CHAPTER II

A NEW CUT FOR INTEGER PROGRAMMING

2.1. Introduction:

An integer linear programming problem is a linear program in which some or all of the variables are restricted to be integer values. A linear programming problem is a mathematical programming problem in which one attempts to optimize a linear objective function subject to a set of linear constraints. In numerous applications, it may be necessary to specify that certain variables can only assume integer values. For example, a decision variable may be used to model the number of vehicles or workers. Clearly, in this case, non-integer solutions would be unacceptable.

Mathematically an integer linear programming may be written as

\[ \text{Maximize } z = cx \]

subject to,

\[ A_{m \times n} x \leq b, \tag{2.1.1} \]

\[ x = (x_1, x_2, \ldots, x_n) \geq 0 \]

and \( x_j \) integers \( j = 1, 2, \ldots, n \).

The linear programming problem derived by omitting all the integer restrictions, on the variables is called the linear programming relaxation. The linear programming relaxation associated with 2.1.1 is simply
Maximize \( z = cx \)

subject to,

\[ Ax \leq b \]  

(2.1.2) 

\[ x \geq 0 \]

The feasible region associated with an integer program is always a subset of the feasible region associated with its linear programming relaxation. Thus, when solving a maximization problem, the optimal objective value of the integer program will always be less than or equal to the optimal objective value of the linear programming relaxation. That is, for a maximization integer program, the linear programming relaxation provides an upper bound for the optimal objective value. The linear programming relaxation is used extensively in constructing solution algorithms for integer programming problems.

There are several algorithms to solve integer programming problems. However, most techniques can be classified as either enumeration techniques or cutting plane methods. Enumeration techniques are designed to exploit the fact that the feasible region of a branched integer program always contains a finite subset of feasible points. Branch and bound enumeration Dakin (1965), Lawler and Wood (1965) and implicit enumeration are the techniques where one attempt to enumerate only a small subset of the feasible integer points, while concluding that the remaining points are inferior to those examined.

Recall that if an optimal solution to a linear programming exists, the simplex algorithm always finds an extreme point optimum. This is the basic
motivation for cutting plane methods. Constraints (or cutting planes) are successively added to the linear programming relaxation of an integer programming problem in such a way that the current non-integer optimal extreme point is cut away or made infeasible. However, this is done so that all the integer points remain feasible. By proceeding in this manner, a new convex set is constructed that eventually has an integer point as an extreme point Gomory (1963). Thus, an optimal integer solution can be found by solving a sequence of linear programs.

We have developed a technique in which a new type of cut is added to the problem after finding the solution to the linear programming relaxation problem. This cut is derived by finding the minimum perpendicular distance from the integer points which are inside the feasible region, to the objective surface passing through the non integer solution. The cut is the hyperplane passing through this point and parallel to the objective function surface. The cut has been designed in such a way that the total number of integer solutions in the resulting feasible region is substantially reduced. After adding this cut the problem is then solved by any enumeration technique.

2.2. Derivation of the Cut:

Consider the pure integer programming problem given by (2.1.1) and the linear programming relaxation associated with this problem given by (2.1.2).

First we solve the linear programming relaxation. Let the solution be $x^*$. If $x^*$ is all integer, then the problem is solved.
Let the $k$th component of $x^*$ be non-integer with value $x^k = a^*_k$.

The nearest integer values to $x^k$ are

$$x^*_k = [a^*_k] \text{ and } x^+_k = [a^*_k] + 1 = \langle a^*_k \rangle, \text{ for } k = 1, 2, \ldots, n$$

where $[t]$ is the largest integer less than or equal to $t$ and $\langle t \rangle$ is the smallest integer greater than or equal to $t$.

With such bifurcations we can find all the $2^n$ integer points in the surrounding of the non integer solution $x^*$. (e.g. in case of two variable problem if $x^* = (2.5, 3.4)$ then there will be $2^2 = 4$ integer points $(2,3)$, $(2,4)$, $(3,3)$ and $(3,4)$ around this $x^*$. Denote the set of indices of these $2^n$ points by $T$.

Let the objective value at $x^*$ be $z^*$. Thus the objective function level plane at $x^*$ will be

$$cx^* = z^* \quad (2.2.1)$$

Now we find the perpendicular distance $d_j$ from the surrounding points, to the objective plane by using the formula Dantzig (1963),

$$d_j = \frac{z^* - cx_i^0}{\sqrt{\sum_{j=1}^{n} c_j^2}}, i \in T \quad (2.2.2)$$

Where $x_i^0$ is an integer point around $x^*$. 
Now we search for the point $x_i^0$, which has a minimum distance from the objective function hyperplane.

