Chapter III
3.1 Introduction

Moments of order statistics are extensively used in characterization of specific distributions. Here, in this chapter a general form of distributions considered by Khan and Abu-Salih (1989) have been characterized through conditional expectations, conditioned on two order statistics and several of its important deductions are discussed. To this end we proceed as follows:

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from a continuous population having probability density function (pdf) \( f(x) \) and distribution function (df) \( F(x) \). Let \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \) be the corresponding order statistics.

Khan and Abu-Salih (1989) characterized general form of distributions:

\[
\begin{align*}
\text{i)} & \quad F(x) = 1 - [ah(x) + b]^c \\
\text{ii)} & \quad F(x) = [ah(x) + b]^c \\
\text{iii)} & \quad F(x) = 1 - be^{-ah(x)} \\
\text{iv)} & \quad F(x) = be^{-ah(x)}
\end{align*}
\]

(3.1.1)

through conditional expectation of functions of order statistics fixing adjacent order statistic.
On characterization of distributions...

Khan and Abouammoh (2000) extended the result of Khan and Abu-Salih (1989) where the conditioned order statistic may not be adjacent one. Other related references are Wu and Ouyang (1996), Blaquez and Rebollo (1997) and Franco and Ruiz (1997).

In this chapter the conditional expectation of functions of order statistics

\[ E[h(X_{j:n})|X_{r:n} = x, X_{s:n} = y], 1 \leq r < j < s \leq n \]

conditioned on two order statistics have been considered to characterize the general form of distributions given in (1.1) at \( j = r + 1 \) and \( j = s - 1 \). It may be noted that, because of the Markovian properties of order statistics, it is of no use to condition it on more than two order statistics. Khan et al. (1988) were perhaps the first to characterize logistic distribution fixing two order statistics. Balasubramanian and Beg (1992) and Balasubramanian and Dey (1997) have also obtained some results fixing two order statistics, but our approach is entirely different to theirs.

To establish our results, we have utilized the relationship between conditional and truncated distributions.

The conditional distribution of \( X_{j:n} \) given \( X_{r:n} = x \) and \( X_{s:n} = y, 1 \leq r < j < s \leq n \) is unconditional distribution of \( X_{j-r:s-r-1} \) truncated to the left at \( x \) and to the right at \( y \).

That is

\[ E[h(X_{j:n})|X_{r:n} = x, X_{s:n} = y] = E[h(X_{j-r:s-r-1})|x \leq X_{j-r:s-r-1} \leq y] \]

(3.1.2)
On characterization of distributions...

and

\[ E[h(X_{r+1:n}) | X_{r:n} = x, X_{r+2} = y] = E[h(X) | x \leq X \leq y] \]  \hspace{1cm} (3.1.3)

The doubly truncated pdf of continuous random variable will be denoted as

\[ \frac{f(x)}{P - Q}, \quad Q_1 < x < P_1 \]  \hspace{1cm} (3.1.4)

where

\[ F(Q_1) = \int_{-\infty}^{Q_1} f(x) \, dx = Q \]

\[ F(P_1) = \int_{-\infty}^{P_1} f(x) \, dx = P \]

and the df is

\[ \frac{F(x) - Q}{P - Q} \]  \hspace{1cm} (3.1.5)

Also, we will denote by convention

\[ X_{0:n} = Q_1 \quad \text{and} \quad X_{n:n-1} = P_1 \]  \hspace{1cm} (3.1.6)

3.2 Characterization Theorems

**Theorem 3.2.1:** For any continuous and differentiable function \( h() \) and \( 1 \leq r < s \leq n, \)

\[ E[h(X_{r+1:n}) | X_{r:n} = x, X_{s:n} = y] \]

\[ = (-1)^m p_2^m h(y) \prod_{i=1}^{m} \frac{ic}{(ic + 1)} + Q_2 h(x) \sum_{i=0}^{m-1} (-1)^i p_2^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c + 1} \]

\[ - \frac{b}{a} \sum_{i=0}^{m-1} (-1)^i p_2^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c + 1} \]  \hspace{1cm} (3.2.1)
if and only if

\[ F(x) = 1 - [ah(x) + b]^c, \alpha \leq x \leq \beta \]  \hspace{1cm} (3.2.2)

where \( m = s - r - 1, P_2 = \frac{1 - P}{P - Q} \) and \( Q_2 = \frac{1 - Q}{P - Q} \),

\[ a \neq 0, (m - i)c \neq 0, (m - i)c + 1 \neq 0 \]

