CHAPTER 5

EXISTENCE OF SOLUTIONS OF ABSTRACT FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS

5.1 INTRODUCTION


5.2 A CAUCHY PROBLEM FOR ABSTRACT FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY

In this chapter, the study has been extended to find the existence of the mild solution of the following Cauchy problem for abstract fractional integrodifferential equation with infinite delay
\[ D^q x(t) = f(t, x_t) + \int_0^t h(t, s, x_s) ds, \quad t \in [0, T] \& x_0 = \phi \tag{5.1} \]

where \(0 < q \leq 1, 0 < T < \infty, \phi \in \mathcal{P}, \mathcal{X} \) is a Banach space, \( \mathcal{P} \) is a phase space and \( f \in C([0, T] \times \mathcal{P}) \) and \( h \in C(\Delta \times \mathcal{P}), \) where \( \Delta = \{(t, s); 0 \leq s \leq t \leq \eta\} \) and \( \eta = [0, T]. \)

5.3 PRELIMINARIES

In this chapter, \((\mathcal{X}, ||\cdot||)\) will be a real Banach space and \(x_t(\cdot) := x(t + \cdot)\) is valid in \(X.\) Let \((\mathcal{P}, ||\cdot||_\mathcal{P})\) be a Banach space consisting of functions from \((-\infty, 0]\) into \(X\) satisfying the following assumptions:

(H1) For any \(t_0 \in \mathbb{R}\) and \(\alpha > 0,\)

if \(\chi: (-\infty, t_0 + \alpha] \to \mathcal{X}\) is continuous on \([t_0, t_0 + \alpha]\) and \(x_0 \in \mathcal{P},\)

then \(x_t \in \mathcal{P}\) and \(x_t\) is continuous in \([t_0, t_0 + \alpha].\)

(H2) There exist nonnegative, measurable, and locally bounded functions \(K(t)\) and \(M(t)\) and \(t \geq 0\) such that

\[ ||x_t||_\mathcal{P} \leq K(t - t_0) \sup_{s \in [t_0, t]} ||x(s)|| + M(t - t_0)||x_{t_0}||_\mathcal{P} \]

for \(t \in [t_0, t_0 + \alpha]\) and \(x\) as in (H1). Phase space is a classical concept in the study of functional differential equations with infinite delay.

Definition 5.3.1

A mild solution of the Equation (5.1) is a function \((-\infty, T] \to \mathcal{X}\) and continuous in \([0, T]\) and satisfies

\[ x(t) = \phi(0) + \frac{1}{\Gamma(q)} \left[ \int_0^t (t - s)^{q-1} f(s, x_s) ds + \int_0^t (t - s)^{q-1} \right. \]

\[ \times \int_0^s h(s, \mu, x_\mu) d\mu ds \bigg], t \in [0, T], \]
\[ & x(t) = \phi(t), \ t \in (-\infty, 0]. \]

(5.2)

### 5.4 MAIN RESULTS

We further assume that

(H3) There is a positive constant \( L \) and \( N \) such that

\[
\| f(t, u_1) - f(t, u_2) \| \leq L(\| u_1 - u_2 \|),
\]

\[
\| h(t, s, u_1) - h(t, s, u_2) \| \leq N(\| u_1 - u_2 \|),
\]

for all \( t \in [0, T], u_1, u_2 \in \mathcal{P}. \)

(H4) There are positive functions \( \mu(t), \phi(t) \in L^2(0, T) \) such that

\[
\| f(t, u_1) - f(t, u_2) \| \leq \mu(t)(\| u_1 - u_2 \|),
\]

\[
\| h(t, s, u_1) - h(t, s, u_2) \| \leq \phi(t)(\| u_1 - u_2 \|),
\]

for all \( t \in [0, T], u_1, u_2 \in \mathcal{P}. \)

(H5) For every \( r > 0 \) there exist constants \( L(r) \) and \( N(r) \) such that

\[
\| f(t, u_1) - f(t, u_2) \| \leq L(r)(\| u_1 - u_2 \|),
\]

\[
\| h(t, s, u_1) - h(t, s, u_2) \| \leq N(r)(\| u_1 - u_2 \|),
\]

for any \( \| u_1 \|, \| u_2 \| \leq r. \) Under condition (H3), a basic theorem has been obtained.

**Theorem 5.4.1**
Let \( f \in C([0, T] \times \mathcal{P}) \) and satisfies (H3). Then Equation (5.1) has a unique mild solution.

