CHAPTER IV
4.1 INTRODUCTION:

The basic equation of continuity and momentum due to closing of penstock valve are approached for solution from classical approach. This requires a general review of different existing solution of the equations. These solutions cover the classical solution, Thoma’s solution, Approximate solution, Graphical solution and modern numerical solution.

4.2 BASIC MATHEMATICAL FORMATION:

The basic equations of continuity and momentum of unsteady flow situation in a conduit due to sudden closure becomes

\[ v = \frac{A_i}{A_i} \frac{dy}{dt} + \frac{Q_i}{A_i} \]  \hspace{1cm} (4.1)

and

\[ \frac{L}{g} \frac{dv}{dt} + y + \frac{fL}{} \frac{|v|^2}{2gD} = 0 \]  \hspace{1cm} (4.2)

If the valve is closed completely, continuity equation becomes

\[ v = \frac{A_i}{A_i} \frac{dy}{dt} \]  \hspace{1cm} (4.3)

Substituting \( v \) from (4.3) in (4.2), the resulting equation becomes
\[
\frac{d^2 y}{dt^2} + \frac{f A_t}{2 g D A_t} \left( \frac{dy}{dt} \right)^2 + \frac{g A_t}{L A_t} y = 0 \quad (4.4)
\]

The resulting equation is nonlinear; therefore, it cannot be solved analytically.

Some classical solutions are available for equation (4.4).

If friction is neglected, it becomes

\[
\frac{d^2 y}{dt^2} + \frac{g A_t}{l A_t} y = 0 \quad (4.5)
\]

Equation (4.5) is now an ordinary second degree differential equation which can be solved analytically. Putting boundary conditions at \( t=0, V=V_0 \) (steady velocity), classical solutions of the equation (4.4) gives,

\[
y = V_0 \sqrt{\frac{L A_t}{g A_t}} \sin \sqrt{\frac{g A_t}{L A_t}} t \quad (4.6)
\]

\[
V = V_0 \cos \sqrt{\frac{g A_t}{L A_t}} t \quad (4.7)
\]

\[
V_s = \frac{A_t}{A_z} V_0 \cos \sqrt{\frac{g A_t}{L A_t}} t \quad (4.8)
\]

This classical solution indicates that surge height fluctuation \( y \) is a sine function whereas unsteady velocity \( V \) in the tunnel and \( V_s \) in the surge tank are cosine function. This classical solution is of no practical importance due to zero friction.
However it gives an idea of the type of actual solution which may be handled numerically with the help of modern computing techniques.

4.3 MAXIMUM VALUES OF PARAMETERS IN CLASSICAL SOLUTION:

In equation (4.6), Y is maximum when \( \sin \sqrt{\frac{gA_t}{LA_t}} \) maximum i.e is.

\[
\sin \sqrt{\frac{gA_t}{LA_t}} = \sin \frac{\pi}{2}
\]

\[
\therefore \sqrt{\frac{gA_t}{LA_t}} = \frac{\pi}{2}
\]

\[
\therefore t = \frac{\pi}{2} \sqrt{\frac{LA_t}{gA_t}}, \text{i.e., } Y \text{ is maximum at this value of } t, \text{ Equation (4.6) may be written as}
\]

\[
t = \frac{\pi}{2} \sqrt{\frac{LA_t}{gA_t}} \text{ as } y_{\text{max}} = V_0 \sqrt{\frac{LA_t}{gA_t}} \quad (4.9)
\]

Similarly, it may be shown that V is maximum at \( t=0 \) i.e., \( V_{\text{max}} = V_0 \) .... (4.10)

and, \( V_{\text{max}} = \frac{A_t}{A} V_0 \) \quad (4.10a)

4.4 THOMA’S SOLUTION:

Making one assumption, Thomas solved the above nonlinear equation for the design of surge tank area only to obtain stability. He assumed that velocity V varies linearly with head H not with square root of head. This assumption of linear variation of velocity head deviates from well known Torricelli’s formula i.e

\[
V = \sqrt{2gH} \quad (4.11)
\]

