genius mathematicians failed in their attempts. Their investigations and probes devoted to the parallel postulate resulted the birth of non-Euclidean geometries, nearly 25 equivalent propositions to the fifth postulate and a proof that it is not merely difficult but impossible to deduce Euclid V from Euclid I-IV.

After Euclid, the great minds like Aristotle, Diodorus, Proclus, Agh_n_s, Ab_‘Al_ ibn S_n, Pietro Antonio Cataldi, Girolamo Saccheri, Farkas Bolyai, Friedrich Ludwig Watcher, Gauss, Bernhard Friedrich

2.1.1 Ptolemy’s Attempt

In the history of the attempts to prove the parallel postulate, Ptolemy occupies an important place in the second century. Ptolemy assumed Playfair’s axiom which states that through a point not on a line there is exactly one and only line parallel to the given line. Since this axiom is an equivalent to the parallel postulate Ptolemy’s proof was incorrect (Marvin 1994).

2.1.2 Proclus’s Attempt

The recorded history tells us that Proclus lived in the 5th century was the second person who attempted to prove the parallel postulate. In his proof he started with assuming that two lines are everywhere the same distance. This assumption is not stated in the first four postulates and also this assumption is equivalent to the fifth postulate. The condition is to deduce Euclid V from Euclid I-IV. And hence Proclus’s assumption is not acceptable (Marvin 1994).
2.1.3 **Al-Gauhary’s Attempt**

An another proof was offered by Al-Gauhary in the 9th century from the proposition that through any point interior to an angle it is possible to draw a line that intersects both sides of the angle. He showed this by assuming that if the alternating angles determined by a line cutting two other lines are equal, then the same will be true for all lines cutting the given two. This was also not acceptable (Marvin 1994).

2.1.4 **Al-Haytham’s Attempt**

Al-Haytham’s Kinematic method was criticized by Omargayam (11th century) whose own proof was published for the first time 1936. To Omar’s credit he thought up a figure that was latter named after Saccheri. Nasir ad-Din at-Tusi (13th century) was more fortunate a latin edition of his work appeared in Europe in 1657. At-Tusi critically analyzed the works of Al-Gauhary, Al-Haytham and Omargayam. In one of his own attempt, At-Tusi tried to prove the postulate by a proof by contradiction. This appears to be first attempt to prove the postulate by deriving a contradiction from the assumption that the fifth postulate is wrong (Marvin 1994).

2.1.5 **Clavius’s Attempt**

In the 17th century Clavius proposed a proof that all the points equidistant from a given straight line on a given side of it constitute a straight line. Since this is also a replica of parallel postulate. This proof was also not agreeable (Marvin 1994).
2.1.6 John Wallis’s Attempt

To the given triangle, we can construct a similar triangle of any size. John Wallis easily proved this proposition in 17\textsuperscript{th} century. Since he assumed the circular reasoning, his proof became null and void (Marvin 1994).

Figure 2.2 Euclidean triangle

Figure 2.3 Euclidean triangle
2.1.7 Saccheri’s Attempt

In the 17th century, the Italian professor Saccheri created his quadrilaterals to prove the 5th postulate by proof by contradiction.

![Saccheri quadrilateral](image)

**Figure 2.4 Saccheri quadrilateral**

In the Figure 2.4, AB and CD are equal in length perpendiculars to the base BC. AB and CD are called lateral sides. AD is known as summit side. The line joining the mid points of base BC and summit AD is parallel to lateral sides and angles at E and F are right. Also, the summit angles at A and D are equal.

Proof for the summit angles BAD and CDA are equal and angle AFE = Angle DFE = angle DEF = angle CEF.

Join F and B; F and C.

By SSS correspondence (Elements I, proposition 8)

Triangles ABF and DCF are congruent. \[(2.1)\]

So, angles ABF and DCF are equal. \[(2.2)\]
and sides BF and CF are equal \hspace{1cm} (2.3)

and also angles AFB and DFC are equal \hspace{1cm} (2.4)

By SSS correspondence (Element I, proposition 8) and also by SAS correspondence (Elements I, proposition 4) triangles BEF and CEF are congruent.