Obviously the negative distances and the distances from the infeasible points should be omitted. We choose the minimum positive distance only from the points, which are feasible.

Let $S$ be the set of indices $i \in T$ for, which $x_i^0$ are feasible.

Let $x^0 = \left\{ x_i^0 / d_k = \min_{i \in S} d_i \right\}$

A plane passing through this integer point and parallel to the objective hyperplane will be $cx^0 = z^0$.

Clearly $z^0 < z^*$.

We will introduce the cut

$$cx^0 \geq z^0.$$  \hspace{1cm} (2.2.3)

Now $z^0$ acts as a lower bound for the integer solution to the problem (2.1.1).

The following theorem will show that this constraint when added to the original problem will not eliminate the integer optimal solution to the (2.1.1).

**Theorem**: The added constraint will not eliminate the optimal integer solution to the original integer programming problem.

**Proof**: Let the solution to the (2.1.2) be attained at $x^*$. The cut is derived passing through the integer point $x^0$, which is at a minimum positive
distance $d_1$ from the objective function level surface passing through the non
integer point $x^*$. Suppose that $x^0$ is not an integer solution. Then there exists another feasible
integer point $x^{**}$ such that

$$z^{**} > z^0$$

Where $z^{**} = cx^{**}$ and $z^0 = cx^0$ are the values of the objective function at
$x^{**}$ and $x^0$ respectively.

Let $x^{**}$ be at a distance $d_2$ from $cx^* = z^*$. Now we want to show that if $z^{**} > z^0$ then $d_2 < d_1$.

(i.e. $x^{**}$ will be above the cut $cx^0 > z^0$)

By assumption

$$cx^{**} > cx^0$$

which gives

$$\frac{z^* - cx^{**}}{\sqrt{\sum_{j=1}^{n} c_j^2}} < \frac{z^* - cx^0}{\sqrt{\sum_{j=1}^{n} c_j^2}}$$

i.e. $d_2 < d_1$.

Hence the added cut will not exclude the integer optimal solution to the
original problem.

This shows that if there is any other integer optimal solution then this
will lie above the curve not below (this as depicted in the figure1 for two
variable problem).
2.3. Procedure for Solving the Problem:

The procedure contains following steps:

Step 1: Solve the problem (2.1.2) using simplex or dual simplex method.

Step 2: If this solution is integer, stop. Otherwise, round off the non integer solution to the nearest integers.

Step 3: Find the minimum perpendicular distance from the integer point which is inside the feasible region on the objective function curve passing through the non integer solution. Derive cut passing through this point and parallel to the objective function curve.

Step 4: Use branch and bound or cutting plane method to find the integer optimum.

Following example will illustrate the procedure.
Example:

Maximize $z = 2x_1 + 3x_2$

subject to

\[5x_1 + 2x_2 \leq 15\]
\[3x_1 + 5x_2 \leq 15\]
\[x_1, x_2 \geq 0 \text{ and integer.}\]

After solving this problem as a non integer problem by using simplex method we get the non integer solution as

\[x_1 = 2.37, \ x_2 = 1.58 \text{ and } z = 9.48\]

So we round off the non integer solution to the nearest four integers points as (2,2), (3,2), (3,1) and (2,1). Now calculate the perpendicular distances from these points by using the distance formula.

The distance from the point (2,2) is $\frac{-0.57}{\sqrt{13}}$.

the distance from the point (3,2) is $\frac{-2.57}{\sqrt{13}}$.

the distance from the point (3,1) is $\frac{+0.43}{\sqrt{13}}$.

and the distance from the point (2,1) is $\frac{+2.43}{\sqrt{13}}$.

We discard those points for which distance is negative and check whether the constraints are satisfied for the points for which the distance is positive. If constraints are not satisfied then discard that point. Now we are left with only one point (2,1) which is in the feasible region.

Now we derive a cut passing through the integer point (2,1) as (figure 2).
\[2x_1 + 3x_2 \geq 7\]

and solving the new problem by using branch and bound method we get the optimal integer solution to the problem as (figure 3)

\[x_1 = 0, x_2 = 3 \text{ and } z = 9.\]
\begin{itemize}
\item \(x_1 \leq 2\)
\item \(x_1 \geq 3\)
\item \(x_2 \leq 1\)
\item \(x_2 \geq 2\)
\item \(x_1 \leq 1\)
\item \(x_1 \geq 2\)
\item \(x_2 \leq 2\)
\item \(x_2 \geq 3\)
\end{itemize}

(Figure 3)