Chapter II
Chapter II

CHARACTERIZATION OF DISTRIBUTIONS THROUGH ORDER STATISTICS

2.1 Introduction

Order statistics have been extensively used in problems on ranges, quasi-ranges, tolerance limits, estimation of parameters, censored samples, selection and ranking problems. Many recurrence relations between moments of order statistics are available in the literature. References may be made to Joshi (1971), Joshi and Balakrishnan (1982), Khan et. al. (1983a, b), Balakrishnan et al. (1988), Kamps (1991), Ali and Khan (1997, 1998) and references therein.

Here in this chapter a general class of distribution function $F(x) = ah(x) + b$ has been characterized through conditional expectation of a function of order statistic, conditioned on two order statistics and then the result is expressed in terms of weighted mean of function of conditioned order statistics. Further some of its important deductions have also been discussed here.

Let $X_1, X_2, ..., X_n$ be a random sample of size $n$ from a continuous population having probability density function $(pdf) f(x)$ and distribution function $(df) F(x)$, over the support
(α, β) and let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be the corresponding order statistics. Balasubramanian and Beg (1992) used the relation

$$E[h(X) \mid x \leq X \leq y] = \frac{h(x) + h(y)}{2} \quad (2.1.1)$$

to characterize some distribution functions where $h(x)$ is a measurable function of $x$.

It may be seen that the conditional distribution of $X_{j:n}$ given $X_{r:n} = x$ and $X_{s:n} = y, 1 \leq r < j < s \leq n$, is unconditional distribution of $X_{j-r:s-r-1}$ truncated to the left at $x$ and to the right at $y$. That is,

$$E[h(X_{j:n}) \mid X_{r:n} = x, X_{s:n} = y] = E[h(X_{j-r:s-r-1}) \mid x \leq X_{j-r:s-r-1} \leq y] \quad (2.1.2)$$

and thus at $j = r + 1$ and $s = r + 2$,

$$E[h(X_{r+1:n}) \mid X_{r:n} = x, X_{r+2:n} = y] = E[h(X_{1:n}) \mid x \leq X_{1:n} \leq y] = E[h(X) \mid x \leq X \leq y] \quad (2.1.3)$$

Further since

$$\sum_{r=1}^{n} E(X_{r:n}) = nE(X) \quad (2.1.4)$$

we have,

$$\frac{1}{s-r-1} \sum_{j=r+1}^{s-1} E[h(X_{j:n}) \mid X_{r:n} = x, X_{s:n} = y] = E[h(X) \mid x \leq X \leq y] \quad (2.1.5)$$
Therefore, result obtained by Balasubramanian and Beg (1992) in terms of (2.1.1) is also true for (2.1.5) for order statistics over summation. However, for the expression

\[ E[h(X_{j:n}) | X_{r:n} = x, X_{s:n} = y] \]

perhaps no such results are available in the literature. We have, therefore, made an attempt to characterize a family of distributions utilizing conditional expectation (2.1.6).

The doubly truncated \textit{pdf} of continuous random variable will be denoted as

\[ \frac{f(x)}{P-Q}, Q \leq x \leq P \]

where \( F(Q) = Q, F(P) = P \)

and the \textit{df} is

\[ \frac{F(x) - Q}{P - Q} \]

Also, we shall use the convention

\[ X_{0:n} = Q, \text{ and } X_{n:n-1} = P \]

\subsection*{2.2 Characterization Theorem}

\textbf{Theorem 2.2.1:} For any continuous and differentiable function \( h() \) and \( m = s - r - 1, 1 \leq r < s \leq n, i = 1, 2, ..., m \)

\[ E[h(X_{r+i:n}) | X_{r:n} = x, X_{s:n} = y] = \frac{(m-i+1)h(x) + ih(y)}{(m+1)} \]
if and only if

\[ F(x) = ah(x) + b, \alpha \leq x \leq \beta \]  \hspace{1cm} (2.2.2)

with \( F(\alpha) = 0 \) and \( F(\beta) = 1 \)

**Proof:** To prove (2.2.2) implies (2.2.1), we have for \( F(x) = ah(x) + b \) (Ali and Khan, 1997),

\[ E[h(X_{i:m})] - E[h(X_{i-1:m})] = \frac{P - Q}{(m + 1)a} \]  \hspace{1cm} (2.2.3)

where \( P = F(y), Q = F(x) \), and \( P - Q = a[h(y) - h(x)] \).