So, angles BEF and CEF are equal \hspace{1cm} (2.5)

and angles BFE and CFE are equal \hspace{1cm} (2.6)

From Equation (2.4) and Equation (2.6), we get that angles AFE and DFE are equal.

i.e. EF is perpendicular to AD. \hspace{1cm} (2.7)

From Equation (2.5), we have EF is perpendicular to EC \hspace{1cm} (2.8)

In the Saccheri quadrilateral ABCD as shown in Figure 2.4, the summit angles BAD and CDA are equal.

Saccheri’s assumption that the summit angles are obtuse led to a contradiction. Saccheri labored a lot for more than 50 long years to get a contradiction for the acute angle hypothesis. But unfortunately, he was not successful. Unknowingly he has found many new results which were too famous in the latter years after the birth of hyperbolic geometry. Even now, if one is able to show that the acute angle hypothesis yields a contradiction immediately it implies that the summit angles of Saccheri’s quadrilateral are right angles. Consequently, this establishes the parallel postulate.
To put in brief Saccheri’s alternative axiom for the parallel postulate is: In a quadrilateral all the interior angles are right angles.

Saccheri shown that his acute angle hypothesis yielded a contradiction by assuming that there is a point at infinity which lies on a plane. He arrived at a conclusion that “Two distinct lines that meet at an infinite distant point can both be perpendicular at that point to the same straight line”. He thought that this was a contradiction to proposition 12 of Elements I. Which stated that there is a unique perpendicular to a line to each point of the line. After analyzing this the geometers later on concluded that this is only a violation of intuitive ideas rather than a logical contradiction.

2.1.8 Clairaut’s Attempt

In the 18th century, Clairaut proved the parallel postulate by assuming that there are rectangles. The sum of the interior angles of a rectangle is equal to four right angles and a diagonal of every rectangle makes two triangles. Since the opposite sides of a rectangle are equal, these two triangles formed by the diagonal are congruent. From this it is easy to establish Legender’s famous proposition which states that the sum of the interior angles of a triangle is a straight angle. Instead of showing that there exist rectangles, Clairaut assumed the existence of rectangle. He assumed what he was going to prove. Hence Clairaut’s proof is in agreeable (Marvin 1994).

![Clairaut quadrilateral](image)

**Figure 2.5 Clairaut quadrilateral**
2.1.9 Farkas’s Attempt

Farkas in the 18\textsuperscript{th} century pinpointed a proof by assuming that any three points are collinear which is also an alternative to the 5\textsuperscript{th} postulate (Marvin 1994).

2.1.10 Playfair’s Axiom

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{euclidean_triangle.png}
\caption{Euclidean triangle}
\end{figure}

Let ABC be the given triangle as shown in Figure 2.6.

According to Playfair’s axiom, to the base BC of the triangle ABC only one parallel line through vertex A of this triangle can be drawn.

Let DAE be the such a parallel line.

Then according to the parallel theory of Euclid

\begin{align}
\text{angle DAB} &= \text{angle ABC} \quad (2.9) \\
\text{and angle EAC} &= \text{angle ACB} \quad (2.10)
\end{align}

Since DAE is a straight angle,

\begin{align}
\text{angle DAB} + \text{angle BAC} + \text{angle EAC} &= 180^\circ \quad (2.11)
\end{align}
Analysing Equation (2.9), Equation (2.10) and Equation (2.11), we obtain that

the sum of the interior angles of the given triangle is 180 degrees.

\[ \text{angle ABC + angle BCA + angle CAB} = 180^\circ \]  

(2.12)

This establishes the parallel postulate.

Playfair obviously assumed that to the straight line BC through point A, only one parallel line can be drawn. In other words, Playfair applied circular reasoning. If John Playfair was successful in showing Equations (2.9) and Equation (2.10) without assuming Euclid’s 5th postulate, then his proof is acceptable (Marvin 1994).

2.1.11 Legendre’s Attempt

The following statements are regarded as equivalent to Euclid’s 5th postulate.

- In any triangle, the three angles sum to two right angles
- In any triangle each exterior angle equals to the sum of the two remote interior angles
- There exists some triangle whose three angles sum to a straight angle
- There exists an isosceles right triangle whose three angles add to 180°
There exists arbitrarily large isosceles right triangle whose three angles add to 180°

Legender analyzed the above theorems elaborately and established that the following simple statement is an alternative candidate to 5th postulate.