**Proof**

It is denoted that

\[
\mathcal{P}^{[0, T]} := \{ x: (-\infty, T] \to X; x \mid_{0, T] \in C([0, T], X) \text{ and } x_0 \in \mathcal{P} \}.
\]

Then, \( \mathcal{P}^{[0, T]} \) is a Banach space under the norm

\[
\|x\|_{\mathcal{P}^{[0, T]}} := \sup_{t \in [0, T]} \|x(t)\| + \|x_0\|_{\mathcal{P}}
\]

For each \( \phi \in \mathcal{P}^{[0, T]} \), let \( \mathcal{P}_\phi^{[0, T]} := \{ x \in \mathcal{P}^{[0, T]} ; x_0 = \phi \} \). Clearly, \( \mathcal{P}_\phi^{[0, T]} \) is a closed convex subset of \( \mathcal{P}^{[0, T]} \). It is defined that the nonlinear operator

\[
(Fx)(t) = \phi(0) + \frac{1}{\Gamma(q)} \left\{ \int_0^t (t - s)^{q-1} f(s, x_s)ds + \int_0^t (t - s)^{q-1} \int_0^s h(s, \mu, \mu_s) d\mu d\mu_s \right\},
\]

\( t \in [0, T] \),

\[
\& (Fx)(t) = \phi(t), \quad t \in (-\infty, 0],
\]

(5.3)

It is easy to see that \( F \) maps \( \mathcal{P}_\phi^{[0, T]} \) into itself.

Moreover, for \( t \in [0, T], x, y \in \mathcal{P}_\phi^{[0, T]} \), it is obtained that

\[
\| (Fx)(t) - (Fy)(t) \|
\]
\[
\begin{align*}
&= \left\| \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} [f(s, x_s) - f(s, y_s)] ds \\
&\quad + \int_0^t (t - s)^{q-1} \left[ \int_0^s \{h(s, \mu, x_s) - h(s, \mu, y_s)\} d\mu \right] ds \right\| \\
&\leq \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \left[ L \| x_s - y_s \|_F + \eta N \left\| [x_\mu - y_\mu] \right\|_p \right] ds \\
&\leq \frac{t^q}{\Gamma(q + 1)} (L( \sup_{t \in [0, T]} K(t)) + \eta N( \sup_{s \in [0, t]} K(t)) \sup_{s \in [0, t]} \|x(s) - y(s)\| \\
&\leq \frac{t^q}{\Gamma(q + 1)} (L + \eta N)( \sup_{t \in [0, T]} K(t)) \sup_{s \in [0, t]} \|x(s) - y(s)\|
\end{align*}
\]

by (H2).

Furthermore, by induction, it is obtained that for \( k = 1, 2, \ldots \)

\[
\|(F^k x)(t) - (F^k y)(t)\| \leq \frac{t^{kq}}{\Gamma(q + 1)} (L^k + \eta^k N^k)( \sup_{t \in [0, T]} K(t)^k) \sup_{s \in [0, t]} \|x(s) - y(s)\|
\]

Since

\[
\lim_{n \to \infty} \frac{t^{kq}}{\Gamma(q + 1)} (L^k + \eta^k N^k)( \sup_{t \in [0, T]} K(t)^k) = 0,
\]

It can be chosen that an integer \( k \) is large enough such that \( F^k \) is a contraction \( \mathcal{F}_p^{[0, T]} \). Therefore, by a well known extension of the contraction mapping theorem, \( F \) has a unique fixed point \( x \in \mathcal{F}_p^{[0, T]} \) which is the mild solution of the Equation (5.1).

**Remark 5.4.2**
This theorem extends the result in Anichini (1997) even in the case that \( X = \mathbb{R} \).

If \( \frac{1}{2} < q \leq 1 \), a more general theorem has been obtained.

**Theorem 5.4.3**

Let \( f \in C([0, T] \times \mathcal{P}) \) and satisfies (H4). If \( \frac{1}{2} < q \leq 1 \), then the Equation (5.1) has a unique mild solution.