Writing (2.2.3) recursively and noting that

\[ E[h(X_{0:m})] = E[h(Q_1)] = h(x) \]  \hspace{1cm} (2.2.4)

and \( E[h(X_{r+i:n})|X_{r:n} = x, X_{s:n} = y] = E[h(X_{i:m})|x \leq X_{i:m} \leq y] \)

(2.2.5)

the result follows.

To prove (2.2.1) implies (2.2.2), we have (Ali and Khan, 1997),

\[ E[h(X_{i:m})] - E[h(X_{i-1:m})] = \left( \binom{m}{i-1} \right) \int_{x}^{y} h'(t) \left[ \frac{F(t) - F(x)}{F(y) - F(x)} \right]^{i-1} \left[ \frac{F(y) - F(t)}{F(y) - F(x)} \right]^{m-i+1} \, dt \]

\[ = \frac{F(y) - F(x)}{(m + 1)a} \]

or,

\[ \frac{(m + 1)!a}{(i-1)!(m-i+1)!} \int_{x}^{y} h'(t)[F(t) - F(x)]^{i-1}[F(y) - F(t)]^{m-i+1} \, dt \]

\[ = [F(y) - F(x)]^{m+1} \]

Differentiating both sides once w.r.t. \( x \), we get
Differentiating again both sides w.r.t. $x$, we have

$$\frac{(m-1)!}{(i-3)!(m-i+1)!} a \int_x^y h'(t)[F(t) - F(x)]^{i-3} dt = [F(y) - F(t)]^{m-i+1} dt = [F(y) - F(x)]^{m-1}$$

Differentiation at the $(i-1)th$ times gives,

$$\frac{(m-i+2)!}{(m-i+1)!} a \int_x^y h'(t)[F(y) - F(t)]^{m-i+1} dt = [F(y) - F(x)]^{m-i+2}$$

Differentiating again w.r.t. $x$, we have

$$(m-i+2)ah'(x)[F(y) - F(x)]^{m-i+1} = (m-i+2)[F(y) - F(x)]^{m-i+1} f(x)$$

$$\Rightarrow ah'(x) = f(x)$$

i.e. $F(x) = ah(x) + b$

where $b$ satisfies initial conditions of a d.f. $F(x)$.

This proves the theorem.

**Remark 2.2.1:** At $i = 1, s = r + 2, m = s - r - 1 = 1$, 

$$E[h(X_{r+1:n})|X_{r:n} = x, X_{r+2} = y] = E[h(X)|x \leq X \leq y] = \frac{h(x) + h(y)}{2}$$

and also in view of (2.1.5),
\[
\frac{1}{(s-r-1)} \sum_{j=r+1}^{s-1} E[h(X_{j:n}) \mid X_{r:n} = x, X_{s:n} = y] = \frac{h(x) + h(y)}{2}
\]

as obtained by Balasubramanian and Beg (1992).

**Remark 2.2.2:** At \( s = n + 1, X_{n+1:n} = y = \beta, m = n - r \)

\[
E[h(X_{r+1:n}) \mid X_{r:n} = x] = \frac{(n-r-i+1)h(x) + ih(\beta)}{(n-r+1)}
\]

(2.2.6)

as given by Franco and Ruiz (1997).