**Proof:** First we will prove that (3.2.2) implies (3.2.1). We have (Ali and Khan, 1997),

\[ E\{h(X_{r:n})\} - E\{h(X_{r:n-1})\} = \]

\[ - \left( \frac{n - 1}{r - 1} \right) \int_{Q_1}^{P_1} h'(x) [F(x)]^r [1 - F(x)]^{n-r} \, dx \]  \hspace{1cm} (3.2.3)

For doubly truncated distribution function (3.2.2) at \( Q_1 = x \) and \( P_1 = y \), it can be seen that

\[ F(x) = Q_2 + \frac{ah(x) + b}{ca h'(x)} f(x) \]  \hspace{1cm} (3.2.4)

Therefore, replacing \( F(x) \) in (3.2.3) by expression as given in (3.2.4), and noting (Ali and Khan, 1997) that

\[ E\{h(X_{r:n})\} - E\{h(X_{r-1:n})\} = \]

\[ = \left( \frac{n}{r - 1} \right) \int_{Q_1}^{P_1} h'(x) [F(x)]^{r-1} [1 - F(x)]^{n-r+1} \, dx \]  \hspace{1cm} (3.2.5)

we have,

\[ E\{h(X_{1:n})\} - E\{h(X_{1:m-1})\} = -Q_2 \{E[h(X_{1:m-1})] - h(Q_1)\} \]

\[ - \frac{1}{mc} E[h(X_{1:m})] - \frac{b}{mca} \]

That is,
On characterization of distributions...

\[ E[h(X_{1:m})] = \frac{mc}{(mc + 1)} P_2 E[h(X_{1:m-1})] + \frac{mc}{(mc + 1)} Q_2 h(x) \]
\[ - \frac{b}{a} \frac{1}{(mc + 1)} \] (3.2.6)

But since from (3.1.1)

\[ E[h(X_{r+1:n}) | X_{r:n} = x, X_{s:n} = y] = E[h(X_{1:m}) | x \leq X_{1:m} \leq y], \quad m = s - r - 1 \] (3.2.7)

Therefore, writing (3.2.6) recursively after noting that \( E[h(X_{1:0})] = h(y) \), we establish (3.2.1).

To prove (3.2.1) implies (3.2.2), we have

\[
\frac{m}{[F(y) - F(x)]^m} \int_x^y h(t)[F(y) - F(t)]^{m-1} f(t) dt = \\
(-1)^m \left[ \frac{1 - F(y)}{F(y) - F(x)} \right]^m h(y) \prod_{i=1}^{m} \frac{ic}{(ic + 1)} \\
+ \left[ \frac{1 - F(x)}{F(y) - F(x)} \right] h(x) \sum_{i=0}^{m-1} (-1)^i \left[ \frac{1 - F(y)}{F(y) - F(x)} \right]^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c + 1} \\
- \frac{b}{a} \sum_{i=0}^{m-1} (-1)^i \left[ \frac{1 - F(y)}{F(y) - F(x)} \right]^i \frac{1}{(m-i)c} \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c + 1}
\]

Differentiating both the sides w.r.t. \( x \), we get

\[
-f(x) h(x) \left[ mc[F(y) - F(x)]^{m-1} - c \sum_{i=0}^{m-1} (-1)^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c + 1} 
\right] \\
[1 - F(y)]^i [F(y) - F(x)]^{m-i-1} - [1 - F(x)] c \\
\sum_{i=0}^{m-1} (-1)^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c + 1} [1 - F(y)]^j [F(y) - F(x)]^{m-j-2} \\
+ \frac{b}{a} \sum_{i=0}^{m-1} (-1)^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c + 1} [1 - F(y)]^i [F(y) - F(x)]^{m-i-1}
\]
\[= [1 - F(x)]c h'(x) \]
\[
\left\{ \sum_{i=0}^{m-1} (-1)^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c+1} [1-F(y)]^i [F(y)-F(x)]^{m-i-1} \right\}
\]

(3.2.8)

Now consider,

\[
mc[F(y) - F(x)]^{m-1}
\]
\[
- c \sum_{i=0}^{m-1} (-1)^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c+1} [1-F(y)]^i [F(y)-F(x)]^{m-i-1}
\]
\[
- [1-F(x)]c \sum_{i=0}^{m-1} (-1)^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c+1} (m-i-1)
\]
\[
[1-F(y)]^i [F(y)-F(x)]^{m-i-2}
\]