**Proof**

Let \( F \) be operator defined above and \( l = \|\mu\|_{L^q(0, T)} \) and \( m = \|\phi\|_{L^q(0, T)} \). Then for any \( t \in [0, T] \), \( x, y \in \mathcal{P}_{\phi}^{[0, T]} \), it is obtained that

\[
\|(F \phi)(t) - (F \gamma)(t)\|
\]

\[
= \left\| \frac{1}{F(q)} \left\{ \int_0^t (t - s)^{q-1} [f(s, x_s) - f(s, y_s)] ds 
+ \int_0^t (t - s)^{q-1} \left\{ \int_0^s \{h(s, \mu, x_s) - h(s, \mu, y_s)\} d\mu ds \right\} \right\}
\]

\[
\leq \frac{1}{F(q)} \left\{ \int_0^t (t - s)^{q-1} [\mu(s)\|x_s - y_s\|_{\mathcal{P}}] ds 
+ \int_0^t (t - s)^{q-1} [\phi(s)\|x_{\mu} - y_{\mu}\|_{\mathcal{P}}] ds \right\}
\]

\[
\leq \frac{1}{F(q)} \left( \sup_{t \in [0, T]} K(t) \sup_{s \in [0, t]} \|x(s) - y(s)\| \left\{ \int_0^t (t - s)^{q-1} \mu(s) ds \right\}
+ \sup_{t \in [0, T]} \eta K(t) \sup_{s \in [0, t]} \|x(s) - y(s)\| \left\{ \int_0^t (t - s)^{q-1} \phi(s) ds \right\} \right)
\]
\[ \leq \frac{1}{\Gamma(q)} \left( \int_{t}^{t+1} \sup_{t \in [0,T]} K(t) \sup_{s \in [0,t]} \|x(s) - y(s)\| \right) + \eta m \sup_{t \in [0,T]} \left( \int_{0}^{t} (t - s)^{2q-1} ds \right)^{\frac{1}{2}} \]

\[ = \frac{\Gamma(2q - 1)^{\frac{1}{2}} t^{-\frac{(2q-1)}{2}}}{\Gamma(q) \Gamma(2q)^{\frac{1}{2}}} \left( \int_{0}^{t} (t - s)^{2q-1} ds \right)^{\frac{1}{2}} \sup_{t \in [0,T]} \|x(s) - y(s)\|. \]

By induction, for \( k = 1, 2, \ldots \), it is obtained that

\[ \|(F^k x)(t) - (F^k y)(t)\| \]

\[ \leq \frac{\Gamma(2q - 1)^{\frac{k}{2}} t^{-\frac{k(2q-1)}{2}}}{\Gamma(q) \Gamma(k(2q) - 1) + 1} \left( \int_{0}^{t} (t - s)^{2q-1} ds \right)^{\frac{k}{2}} \sup_{t \in [0,T]} \|x(s) - y(s)\|. \]

It shows that \( F \) has a unique fixed point in \( \mathcal{F}_{\phi}^{[0,T]} \) which is the mild solution of Equation (5.1).

**Theorem 5.4.4**

Let \( f \in C([0,T] \times \mathcal{P}) \) and \( f \) satisfies (H5). Then, there exists a real number \( T' \in (0, T) \) such that Equation (5.1) has a unique mild solution on the interval \( (0, T') \).

**Proof**

Take a real number \( a > 0 \), Let

\[ \mathcal{F}_{\phi, a}^{[0,T]} := \{ x \in \mathcal{F}_{\phi}^{[0,T]}, \sup_{t \in [0,T]} \|x(t)\| \leq \|\phi(0)\| + a \} \]

Set

\[ b = \sup_{t \in [0,T]} \{ K(t), M(t), \|\phi(0)\|, \|\phi\|_{\mathcal{P}} \}, r = \max\{a + b, b(a + 2b)\} \]
It is clear that, for $x \in \mathcal{F}_{\phi, \alpha}^{[0, T]}$,

$$\max_{t \in [0, T]} \{\|x(t)\|, \|x_t\|\} \leq r.$$ 

For $t \in [0, T]$ and $x \in \mathcal{F}_{\phi, \alpha}^{[0, T]}$, it is obtained that

$$\|F x(t) - \phi(0)\|$$

$$= \left\| \phi(0) + \frac{1}{F(q)} \int_0^t (t - s)^{q-1} [f(s, x_s) + \int_0^s h(s, \mu, x_\mu) d\mu] ds - \phi(0) \right\|$$

$$= \left\| \frac{1}{F(q)} \int_0^t (t - s)^{q-1} [f(s, x_s) + \int_0^s h(s, \mu, x_\mu) d\mu] ds \right\|$$

$$\leq \frac{t^q}{F(q + 1)} \left\| f(s, x_s) - f(s, 0) + f(s, 0) \right\|$$

$$+ \eta \left\| h(s, \mu, x_\mu) - h(s, \mu, 0) + h(s, \mu, 0) \right\|$$

$$\leq \frac{t^q}{F(q + 1)} \left[ r L(r) + \|f(t, 0)\| + \eta r N(r) + \|h(t, \mu, 0)\| \right]$$