Therefore, in the design of surge tank area, although Thomas formula is popular, a higher factor of safety is essential which may make the design uneconomic.
Consider any instant after partial valve closure. Take $V_p$ as the velocity in the penstock, $H$ as the net head on turbine and $Y$ as the negative head built up in the surge tank. Then ratio of velocity in penstock and steady velocity in tunnel may be written according Thoma's assumption as:

$$\frac{V_p}{V_0} = \frac{H_s - Y}{H_s}$$

$$\therefore V_p = V_0 \left[ \frac{H_s - Y}{H_s} \right] \quad (4.12)$$

Putting this simplified velocity $V_p$ in continuity equation

$$A_i V = A_i V_s A_1 V_0 H_s - Y \quad (4.13)$$

Squaring,

$$V^2 = \left( \frac{A_2 V_s}{A_t} \right)^2 + V_0^2 + \left( -\frac{V_0 Y}{H_s} \right)^2 + 2 \frac{A_2}{A_t} V_s V_0 - 2 V_0 \left( \frac{V_0 Y}{H_s} \right) - 2 \left( \frac{V_0 Y}{H_s} \right) A_t V_s$$

Thomas assumed $V_s$ and $y/H_s$ to be small. Therefore, $V_s^2$ in first term, $Y^2/H_s^2$ in second term and $V_s (Y/H_s)$ in third term of R.H.S. almost tend to zero. Hence neglecting those term, $V^2$ reduces to:

$$V^2 = \left( V_0^2 + \frac{2 A_2 V_s V_0}{A_t} - \frac{2 V_0^2 Y}{H_s} \right) \quad (4.14)$$

Putting $V$ and $V^2$ from (4.13) and (4.14) in the dynamic equation (4.2)

$$\frac{L}{g} \frac{d}{dt} \left( \frac{A_2 V_s}{A_t} + V_0 \frac{V_0 Y}{H_s} \right) + Y + \frac{fL}{2gD} \left( V_0^2 + \frac{2 A_2 V_s V_0}{A_t} - \frac{2 V_0^2 Y}{A_t} \right) = 0$$

Simplifying further, this may be written as:
\[
\frac{d^2 y}{dy^2} - \beta \frac{dy}{dy} + 0y + \alpha = 0 \tag{4.15}
\]

Where,

\[
\beta = \left( \frac{AV_0}{A_L V_t} - \frac{2CV_v g}{L} \right),
\]

Where, \(h_f = CV_v^2\) i.e. \(C = \left( \frac{fL}{2gD} \right)\)

\[
\theta = \left( \frac{gA_L - 2CV_v^2 A_t g}{A_L H_t} \right), \text{and}
\]

\[
\alpha = \frac{CA_t g V_v^2}{A_L L}
\]

Equation (4.15) is now a linear ordinary differential equation which could be handled with exact mathematics. Solving equation (4.10a) with boundary condition at \(t=0, y=0\), it is obtained that:

\[
Y = C, \left[ e^{(\beta t)/2} \right] \sin \left[ \sqrt{\left( \beta^2 - 4\theta \right)} t \right] \tag{4.17}
\]

Which shows \(Y\) is periodic sine function which is similar to that of classical solution discussed above. For practical purpose, the oscillations of this sine function must damp down with increase of time and this is only possible if \(\beta\) in the hyperbolic function is negative.

i.e., \(\beta \leq 0\)

i.e., \(\frac{AV_0}{A_L H_t} - \frac{2CV_v g}{L} \leq 0\)
Equation (4.18) is the Thoma formula for minimum surge tank area for the damping down of the surges produced due to water hammer pressure.

Pressel presented a numerical step integration method of solution of continuity and momentum equation in which the surface level in a surge tank can be determined for a known penstock flow rate. He used a constant value for turbulent friction factor.