In the above expression write \([1-F(x)]\) as \([F(y)-F(x)] + [1-F(y)]\), which on solving equates to

\[
\sum_{i=0}^{m-1} (-1)^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c+1} [1-F(y)]^i [F(y)-F(x)]^{m-i-1}
\]

and hence from (3.2.8) we have,

\[
-f(x) \left[ h(x) + \frac{b}{a} \right] = [1-F(x)]c h'(x)
\]
\[
\frac{f(x)}{1-F(x)} = \frac{cah'(x)}{ah(x)+b}
\]

Therefore \(F(x) = 1-[ah(x)+b]^c\)
Examples on Theorem 3.2.1:

Proper choice of \( a, b \), and \( h(x) \) characterize the distributions as given in the table.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( F(x) )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( h(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Power function</td>
<td>( a^{-p} x^p, 0 \leq x \leq a )</td>
<td>( -a^{-p} )</td>
<td>1</td>
<td>1</td>
<td>( x^p )</td>
</tr>
<tr>
<td>2. Pareto</td>
<td>( 1 - a P x^{-p}, a \leq x \leq \infty )</td>
<td>( a^p )</td>
<td>0</td>
<td>1</td>
<td>( x^{-p} )</td>
</tr>
<tr>
<td></td>
<td>( a )</td>
<td>0</td>
<td>( p )</td>
<td>( x^{-1} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( a^{-p} )</td>
<td>0</td>
<td>( -1 )</td>
<td>( x^p )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( a^{-1} )</td>
<td>0</td>
<td>( -p )</td>
<td>( x )</td>
<td></td>
</tr>
<tr>
<td>3. Beta of the first kind</td>
<td>( 1 - (1 - x)^p, 0 \leq x \leq 1 )</td>
<td>( -1 )</td>
<td>1</td>
<td>( p )</td>
<td>( x )</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>( p )</td>
<td>( 1 - x )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( (1 - x)^p )</td>
<td></td>
</tr>
<tr>
<td>4. Weibull</td>
<td>( 1 - e^{-\theta x^p}, 0 \leq x &lt; \infty )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( e^{-\theta x^p} )</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>( \theta )</td>
<td>( e^{-x^p} )</td>
<td></td>
</tr>
<tr>
<td>5. Inverse Weibull</td>
<td>( e^{-\theta x^{-p}}, 0 \leq x &lt; \infty )</td>
<td>( -1 )</td>
<td>1</td>
<td>1</td>
<td>( e^{-\theta x^{-p}} )</td>
</tr>
<tr>
<td>6. Burr type II</td>
<td>( (1 + e^{-x})^{-k}, \infty &lt; x &lt; \infty )</td>
<td>( -1 )</td>
<td>1</td>
<td>1</td>
<td>( (1 + e^{-x})^{-k} )</td>
</tr>
<tr>
<td>7. Burr type III</td>
<td>( (1 + x^{-c})^{-k}, 0 \leq x &lt; \infty )</td>
<td>( -1 )</td>
<td>1</td>
<td>1</td>
<td>( (1 + x^{-c})^{-k} )</td>
</tr>
<tr>
<td>8. Burr type IV</td>
<td>[ 1 + \left( \frac{c-x}{x} \right)^{1/c} \right]^{-k}, 0 \leq k \leq c ]</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>[ 1 + \left( \frac{c-x}{x} \right)^{1/c} \right]^{-k}</td>
</tr>
<tr>
<td>9. Burr type V</td>
<td>[ 1 + ce^{-\tan x} \right]^{-k}, -\pi/2 \leq x \leq \pi/2 ]</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>[ 1 + ce^{-\tan x} \right]^{-k}</td>
</tr>
<tr>
<td>10. Burr type VI</td>
<td>[ 1 + ce^{-k \sinh x} \right]^{-k}, -\infty &lt; x &lt; \infty ]</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>[ 1 + ce^{-k \sinh x} \right]^{-k}</td>
</tr>
<tr>
<td>11. Burr type VII</td>
<td>( 2^{-k} (1 + \tanh x)^k, \infty &lt; x &lt; \infty )</td>
<td>-2^{-k}</td>
<td>1</td>
<td>1</td>
<td>( (1 + \tanh x)^k )</td>
</tr>
<tr>
<td>12. Burr type VIII</td>
<td>[ \frac{2}{\pi} \left( \tan^{-1} e^x \right)^k, -\infty &lt; x &lt; \infty ]</td>
<td>-\left( \frac{2}{\pi} \right)^k</td>
<td>1</td>
<td>1</td>
<td>( (\tan^{-1} e^x)^k )</td>
</tr>
<tr>
<td>13. Burr type IX</td>
<td>[ \frac{c}{2} \left( (1 + e^x)^k - 1 \right) + 2, -\infty &lt; x &lt; \infty ]</td>
<td>\frac{c}{2}</td>
<td>1 - \frac{c}{2}</td>
<td>-1</td>
<td>( (1 + e^x)^k )</td>
</tr>
<tr>
<td>14. Burr type X</td>
<td>( (1 - e^{-x^2})^k, 0 \leq x &lt; \infty )</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>( (1 - e^{-x^2})^k )</td>
</tr>
</tbody>
</table>
### Corollary 3.2.1: For any continuous and differentiable function $h()$ and $1 \leq r < j < s \leq n$;

