Chapter V
5.1. Introduction

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from a continuous population having probability density function (pdf) \( f(x) \) and distribution function (df) \( F(x) \), over the support \((\alpha, \beta)\) and let \( X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n} \) be the corresponding order statistics.

Khan and Abu-Salih (1989) have characterized general form of distributions through conditional expectations of function of order statistics and established

\[
E[h(X_{r+1:n})|X_{r:n} = x] = \frac{ac(n-r)h(x) - b}{a[(n-r)c + 1]}
\]

if and only if

\[
F(x) = 1 - [ah(x) + b]^c
\]

and

\[
E[h(X_{r:n})|X_{r+1:n} = x] = \frac{acrh(x) - b}{a(rc + 1)}
\]

if and only if
\[ F(x) = [ah(x) + b]^c \] (5.1.2)

Here it may be noted that expectation of \( h(X) \), a function in \( F(x) \) is considered. Further, most of the distributions considered by Khan and Abu-Salih (1989) are obtained at \( c = 1 \).

Therefore, we have made an attempt to characterize

\[ F(x) = ah(x) + b \]

combining the two general form of distributions, through the conditional moments

\[ E[g(X_{r+1:n}) \mid X_{r:n} = x] \quad \text{and} \quad E[g(X_{r:n}) \mid X_{r+1:n} = x] \]

where \( g(x) = e^{-h(x)} \), different from \( h(x) \).

To this end, we note that (David, 1981)

The conditional distribution of \( X_{s:n} \) given \( X_{r:n} = x, 1 \leq r < s \leq n \) is unconditional distribution of \( X_{s-r:n-r} \) truncated to the left at \( x \).

That is,

\[ E[h(X_{s:n}) \mid X_{r:n} = x] = E[h(X_{s-r:n-r}) \mid X_{s-r:n-r} \geq x] \] (5.1.3)

and \[ E[h(X_{n:n}) \mid X_{n-1:n} = x] = E[h(X) \mid X \geq x] \] (5.1.4)

The conditional distribution of \( X_{r:n} \) given \( X_{s:n} = y, 1 \leq r < s \leq n \) is unconditional distribution of \( X_{r:s-1} \) truncated to the right at \( y \).

That is,
The doubly truncated pdf of continuous random variable is denoted as

\[
\frac{f(x)}{P-Q}, \quad Q_1 < x < P_1
\]

(5.1.7)

where \( F(Q_1) = Q, \) \( F(P_1) = P \)

and the df is

\[
\frac{F(x) - Q}{P - Q}
\]

(5.1.8)

Also, we shall use the convention

\[X_{0:n} = Q_1 \text{ and } X_{n:n-1} = P_1, \quad n = 1, 2, 3, \ldots\]

### 5.2 Characterization Theorem

**Theorem 2.1:** For any continuous and differentiable function

\[g() = e^{-h()} \text{ and } m = n - r, \quad 1 \leq r \leq n.\]

\[
E[g(X_{r+1:n})|X_{r:n} = x] = \sum_{i=1}^{m} (-1)^{i+1} \frac{m!}{(m-i)!} \frac{a^i}{[1-F(x)]^i} + g(\beta)(-1)^m \frac{a^m}{[1-F(x)]^m}
\]

(5.2.1)

if and only if

\[F(x) = ah(x) + b, \quad \alpha \leq x \leq \beta\]

(5.2.2)
with $F(\alpha) = 0, \ F(\beta) = 1$ and $g(\beta) = \exp\left((b-1)/a\right)$

**Proof:** To prove (5.2.2) implies (5.2.1), we have for

$$F(x) = ah(x) + b \text{ and } g(x) = e^{-h(x)} \text{ (Ali and Khan, 1997)},$$

$$E[g(X_r:m)] = \frac{am}{(P - Q)} \left[ E[g(X_{r-1}:m-1)] - E[g(X_{r:m-1})] \right] \quad (5.2.3)$$

since $Z(x) = \frac{g'(x)}{h'(x)} = -h(x)$ at $c = 1$

Writing (5.2.3) recursively after noting that

$$E[g(X_{r+1}:m)|X_r:n = x] = E[g(X_{1:m})|X_{1:m} \geq x] \quad (5.2.4)$$

$P = 1, Q = F(x), E[g(X_0:m-1)] = E[g(Q_1)] = g(x)$

and $E[g(X_{1:0})] = E[g(\beta)] = g(\beta)$

the relation (5.2.1) is established.