Jaeger has recommended the use of the following approximate formula for calculation of upsurge in cases where friction is taken into account.

\[
Y_{\text{max}} = V_0 \sqrt{\frac{LA_t}{gA_t}} - 2h_i + \frac{1}{9} \frac{k_t^2}{V_0 \sqrt{LA_t}}
\]  

(4.19)

In a similar manner to obtain the value of for lowest water level i.e., for maximum down charge

\[
Y_{\text{mn}} = -V_0 \sqrt{\frac{LA_t}{gA_t}} + 2h_i
\]  

(4.20)

4.5 GRAPHICAL SOLUTION:

In consequence of the non-linearity of the equations, various graphical methods of solution had been developed before the advent of computers. Those are
still available in the literature. Notable works in this line are due to Calame and Gaden, Schoklitchs, Escande, Jaeger and others.

Graphical methods are favoured by some designers in this field. The method advocated by Calame and Gaden is very convenient and is frequently employed to solve surge tank problems.

The Schoklitsch method is based on the finite difference equation of Hudson and Hunter.

For more precise calculation, Escande had recommended his graphical method extending to cover complex cases of loading. He also observed that at the beginning of oscillation, there is a rapid change of \( y \) and much more gradual change in \( v \). On the other hand, when maximum upsurge is approached, \( v \) changes quickly and \( y \) changes gradually. Based on this principle, he also developed a numerical method. Jaeger had discussed in detail the application of graphical methods to different types of surge system. Thus graphical solutions were developed initially with great skill. These have been popular for quick analysis.

4.6 NUMERICAL SOLUTION:

Elsden suggested an empirical rule for first down surge. He also demonstrated a numerical approach in which maximum upsurge is been determined by different analytical and graphical methods. The purpose was to give an idea of relative accuracy of different methods. In his numerical examples, the maximum upsurge calculated by different method varies from 21.81 feet to 25.35 feet. He concluded after this analysis that there is little choice between various methods unless they are verified by physical model studies.
4.7 JAKOBSEN'S METHOD:

Jakobsen used a finite difference method to solve the equations (4.1) and (4.2) for complete sudden closure. He expressed these two equations in finite difference form as:

\[ \frac{\Delta y}{\Delta t} = \frac{A_t}{A_i} \Delta y \] \hspace{1cm} (4.21)

and

\[ \frac{L}{g} \frac{\Delta V}{\Delta t} + \frac{y + \frac{fL}{2gD} V \sqrt{V}}{2} = 0 \] \hspace{1cm} (4.22)

Taking \( \Delta t \) to be very small, linear variation of \( V \) and \( Y \) during his short interval is assumed.

Therefore,

\[ \bar{y} = \frac{y_0 + y_1}{2} = \frac{y_0 + (y_0 + \Delta y)}{2} = y_0 + \frac{1}{2} \Delta y \] \hspace{1cm} (4.23)

Similarly,

\[ \bar{V} = V_0 + \frac{1}{2} \Delta V \] \hspace{1cm} (4.24)

Putting \( Y \) and \( V \) in equation (4.21) and (4.22)

\[ \frac{L}{g} \frac{\Delta V}{\Delta t} + (y_0 + \frac{1}{2} \Delta y) + \frac{fL}{2gD} (V_0 + \frac{1}{2} \Delta V) \left| V_0 + \frac{1}{2} \Delta V \right| = 0 \]

\[ \Rightarrow \frac{L}{g} \frac{\Delta V}{\Delta t} + y_0 + \frac{1}{2} \Delta y + \frac{fL}{2gD} V_0 |V_0| + \frac{fL}{2gD} (2|V_0| \frac{1}{2} \Delta V + \frac{1}{4} \Delta V^2) = 0 \] \hspace{1cm} (4.25)

Since \( \Delta V \) is very small, Jakobsen neglected the last term (1/4) and solution for \( \Delta V \) is:

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\[ \Delta V = \left[ -(y_0 + \frac{1}{2} \Delta y) + \frac{fL}{2gD} V_0 |V_0| + \frac{gL}{2gD} |V_0| \Delta y \right] + \frac{g \Delta t}{L} \] (4.26)

Again from equation (3.1g)

\[ V_0 + \frac{1}{2} \Delta V = \frac{A_f}{A_t} \frac{\Delta y}{\Delta t} \]

\[ \Rightarrow \Delta y = (V_0 \frac{1}{2} \Delta V) \frac{A_f}{A_t} \Delta t \] (4.27)