$$E[h(X_{j:n})|X_{r:n} = x, X_{s:n} = y]$$

$$= E[h(X_{j-r:s-r-1}), x \leq X_{j-r:s-r-1} \leq y]$$

$$= (s - j)\frac{1}{(s - j + l)}E[h(X_{1:s-j+l}), x \leq X_{1:s-j+l} \leq y]$$

**Proof:** In view of equation (1.5.5), we have

$$E[h(X_{j:n})|X_{r:n} = x, X_{s:n} = y]$$

$$= \frac{(s - r - 1)!}{(j - r - 1)!(s - j - 1)!}$$

$$\int_x^y h(t)[F(t) - F(x)]^{j-r-1}[F(y) - F(t)]^{s-j-1} f(t)dt$$

Now writing the term $[F(t) - F(x)]^{j-r-1}$ in the integrand as

$$[(F(y) - F(x)) - (F(y) - F(t))]^{j-r-1}$$

and then expand it binomially, we get
On characterization of distributions...

\[ E[h(x_{j:n}) | X_{r:n} = x, X_{s:n} = y] \]
\[ = \frac{(s-r-1)!}{(j-r-1)!(s-j-1)!} \sum_{l=0}^{j-r-1} (-1)^l \binom{j-r-1}{l} \int_{x}^{y} \frac{h(t) [F(y) - F(t)]^{s-j+l-1}}{[F(y) - F(x)]^{s-j+l}} f(t) dt \]

and hence the result

**Remark 3.2.1:** At \( s = n+1 \), \( X_{n+1:n} = y = \beta, P = 1 \)

Therefore we have \( P_2 = 0, Q_2 = 1, m = n - r \) and

\[ E[h(x_{r+1:n}) | X_{r:n} = x] = 0 + h(x) \frac{mc}{(mc + 1)} - \frac{b}{a} \frac{1}{mc} \frac{mc}{(mc + 1)} \]
\[ = \frac{acmh(x) - b}{a(mc + 1)} \]

as obtained by Khan and Abu-Salih (1989).

**Remark 3.2.2:** At \( r = 0 \), we have \( X_{0:n} = x = \alpha, Q = 0 \) and \( m = s - 1 \)

Therefore, if \( F() \) is replaced by \( 1 - F() \), then

\[ Q_2 = \frac{1 - Q}{P - Q} = \frac{Q}{Q - P} = 0 \]
\[ P_2 = \frac{1 - P}{P - Q} = \frac{P}{Q - P} = -1 \]

Now replacing \( l \) by \( (s - l) \), \( F() \) by \( 1 - F() \), we get

\[ E[h(x_{1:n}) | X_{0:n} = \alpha, X_{s:n} = y] = E[h(x_{1:n}) | X_{s:n} = y] \]
On characterization of distributions...

$$
= h(y) \prod_{i=1}^{s-1} \frac{(s-i)c}{(s-i)c+1} + \frac{b}{a} \sum_{i=0}^{s-2} \frac{1}{(s-(m-i))c} \prod_{j=0}^{i} \frac{c}{c+1}
$$

$$
= h(y) \prod_{i=0}^{s-2} \frac{(s-1-i)}{(s-1-i)c+1} - \frac{b}{a} \sum_{i=0}^{s-2} \frac{1}{(i+1)c} \prod_{j=0}^{i} \frac{(j+1)c}{(j+1)c+1}
$$

if and only if

$$
F(x) = 1 - [1 - (ah(x) + b)^c] = (ah(x) + b)^c
$$

This characterization result was given by Khan and Abouammoh (2000, Theorem 2.2, \( r = 1 \)). Power function distribution \((a = a^{-p}, b = 0, c = 1, h(x) = x^p)\) was characterized by Khan and Ali (1987).