Further at \( y = \beta \),

\[
F(\beta) = ah(\beta) + b = 1, \quad h(\beta) = \frac{1-b}{a}
\]

Therefore r.h.s. of (2.2.6) is

\[
\frac{(n-r-i+1)h(x)}{(n-r+1)} + \frac{i}{n-r+1} \frac{(1-b)}{a}
\]

(2.2.7)

if and only if

\[
F(x) = -ah(x) + (1-b)
\]

\[
= 1 - [ah(x) + b]
\]

as obtained by Khan and Abouammoh (2000). At \( i = 1 \), the result was given by Khan and Abu-Salih (1989).

**Remark 2.2.3:** At \( r = 0, X_{0:n} = x = \alpha, m = s - 1 \)

\[
E[h(X_{i:n}) \mid X_{s:n} = y] = \frac{(s-i)h(\alpha) + ih(y)}{s}
\]
as given by Franco and Ruiz (1997).

Now since \( F(\alpha) = ah(\alpha) + b = 0 \)

Hence, \( E[h(X_{i:n}) | X_{s:n} = y] = \frac{i}{s} h(y) - \frac{(s-i)b}{a} \)

as obtained by Khan and Abouammoh (2000) at \( c = 1 \).

Further, at \( s = i + 1 \), the result was derived by Khan and Abu-Salih (1989).

**Remark 2.2.4:** Beg and Balasubramanian (1990) have characterized \( df F(x) \) through

\[
E \left[ \frac{1}{s-1} \sum_{i=1}^{s-1} h(X_{i:n}) | X_{s:n} = x \right] = \frac{1}{2} (h(x) + h(a+))
\]

using order statistics. But this could have been obtained rather easily without using order statistics simply by noting that

\[
E \left[ \sum_{i=1}^{s-1} h(X_{i:n}) | X_{s:n} = x \right] = E \left[ \sum_{i=1}^{s-1} h(X_{i:s-1}) | X_{i:s-1} \leq x \right]
\]

\[
= (s-1) E[h(X) | X \leq x]
\]

\[
= (s-1) E[h(X_{1:n}) | X_{2:n} = y]
\]

and therefore the result given by Beg and Balasubramanian (1990) can be obtained by putting \( i = 1 \) and \( s = 2 \) in Remark 2.2.3.
2.3 Examples

Proper choice of $a$, $b$ and $h(x)$ characterize the distributions as given below:

<table>
<thead>
<tr>
<th>Distributions</th>
<th>$F(x)$</th>
<th>$a$</th>
<th>$b$</th>
<th>$h(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Power function</td>
<td>$a^{-p}x^p$, $0 \leq x \leq a$</td>
<td>$a^{-p}$</td>
<td>0</td>
<td>$x^p$</td>
</tr>
<tr>
<td>2. Pareto</td>
<td>$1-a^p x^{-p}$, $a \leq x &lt; \infty$</td>
<td>$-a^p$</td>
<td>1</td>
<td>$x^{-p}$</td>
</tr>
<tr>
<td>3. Weibull</td>
<td>$1-e^{-\theta x^p}$, $0 \leq x &lt; \infty$</td>
<td>$-1$</td>
<td>1</td>
<td>$e^{-\theta x^p}$</td>
</tr>
<tr>
<td>4. Inverse Weibull</td>
<td>$e^{-\theta x^{-p}}$, $0 \leq x &lt; \infty$</td>
<td>1</td>
<td>0</td>
<td>$e^{-\theta x^{-p}}$</td>
</tr>
<tr>
<td>5. Beta of first kind</td>
<td>$1-(1-x)^p$, $0 \leq x \leq 1$</td>
<td>$-1$</td>
<td>1</td>
<td>$(1-x)^p$</td>
</tr>
<tr>
<td>6. Beta of second kind</td>
<td>$1-(1+x)^{-1}$, $0 \leq x \leq 1$</td>
<td>$-1$</td>
<td>1</td>
<td>$(1+x)^{-1}$</td>
</tr>
<tr>
<td>7. Extreme value</td>
<td>$1-\exp(-e^x)$, $-\infty &lt; x &lt; \infty$</td>
<td>$-1$</td>
<td>1</td>
<td>$\exp(-e^x)$</td>
</tr>
<tr>
<td>8. Cauchy</td>
<td>$\frac{1}{\pi} \tan^{-1} \left( \frac{x}{2} \right)$, $-\infty &lt; x &lt; \infty$</td>
<td>$\frac{1}{\pi}$</td>
<td>$\frac{1}{2}$</td>
<td>$\tan^{-1} x$</td>
</tr>
<tr>
<td>9. Gumbel</td>
<td>$\exp(-e^{-x})$, $-\infty &lt; x &lt; \infty$</td>
<td>1</td>
<td>0</td>
<td>$\exp(-e^{-x})$</td>
</tr>
<tr>
<td>10. Burr type II</td>
<td>$(1+e^{-x})^{-k}$, $-\infty &lt; x &lt; \infty$</td>
<td>1</td>
<td>0</td>
<td>$(1+e^{-x})^{-k}$</td>
</tr>
<tr>
<td>11. Burr type III</td>
<td>$(1+x^{-c})^{-k}$, $0 \leq x &lt; \infty$</td>
<td>1</td>
<td>0</td>
<td>$(1+x^{-c})^{-k}$</td>
</tr>
<tr>
<td>12. Burr type IV</td>
<td>$\left[ 1+\left( \frac{c-x}{x} \right)^{1-c} \right]^{-k}$, $0 \leq x \leq c$</td>
<td>1</td>
<td>0</td>
<td>$\left[ 1+\left( \frac{c-x}{x} \right)^{1-c} \right]^{-k}$</td>
</tr>
<tr>
<td>13. Burr type V</td>
<td>$[1+ce^{-\tan x}]^{-k}$, $\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$</td>
<td>1</td>
<td>0</td>
<td>$[1+ce^{-\tan x}]^{-k}$</td>
</tr>
<tr>
<td>14. Burr type VI</td>
<td>$[1+ce^{-k \sinh x}]^{-k}$, $-\infty &lt; x &lt; \infty$</td>
<td>1</td>
<td>0</td>
<td>$[1+ce^{-k \sinh x}]^{-k}$</td>
</tr>
<tr>
<td>15. Burr type VII</td>
<td>( 2^{-k} (1 + \tanh x)^k ), (-\infty &lt; x &lt; \infty)</td>
<td>2(^{-k})</td>
<td>0</td>
<td>((1 + \tanh x)^k)</td>
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<tr>
<td>16. Burr type VIII</td>
<td>( \left( \frac{2}{\pi} \tan^{-1} e^x \right)^k ), (-\infty &lt; x &lt; \infty)</td>
<td>( \left( \frac{2}{\pi} \right)^k )</td>
<td>0</td>
<td>((\tan^{-1} e^x)^k)</td>
</tr>
<tr>
<td>17. Burr type IX</td>
<td>(1 - 2[c(1 + e^x)^k - 1] + 2]^{-1}), (-\infty &lt; x &lt; \infty)</td>
<td>-2</td>
<td>1</td>
<td>(c(1 + e^x)^k - 1 + 2]^{-1})</td>
</tr>
<tr>
<td>18. Burr type X</td>
<td>((1 - e^{-x^2})^k, x \geq 0)</td>
<td>1</td>
<td>0</td>
<td>((1 - e^{-x^2})^k)</td>
</tr>
<tr>
<td>19. Burr type XI</td>
<td>(\left( x - \frac{1}{2\pi} \sin 2\pi x \right)^k), (0 \leq x \leq 1)</td>
<td>1</td>
<td>0</td>
<td>((x - \frac{1}{2\pi} \sin 2\pi x)^k)</td>
</tr>
<tr>
<td>20. Burr type XII</td>
<td>(1 - (1 + \theta x^p)^{-m}), (x \geq 0)</td>
<td>-1</td>
<td>1</td>
<td>((1 + \theta x^p)^{-m})</td>
</tr>
</tbody>
</table>