There is at least one triangle (not the given triangle) whose interior angles sum to 180°. This statement can be easily proved by Playfair’s axiom (Marvin 1994).

**Legender’s another proposition**

Legender made attempts to prove the fifth postulate. His false proof linked the parallel line axiom to the whole of classical geometry. Besides interior angle sum axiom, the following axiom is also famous. For any acute angle A and any point D in the interior angle A there exists a line through D and not through A which intersects both sides of angle A (Marvin 1994).

![Figure 2.7 Euclidean angle](image-url)
2.1.12 Lambert’s Attempt

A Lambert quadrilateral is a quadrilateral where three of interior angles are right angles as shown in Figure 2.8. It is a special case of Saccheri’s quadrilateral. Like Saccheri, Lambert succeeded in establishing that the obtuse angle hypothesis of angle ADC is not acceptable. But he could not show that angle ADC is not acute. If this was carried out by Lambert he would have been successful in his efforts to show that angle ADC is right angle. Lambert also worked on this problem day and night for half a century. In brief, Lambert’s alternative for the parallel lines is: In a Lambert quadrilateral the fourth angle is a right angle (Marvin 1994).

Fortunately his prolonged deep research produced the following lemma:

“The angle sum of a triangle increases when area decreases.”

2.1.13 Farkas Axiom

Three non collinear points always lie on a circle (Marvin 1994).
Farkas Bolyai who was the father of Janose Bolyai mentioned above proposed this equivalent proposition to the parallel postulate.

The replica of this axiom is:

1. a triangle can be circumscribed.
2. any three points not lying on a line are co-cyclic.

![Figure 2.9 Points in euclidean space](image)

![Figure 2.10 Euclidean triangle and circle](image)

More advances have been made to Euclidean geometry by Hilbert, Birkhoff and Decardes.
2.1.14 **Pythagorean Theorem**

Pythagorean Theorem is an alternative equivalent to the 5\textsuperscript{th} postulate.

![Euclidean right angled triangle](image)

**Figure 2.11 Euclidean right angled triangle**

Let BC = a, AC = b and AB = c. In a right angled triangle the square on the hypotenuse is equal to the sum of the squares of the other two sides.

i.e. \( b^2 = a^2 + c^2 \)

If we can show the above relation either geometrically or algebraically, then the parallel postulate holds.

2.1.15 **Hilbert’s Parallel Axiom**

There can be drawn through any point A lying outside of a line one and only one line that does not intersect the given line (Marvin 1994).
The Hilbert axiom says that through point A one and only one straight line can be drawn which does not cut the given line $L_1$.

2.1.16 Janos Bolyai Attempt

In the history of attempts to prove the parallel postulate, one cannot ignore the Hungarian research worker Janos Bolyai. Farkas Bolyai, father of Janos Bolyai, was the main cause and source for Janos in the beginning years. But the father lamented in later years: “Janos, give up this: otherwise you will become mad”. But Janos Bolyai paid no attention to his father’s request and advice. He went on working on this problem and spent sleepless nights. He found many theorems which were proved in later years as hyperbolic propositions in hyperbolic geometry. He wrote his findings to Gauss in Germany and Gauss replied that he too had found that idea (Marvin 1994).

2.1.17 Gauss’s Attempt

Gauss is called the prince of mathematics. He was an expert in algebra, geometry, number theory, calculus, statistics and in astronomy. Both Newton in Britain and Leibnitz in Germany had developed
independently their types of calculus and differential equation at the platform of Gauss. He was such a genius that at the age of seven, while Gauss was studying 3rd standard he invented the famous formula in number theory for the sum of first n natural numbers. i.e. \( s = \frac{n(n + 1)}{2} \). At the age of 15, Gauss started his work to prove the parallel postulate. He worked non-stop on this problem for more than 40 years. By 1817, however Gauss felt that the fifth postulate was independent of the other four axioms. Also, he has found an idea for the birth of hyperbolic geometry. But afraid of being ridiculed by fellow mathematicians, Gauss kept silence and never dared to publish his findings. Only after his death all his findings were revealed to the research community (Marvin 1994).