Further, at \( s = 2 \), we have

$$
E[h(X_{1:n}) | X_{2:n} = y] = E[h(X) | X \leq y] = \frac{ach(y) - b}{a(c + 1)}
$$

if and only if \( F(x) = [ah(x) + b]^c \) \[ Khan and Abu-Salih, 1989 \].

**Theorem 3.2.2:** Under the conditions given in Theorem 2.1, and \( 1 \leq r < s \leq n, m = s - r - 1 \)

$$
E[h(X_{s-1:n}) | X_{r:n} = x, X_{s:n} = y] = \frac{(-1)^m Q_3^m h(x) \prod_{i=1}^{m} \frac{ic}{(ic + 1)} + P_3 h(y) \sum_{i=0}^{m-1} (-1)^i Q_3^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c+1}}{-\frac{b}{a} \sum_{i=0}^{m-1} (-1)^i Q_3^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c+1}}
$$

(3.2.9)

if and only if

$$
F(x) = (ah(x) + b)^c, \quad \alpha \leq x \leq \beta
$$

(3.2.10)
where $Q_3 = \frac{Q}{P-Q}$, $P_3 = \frac{P}{P-Q}$

**Proof:** The Theorem can be proved on the lines of Theorem 3.2.1 or else by noting the fact that the conditional distribution of $X_{r+1:n}$ given $X_{r:n} = x$ and $X_{s:n} = y$ from $F(.)$ is the same as the conditional distribution of $X_{n-r:n}$ given $X_{n-s+1:n} = x$ and $X_{n-r+1:n} = y$ from $1-F(.)$. Therefore, replacing $1-P$ by $Q$, $1-Q$ by $P$, $x$ by $y$ and $y$ by $x$ in Theorem 3.2.1, we have

$$E[h(X_{n-r:n}) | X_{n-s+1:n} = x, X_{n-r+1:n} = y] = (-1)^m Q_3^m h(x) \prod_{i=1}^{m} \frac{ic}{(ic+1)} + P_3 h(y) \sum_{i=0}^{m-1} (-1)^i Q_3^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c+1}$$

if and only if

$$F(x) = 1-[1-(ah(x) + b)^c] = [ah(x) + b]^c$$

Now replacing $(n-s+1)$ by $r$ and $(n-r+1)$ by $s$, the Theorem is proved.

**Examples on Theorem 3.2.2:**

Proper choice of $a$, $b$, and $h(x)$ characterize the distributions as given in the table.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$h(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Power function</td>
<td>$a^{-p}$</td>
<td>0</td>
<td>1</td>
<td>$x^p$</td>
</tr>
<tr>
<td></td>
<td>$a^{-1}$</td>
<td>0</td>
<td>$p$</td>
<td>$x$</td>
</tr>
<tr>
<td></td>
<td>$a^p$</td>
<td>0</td>
<td>$-1$</td>
<td>$x^{-p}$</td>
</tr>
<tr>
<td></td>
<td>$a$</td>
<td>0</td>
<td>$-p$</td>
<td>$x^{-1}$</td>
</tr>
<tr>
<td>2. Pareto</td>
<td>$-a^p$</td>
<td>1</td>
<td>1</td>
<td>$x^{-p}$</td>
</tr>
</tbody>
</table>
3. Beta of the first kind  | -1 | 1 | 1 | \((1-x)^p\)
4. Weibull       | -1 | 1 | 1 | \(e^{-\theta x^p}\)
5. Inverse Weibull      | 1 | 0 | 1 | \(e^{-\theta x^{-p}}\)
6. Burr type II      | 1 | 1 | -k | \(e^{-x}\)
7. Burr type III     | 1 | 1 | -k | \(x^{-c}\)
8. Burr type IV      | 1 | 1 | -k | \(\left(\frac{c-x}{x}\right)^{1/c}\)
9. Burr type V       | c | 1 | -k | \(e^{-\tan x}\)
10. Burr type VI     | c | 1 | -k | \(e^{-k \sinh x}\)
11. Burr type VII    | \frac{1}{2} | \frac{1}{2} | k | \tanh x
12. Burr type VIII   | \frac{2}{\pi} | 0 | k | \tan^{-1} e^x
13. Burr type IX     | -1 | 1 | 1 | \([c((1+e^x)^k - 1) + 2]^{-1}\)
14. Burr type X      | -1 | 1 | k | \(e^{-x^2}\)
15. Burr type XI     | 1 | 0 | k | \(\left(x - \frac{1}{2\pi} \sin 2\pi x\right)\)
16. Burr type XII    | -1 | 1 | 1 | \((1+\theta x^p)^{-m}\)
17. Cauchy           | \frac{1}{\pi} | \frac{1}{2} | 1 | \tan^{-1} x