Putting \( \Delta y \) from (3.22) in (3.21) and simplifying

\[ \Delta V = \frac{y_0 + \frac{fL}{2gD} V_0 |V_0| + \frac{1}{2} V_0 \Delta t^2 \left( \frac{A_f}{A_t} \right) - L}{g + \frac{1}{4} \left( \frac{A_f}{A_t} \right)^2 \Delta t^2 + \frac{fL}{2gD} V_0 |V_0| \Delta t} \] (4.28)

Snow putting \( \Delta V \) from (4.28) in (4.27), \( \Delta y \) is obtained. When \( \Delta V \) and \( \Delta y \) are calculated knowing initial values of \( y_0 \) and \( V_0 \) at \( t=y \) and \( V \) in next time step \( \Delta t \) are calculated from equations

\[ y_n = y_{n-1} + \Delta y \] (4.29)

\[ V_n = V_{n-1} + \Delta V \] (4.30)

Thus the above techniques and equations are involved in lying the finite difference method of Jakobsen. Although Jakobsen neglected a term \((1/4) \Delta V^2\), Pickford advocated that obsen’s method seems to be more accurate than other numerical methods such as Escande’s, Pressel’s and simple arithmetic methods.

AIT, Bangkok presented a numerical solution similar to Jakobsen. In the case presented solution was advance for 23 sec only with an integration step of 1 second. It gave only the first maximum upsurge and first minimum downsurge. The
solution was compared with laboratory data produced in a small surge tank of 4.5 inches diameter and a small pipe of length 28.76 ft with diameter 2 inches.

Chatterjee developed direct step by step finite difference integration of the equations with the help of computer. He presented his solution of surge height upto time of 135 seconds. His $\Delta t$ for integration step was 5 seconds. He did not compare his solution with any model data.

4.8 APPROXIMATE METHOD:

An approximate method or equation is suggested by Pearsall as:

$$Y_{r1} = \left(1 - \frac{1}{2} F_r\right)^2 \quad (4.30)$$

Where, $Y_{r1} = \frac{Y_1}{Y_0}$; $Y_0 = \sqrt{\frac{L}{g}} \sqrt{\frac{A_t}{A_i}} \quad (4.31)$

$$F_r = \frac{gLV^2}{Y_0}$$

Sulton has given an approximate series solution as:

$$y_{r1} = 1 - \frac{2}{3}F_r + \frac{1}{9}F_r^2 + \frac{1}{135}F_r^3 + \frac{1}{270}F_r^4 \quad (4.32)$$

A number of other approximate methods have been developed by Journey, Prasil, Warren and others. Johnson, Rich, and Parmakian derived charts for approximate solution with sudden complete closure.
4.9 CONCLUSIONS:

A number of varied conclusions can be drawn from the above study. The classical solutions have found places in various practical applications. They have not included the effects of friction on the flow and predicted only the type and fluctuations of the flow.

The Thoma's solution is used for surge tank design with the limitation of maximum surge height for complete load failure. A factor of safety of nearly 2 to 4 is required to assume in the design where Thoma's solution is to be utilized.

Pressel's solution has not considered the variation of friction factor with increasing Reynolds's number. Instead it has used a turbulent friction factor, which becomes a major drawback.

Jaeger's solution gives only the approximate values of maximum upsurge and down surge. Graphical methods require interpolation of value positively or negatively if it doesn't lie exactly over the pre drawn curves. Some solution also suggest charts for ready reference.

Jakobsen's numerical method appear to be better than other methods. The inference can be drawn in the line that a numerical solution with modern computers will be the best solution for this situation. Physical model study will help if the data can be inserted to train the model for better solution.
The graphical method employs interpolation technique which is mostly approximate and don’t give exact values. This method was popular before the advent of modern computers. The graphical methods are tedious when the friction factor is to be taken into consideration.

Therefore, the numerical solution with the computers will give the best result. The data from the experimental results may be compared with physical data of the phenomenon. This work includes this approach to compare the result with other available solutions.