Before Bolyai and Lobachevsky, Gauss came to a conclusion that the parallel postulate was independent of the other four postulates. Gauss replaced the fifth postulate by the following postulate: “Through a given point there could be drawn more than one parallel line”.

2.1.18 Lobachevsky’s Attempt

This Russian mathematician also tried his best to deduce Euclid V from Euclid I to IV for more than 3 decades. Ultimately in 1829, he found that there is more than one parallel line to the given line. By this assumption he was very successful in creating hyperbolic geometry. His result was rejected by the mathematical community even after the publication for more than 30 long years. Only Gauss stood by him. He was dismissed of his post as Rector of the Kazan University for his publication entitled “On geometric fundamentals”. After this, Lobachevsky became blind and died in complete sorrowness. Lobachevsky was internationally recognized only in 1915, after the publication on Einstein’s general
relativity which is Lobacheskian geometric interpretation of space-time. Also, the formulae of hyperbolic geometry are widely used to study the properties of atomic objects in quantum physics.

It is important that Gauss, Bolyai and Lobachesky found their results independently. So, the suitable replacement for hyperbolic geometry is Gauss-Bolyai-Lobachesky model of geometry.

2.1.19 Riemann’s Attempt

Riemann was Gauss’s beloved student. He attempted to show the fifth postulate for several decades. One fine evening, Riemann accidentally noticed the school children playing foot ball. A lightning sparkled in Riemann’s thoughts. He immediately ran to the play ground and grabbed the ball from the students. He then made real experiment which forced Riemann to arrive at the following conclusion: “In sphere great circles move through the poles. And in the quadrature formed by the great circles (spherical triangle) all the three angles are right angles”. Conversely, this statement can be written as, two perpendicular segments to the base of an another segment will intersect at the poles. Thus in Riemann (Elliptic) geometry there are no parallel lines at all. This idea gave rise to differential geometry which was applied by Einstein in 1915 to formulate general theory of relativity. In GTR, there is no relativity at all. It is the theory of gravitation only. The principles of Elliptic geometry are applied in differential equations, astronomy and in navigation.

Although, the four genius mathematicians, mentioned above, were successful in creating non-Euclidean models of geometries, they have not established this geometrical consistency. It was Beltrami who showed mathematically that these non-Euclidean models of geometry are
mathematically consistent. Not only that, the great geometrical battle came to an end and peace was restored when Beltrami put a full stop by mathematically putting that it is impossible to prove Euclid’s 5th postulate as a theorem. The parallel postulate is a special case since it is valid only in Euclidean space.

Figure 2.13 Euclidean

Figure 2.14 Hyperbolic

Figure 2.15 Elliptic

Euclidean: Only one parallel concept
Lobachevskyan: More than one parallel concept

Riemannian: No parallel concept

2.2 CLASSICAL ATTEMPTS

2.2.1 Euclid’s Attempt

Let ABC be a triangle, and let one side of it BC be produced to D. Draw CE parallel to AB (Proposition I.31). Since AB parallel to CE and AC has fallen upon them, the alternate angles BAC and ACE are equal (Proposition I.29). Also by Proposition I.29, since AB parallel to CE and BD has fallen upon them, the exterior angle ECD is equal to the interior and opposite angle ABC. It follows that the exterior angle ACD is equal to the sum of two interior and opposite angles (in triangle ABC) BAC and ABC:

\[
\text{Angle ACD} = \text{angle CAB} + \text{angle ABC}
\]

Add on both sides angle ACB. On the left we get two right angles; on the right the sum of the angles in DABC and hence the proof (Marvin 1994).
2.2.2 The Pythagorean’s Attempt

![Figure 2.17 Euclidean triangle]

The Pythagorean proof is even simpler. A line parallel to the base BC is drawn through the vertex A, which gets us two pairs of alternate angles: CBA, DAB and BCA, EAC. Now, angle BAC complements, on the one hand, angle CBA + angle BCA to the sum of the angles of DABC, and, on the other, angle DAB + angle EAC to two right angles (Marvin 1994).