**Corollary 3.2.2:** For any continuous and differentiable function \(h()\) and \(1 \leq r < j < s \leq n\):

\[E[h(X_{j:n}) | X_{r:n} = x, X_{s:n} = y] = E[h(X_{j-r:s-r-1}), x \leq X_{j-r:s-r-1} \leq y]\]

\[= (s-j) \left(\frac{s-r-1}{j-r-1}\right)^{s-j-1} \sum_{l=0}^{s-j-1} (-1)^l \binom{s-j-1}{l} \frac{1}{(j-r+l)} E[h(X_{j-r+l:j-r+l}), x \leq X_{j-r+l:j-r+l} \leq y]\]

**Proof:** This may be proved on the lines on Corollary 3.2.1, after expressing the term \([F(y) - F(t)]^{s-j-1}\) as \([\{F(y) - F(x)] - \{F(t) - F(x)]\}^{s-j-1}\).
Remark 3.2.3: At \( r = 0 \), we have

\[ X_{0:n} = x = \alpha, Q = 0, Q_3 = 0, P_3 = 1 \] and \( m = s - 1 \)

Therefore,

\[
E[h(X_{s-1:n}) \mid X_{s:n} = y] = h(y) \frac{mc}{(mc + 1)} - \frac{b}{a} \frac{1}{mc + 1}
\]

and

\[
E[h(X_{r:n}) \mid X_{r+1:n} = y] = \frac{arch(y) - b}{a(rc + 1)}
\]

as obtained by Khan and Abu-Salih (1989).

Remark 3.2.4: At \( s = n + 1 \), we have \( X_{n+1:n} = y = \beta, P = 1 \) and \( m = n - r \). Now if \( F() \) is replaced by \( 1 - F() \), then

\[
Q_3 = \frac{Q}{P - Q} = \frac{1 - Q}{Q - P} = -1
\]

\[
P_3 = \frac{P}{P - Q} = \frac{1 - P}{Q - P} = 0
\]

Therefore proceeding as in Remark 3.2.2, we get

\[
E[h(X_{n:n}) \mid X_{r:n} = x] = h(x) \prod_{i=0}^{n-r-1} \frac{(n-r-i)c}{(n-r-i)c + 1} - \frac{b^{n-r-1}}{a} \sum_{i=0}^{r-1} \frac{1}{(i + 1)} \prod_{j=0}^{i} (j + 1)c + 1
\]

if and only if \( F(x) = 1 - [ah(x) + b]^c \)

This result was given by Khan and Abouammoh (2000, Theorem 2.1) for \( s = n \).

At \( r = n - 1 \),
\[ E[h(X_{n;n}) \mid X_{n-1;n} = x] = E[h(X) \mid X \geq x] \]
\[ = \frac{ach(x) - b}{a(c + 1)} \]

as given by Khan and Abu-Salih (1989).