To prove (5.2.1) implies (5.2.2), we have

$$\frac{m}{[1 - F(x)]^m} \int_x^\beta g(t)[1 - F(t)]^{m-1} f(t) dt$$

$$= g(x) \sum_{i=1}^{m} (-1)^{i+1} \frac{m! a^i}{(m - i)! [1 - F(x)]^i} + g(\beta)(-1)^m m! \frac{a^m}{[1 - F(x)]^m}$$

or,

$$m \int_x^\beta g(t)[1 - F(t)]^{m-1} f(t) dt$$

$$= g(x) \sum_{i=1}^{m} (-1)^{i+1} \frac{m! a^i [1 - F(x)]^{m-i}}{(m - i)!} + g(\beta)(-1)^m m! a^m$$

Differentiating both the sides w.r.t $x$, we have
On characterization of probability distributions...

\[ -mg(x)[1 - F(x)]^{m-1} f(x) \]

\[ = -g(x)\sum_{i=1}^{m} (-1)^{i+1} \frac{m!}{(m-i)!} a^i (m-i)[1 - F(x)]^{m-i-1} f(x) \]

\[ + g'(x)\sum_{i=1}^{m} (-1)^{i+1} \frac{m!}{(m-i)!} a^i [1 - F(x)]^{m-i} \]

Dividing both sides by \(-g(x)\) and after noting that \(g'(x) = -h'(x)g(x)\), we get

\[ m[1 - F(x)]^{m-1} f(x) \]

\[ = \sum_{i=1}^{m} (-1)^{i+1} \frac{m!}{(m-i)!} a^i (m-i)[1 - F(x)]^{m-i-1} f(x) \]

\[ + h'(x)\sum_{i=1}^{m} (-1)^{i+1} \frac{m!}{(m-i)!} a^i [1 - F(x)]^{m-i} \]

\[ \Rightarrow ah'(x)\sum_{i=1}^{m} (-1)^{i+1} \frac{m!}{(m-i)!} a^{i-1}[1 - F(x)]^{m-i} \]

\[ = f(x)\sum_{i=1}^{m} (-1)^{i+1} \frac{m!}{(m-i)!} a^{i-1}[1 - F(x)]^{m-i} \]

That is,

\[ f(x) = ah'(x) \]

or,

\[ F(x) = ah(x) + b \]

Hence the theorem.

**Theorem 5.2.2:** For any continuous and differentiable function

\[ g() = e^{-h()} \] and \(1 \leq r \leq n\).
On characterization of probability distributions...

\[ E[g(X_{r:n})|X_{r+1:n} = y] = g(\alpha) \frac{r!a^r}{[F(y)]^r} - g(y) \sum_{i=1}^{r} \frac{r!}{(r-i)!} \frac{a^i}{[F(y)]^i} \]

(5.2.5)

if and only if

\[ F(x) = ah(x) + b, \ \alpha \leq x \leq \beta \]  

(5.2.6)

with \( F(\alpha) = 0, \ F(\beta) = 1 \) and \( g(\alpha) = e^{b/a}. \)

**Proof:** The theorem can be proved on the lines of Theorem 5.2.1 after noting that

\[ E[g(X_{r:n})|X_{r+1:n} = y] = E[g(X_{r:r})|X_{r:r} \leq y] \]

\[ P = F(y), \ \Omega = 0, \ E[g(X_{r:r-1})] = E[g(P_1)] = g(y) \]

and \( E[g(X_{0:0})] = E[g(\alpha)] = g(\alpha) \)

**Remark 5.2.1:** From Theorem 5.2.1 and equation (5.1.4), we have at \( r = n - 1 \)

\[ E[g(X)|X \geq x] = \frac{a[g(x) - g(\beta)]}{[1 - F(x)]} \]

if and only if

\[ F(x) = ah(x) + b, \ \text{where} \ g(x) = e^{-h(x)}. \]

**Remark 5.2.2:** From Theorem 5.2.2 and equation (5.1.6), we deduce at \( r = 1 \)

\[ E[g(Y)|Y \leq y] = \frac{a[g(\alpha) - g(y)]}{F(y)} \]
if and only if

\[ F(x) = ah(x) + b, \text{ where } g(y) = e^{-h(y)}. \]

For the examples of above distribution one may refer to Chapter II.