2.2.3 Thibaut Attempt

Since the assertion to be proved is equivalent to the Parallel postulate, the proof below should be considered as an attempt to prove the later (Marvin 1994).
(Heath) Suppose CB produced to D, and let BD (produced to any necessary extent either way) revolve in one direction (say clockwise) first about B into the position BA, then about A into the position AC produced both ways, and lastly about C into the position CB produced both ways.

The argument then is that the straight line BD has revolved through the sum of the three exterior angles of the triangle. But, since it has at the end of the revolution assumed a position in the same straight line with the original position, it must have revolved through four right angles.

Therefore the sum of the three exterior angles is equal to four right angles from which it follows that the sum of the three angles of the triangle is equal to two right angles.

As we now know, something is wrong in Thibaut's argument. But what?
Even on a sphere, where the parallel postulate does not hold and the sum of angles in a triangle need not be 180°, a complete revolution around a point measures four right angles. However, three successive rotations around the vertices of a triangle do not necessarily cause a line to rotate four right angles! To see that, imagine a spherical equilateral triangle with all three angles right.

The fact is that the successive execution of rotations around the vertices of a triangle results in a rotation combined with a translation. The line BC indeed returns to its original position, but not point wise. It is also shifted along the way. Somewhere in establishing the fact that the resulting rotation equals four right angles the Parallel postulate is bound to crop in.

2.3 CONSTANT HYPOTHESIS FOR THE SUM OF THE ANGLES OF A TRIANGLE

Proof that the sum of the angles in a triangle is 180 degrees http://www.apronus.com/geometry/triangle.htm.

Theorem

If ABC is a triangle then angle ABC + angle BCA + angle CAB = 180 degrees.

Proof

Draw line a through points A and B. Draw line b through point C and parallel to line a.
Figure 2.19 Euclidean triangle

Since lines $a$ and $b$ are parallel, angle $BAC = \angle B'CA$ and angle $ABC = \angle BCA'$.

It is obvious that

\[ \angle B'CA + \angle ACB + \angle BCA' = 180^\circ \]

Thus angle $ABC + \angle BCA + \angle CAB = 180^\circ$.

Lemma

If $ABCD$ is a quadrilateral and angle $CAB = \angle DCA$ then $AB$ and $DC$ are parallel.

Proof

Assume to the contrary that $AB$ and $DC$ are not parallel.

Draw a line trough $A$ and $B$ and draw a line trough $D$ and $C$. 
These lines are not parallel so they cross at one point. Call this point E.

\[ \text{Figure 2.20 Euclidean triangle and quadrilateral} \]

Notice that angle AEC is greater than 0.

Since angle CAB = angle DCA, angle CAE + angle ACE = 180 degrees, angle AEC + angle CAE + angle ACE is greater than 180 degrees.

This is a contradiction. This completes the proof (Marvin 1994, Trudeau 1987).

2.4 THE FAMOUS CLASSICAL MATHEMATICAL IMPOSSIBILITIES

Besides the parallel postulate problem, there are the well known mathematical impossibilities such as:

1. Trisection of the general angle by using ruler and compass only.
2. Squaring a circle.

3. Duplicating a cube.

4. To draw a regular septagon.

5. To find out the solution formula for fifth degree algebraic polynomial equation.

6. And the general formula for prime numbers.

7. There is no way to draw a straight line using only a compass and straight edge.

8. There is no way to draw a cube without poking a pencil through the paper.

9. It is not possible to prove the Pythagorean theorem if asked to do so on a test.

10. A circle has no sides or corners it also has infinitely many sides and corners this is clearly impossible, because one cannot count to zero. (Carrol 1973, Swetz 1996, Trudeau 1987).

Most of the famous mathematicians worked on the above problems. The quest and search motivated for the parallel postulate problem gave birth to two types of non-Euclidean geometries with tremendous physical applications. On the contrary the studies of the above problems did not yield anything at all. At the latter half of the 19th century the application of abstract algebra established once for all that it is impossible to solve the above problems.
Regarding the above problems Dudley writes:

“The sum of two odds is always an even; the square root 2 is always an irrational; one cannot find more than two roots for a polynomial quadratic equation; and so. If anybody offers that the square root of 2 is rational you must simply say, No, it cannot be so. I need not examine your proof. There must be an error”.

- Underwood Dudley