**Theorem 3.2.3:** Under the conditions given in Theorem 2.1, and for \(1 \leq r < s \leq n, m = s - r - 1,\)

\[ E[h(X_{r+1;n}) \mid X_{r:n} = x, X_{s:n} = y] = (-1)^m P^m_2 h(y) + Q_2 h(x) \sum_{i=0}^{m-1} (-1)^i P^i_2 + \frac{1}{a} \sum_{i=0}^{m-1} (-1)^i \frac{1}{(m-i)} P^i_2 \]
\[ (3.2.11) \]

if and only if

\[ F(x) = 1 - be^{-ah(x)}, \alpha \leq x \leq \beta \]
\[ (3.2.12) \]

for \(a \neq 0, (m-i) \neq 0, be^{ah(\alpha)} = 1,\)

**Proof:** To prove necessity, note that

\[ F(x) = Q_2 - \frac{f(x)}{ah(x)} \]

Using the relations (3.2.3) and (3.2.5), it can be shown that

\[ E[h(X_{1;m})] = -P_2 E[h(X_{1;m-1})] + Q_2 h(x) + \frac{1}{ma} \]

which on writing recursively and noting the relation given in (3.2.7), one can establish (3.2.11).
To prove sufficiency, proceed on the lines of Theorem 3.2.1 and differentiate both sides w.r.t. \( x \), to get

\[
-mh(x)[F(y) - F(x)]^{m-1} f(x) = 0 + [1 - F(x)]h'(x) \sum_{i=0}^{m-1} (-1)^i [1 - F(y)]^i [F(y) - F(x)]^{m-i-1}
\]

\[
-f(x)h(x) \sum_{i=0}^{m-1} (-1)^i [1 - F(y)]^i [F(y) - F(x)]^{m-i-1}
\]

\[
-[1 - F(x)]h(x) f(x) \sum_{i=0}^{m-1} (-1)^i (m - i - 1)[1 - F(y)]^i
\]

\[
[F(y) - F(x)]^{m-i-2} - \frac{f(x)}{a} \sum_{i=0}^{m-1} (-1)^i [1 - F(y)]^i [F(y) - F(x)]^{m-i-1}
\]

\[
\frac{f(x)}{a} \left\{ -m h(x)[F(y) - F(x)]^{m-1} + ah(x) \sum_{i=0}^{m-1} (-1)^i [1 - F(y)]^i
\right. \\

\left. [F(y) - F(x)]^{m-i-1} + ah(x)[1 - F(x)] \sum_{i=0}^{m-1} (-1)^i (m - i - 1)
\right.

\left. + \sum_{i=0}^{m-1} (-1)^i [1 - F(y)]^i [F(y) - F(x)]^{m-i-1} \right
\]

\[
= h'(x)[1 - F(x)] \left\{ \sum_{i=0}^{m-1} (-1)^i [1 - F(y)]^i [F(y) - F(x)]^{m-i-1} \right
\]

\[
(3.2.13)
\]

In the above equation consider the term

\[
-mah(x)[F(y) - F(x)]^{m-1}
\]

\[
+ ah(x) \sum_{i=0}^{m-1} (-1)^i [1 - F(y)]^i [F(y) - F(x)]^{m-i-1} + ah(x)[1 - F(x)]
\]

\[
\sum_{i=0}^{m-1} (-1)^i (m - i - 1)[1 - F(y)]^i [F(y) - F(x)]^{m-i-2}
\]

and express the \([1 - F(x)]\) as \([\{F(y) - F(x)\} + \{1 - F(y)\}]\), which on solving equates to zero.
Hence from (3.2.13), we have

\[- \frac{f(x)}{1 - F(x)} = -ah'(x)\]

giving

\[F(x) = 1 - be^{-ah(x)}\]

**Examples on Theorem 3.2.3:**

<table>
<thead>
<tr>
<th>Distribution</th>
<th>(a)</th>
<th>(b)</th>
<th>(h(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Pareto</td>
<td>(p)</td>
<td>(p \ln a)</td>
<td>(\ln x)</td>
</tr>
<tr>
<td>2. Beta of the first Kind</td>
<td>(-p)</td>
<td>0</td>
<td>(\ln (1 - x))</td>
</tr>
<tr>
<td>3. Weibull</td>
<td>(\theta)</td>
<td>0</td>
<td>(x^p)</td>
</tr>
<tr>
<td>4. Burr type XII</td>
<td>(m)</td>
<td>0</td>
<td>(\ln (1 + \theta x^p))</td>
</tr>
</tbody>
</table>

**Remark 3.2.5:** At \(s = n + 1\), it reduces to the Theorem 2.2 of Khan and Abu-Salih (1989) with \(b\) replaced by \(e^b\).

**Remark 3.2.6:** At \(r = 0\), proceeding on the lines of Remark 2.2, it can be shown that

\[E[h(X_{1:n}) | X_{s:n} = y] = h(y) + \frac{1}{a} \sum_{j=1}^{s-1} \frac{1}{j}\]

as obtained by Khan and Abouammoh (2000) for \(r = 1\).

For \(s = 2\),

\[E[h(X_{1:n}) | X_{2:n} = y] = E[h(X) | X \leq y]\]

\[= h(y) + \frac{1}{a}\]
if and only if \( F(x) = be^{-ah(x)} \)


**Theorem 3.2.4:** Under the conditions given in Theorem 3.2.1 and for \( 1 \leq r < s \leq n \),

\[
E[h(X_{s-1:n}) | X_{r:n} = x, X_{s:n} = y] = (-1)^m Q_3^m h(x) + P_3 h(y) + \sum_{i=0}^{m-1} (-1)^i Q_3^i + \frac{1}{a} \sum_{i=0}^{m-1} (-1)^i \frac{1}{(m-i)} Q_3^i
\]

if and only if

\[
P_3 h(y) = \gamma \eta^m \mu^{m-1} \n
\]

for \( a \neq 0, m - i \neq 0, b = e^{ah(\beta)}, m = s - r - 1 \)

**Proof:** To prove the Theorem, proceed on the lines of Theorem 3.2.2. Also please note that at \( r = 0 \), it reduces to Lemma 2.2 of Khan and Abu-Salih (1989) with \( b \) replaced by \( e^b \). Further, at \( s = n + 1 \)

\[
E[h(X_{n:n}) | X_{r:n} = x] = h(x) + \frac{1}{a} \sum_{j=r}^{n-1} \frac{1}{(n-j)}
\]


Further, at \( r = n - 1 \),

\[
E[h(X_{n:n}) | X_{n-1:n} = x] = E[h(X) \mid X \geq x]
\]
On characterization of distributions...

\[ h(x) + \frac{1}{a} = \]

if and only if \( F(x) = 1 - be^{-ah(x)} \)

**Examples on Theorem 3.2.4:**

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( a )</th>
<th>( b )</th>
<th>( h(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Power function</td>
<td>(-p)</td>
<td>( \ln a^{\frac{1}{p}} )</td>
<td>( \ln x )</td>
</tr>
<tr>
<td>2. Inverse Weibull</td>
<td>( \theta )</td>
<td>0</td>
<td>( x^{-\theta} )</td>
</tr>
<tr>
<td>3. Burr type II</td>
<td>( k )</td>
<td>0</td>
<td>( \ln (1 + e^{-x}) )</td>
</tr>
<tr>
<td>4. Burr type III</td>
<td>( k )</td>
<td>0</td>
<td>( \ln (1 + x^{-c}) )</td>
</tr>
<tr>
<td>5. Burr type IV</td>
<td>( k )</td>
<td>0</td>
<td>( \ln \left[ 1 + \left( \frac{c-x}{x} \right)^{1/c} \right] )</td>
</tr>
<tr>
<td>6. Burr type V</td>
<td>( k )</td>
<td>0</td>
<td>( \ln (1 + ce^{-\tan x}) )</td>
</tr>
<tr>
<td>7. Burr type VI</td>
<td>( k )</td>
<td>0</td>
<td>( \ln (1 + ce^{-\sinh x}) )</td>
</tr>
<tr>
<td>8. Burr type VII</td>
<td>(-k)</td>
<td>0</td>
<td>( \ln \left( \frac{1 + \tanh x}{2} \right) )</td>
</tr>
<tr>
<td>9. Burr type VIII</td>
<td>(-k)</td>
<td>0</td>
<td>( \ln \left( \frac{2}{\pi} \tan^{-1}e^x \right) )</td>
</tr>
<tr>
<td>10. Burr type X</td>
<td>(-k)</td>
<td>0</td>
<td>( \ln (1 + e^{-x^2}) )</td>
</tr>
<tr>
<td>11. Burr type XI</td>
<td>(-k)</td>
<td>0</td>
<td>( \ln \left( x - \frac{1}{2\pi} \sin 2x \right) )</td>
</tr>
<tr>
<td>12. Cauchy</td>
<td>(-1)</td>
<td>0</td>
<td>( \ln \left( \frac{1 + \frac{1}{\pi} \tan^{-1}x}{2} \right) )</td>
</tr>
</tbody>
</table>