Moreover, for $x, y \in \mathcal{F}_{\phi, \alpha}^{[0, T]}$, and $t \in [0, T]$,

$$\|F x(t) - F y(t)\| \leq \frac{t^q}{F(q + 1)} \left[ L(r) + \eta N(r) \right] b \sup_{s \in [0, t]} \|x(s) - y(s)\|$$

Therefore, for any given $0 < \epsilon < 1$, there is a real number $T' \in (0, T)$ such that for $x, y \in \mathcal{F}_{\phi, \alpha}^{[0, T]}$,
$$\max_{t \in [0, \tau]} \|Fx(t) - \phi(0)\| < a,$$

$$\|Fx - Fy\|_{\mathcal{P}[0, \tau']} \leq \varepsilon \sup_{s \in [0, t]} \|x(s) - y(s)\|_{\mathcal{P}[0, \tau']}.$$  

By a contraction mapping theorem, $F$ has a unique fixed point in $\mathcal{P}_{\phi, a}$, which is the mild solution of Equation (5.1) on $(-\infty, \tau').$

**Theorem 5.4.5**

Let $f \in C([0, T] \times \mathcal{P})$ and $f$ satisfies (H5). A mild solution of Equation (5.1) is unique on any interval $(-\infty, \tau] \subset (-\infty, T]$, if it exists.

**Proof**

Let $x, y$ be mild solutions of Equation (5.1) on $(-\infty, \tau]$. Set

$$t_0 = \max \{t \in [0, T]; u|_{[0, t]} \equiv v|_{[0, t]} \}.$$

If $t_0 < \tau$, Let the equation be

$$D^q x(t) = f(t, x_t) + \int_{t_0}^{t} h(t, s, x_s) ds, \quad t \in [t_0, \tau], \quad x_{t_0} = u_{t_0}. \quad (5.4)$$

Similarly as the proof of Theorem 5.4.4, it can be found that $\gamma \in (\tau - t_0)$ such that Equation (5.4) has a unique solution on $[t_0, t_0 + \gamma]$ which contradicts the definition of $t_0$. Thus, it is obtained that $T_0 = \tau$ and it completes the proof.

**Definition 5.4.6**

A function $x$ is a maximum mild solution of Equation (5.1) on $(-\infty, \tau]$, moreover, for any $y$ is a mild solution of Equation (5.1) on $(-\infty, \tau]$, then $\tau \leq t$. 
Next, it is extended to establish the mild solution to the maximum one.

**Theorem 5.4.7**

Let \( f \in C([0, T] \times P) \) and \( f \) satisfies (H5). If \( x \) is the mild solution of Equation (5.1) on \((\mathcal{C}, T_0, T)\), then any one of the following conditions holds.

(C1) \( T_0 = T \).

(C2) \( \limsup_{t \to T_0^-} \|x(t)\| = \infty \).

**Proof**

Let \( T_0 < T \) and \( 0 < c < T - T_0 \). If \( \limsup_{t \to T_0^-} \|x(t)\| < \infty \), then there exists a constant \( b_1 \) such that

\[
\max \left\{ \sup_{t \in [0, T_0 + c]} K(t), M(t), \sup_{t \in [0, T_0]} \|x(t)\|, \|\phi\|_{\mathcal{P}} \right\} \leq b_1.
\]

Set

\[
P_{\phi, a_1}^{[T_0, T_0 + c]} = \left\{ y \in P_{\phi}^{[0, T_0 + c]}, \ y|_{(-\infty, T_0]} = x|_{(-\infty, T_0]}, \ \max_{t \in [T_0, T_0 + c]} \|y(t)\| \leq \|x(T_0)\| + a_1 \right\}.
\]

Choose \( r_1 = \max \{a_1 + b_1, b_1 (a_1 + 2b_1)\} \). Then for any \( y \in P_{\phi, a_1}^{[T_0, T_0 + c]} \),

\[
\max_{t \in [0, T_0]} \{\|y(t)\|, \|y_t\|_{\mathcal{P}}\} \leq r_1.
\]
Let the nonlinear operator be

\[(Fy)(t) = x(T_0) + \frac{1}{F'(q)} \left\{ \int_{T_0}^{T_0 + c} (t - s)^{q-1} f(s, x_s) ds \right\} + \int_{T_0}^{T_0 + c} (t - s)^{q-1} \left[ \int_{0}^{s} h(s, \mu, x_{\mu}) d\mu \right] ds, t \in [T_0, T_0 + c], \]

\& (Fy)(t) = x(t), t \in (-\infty, T_0]. \tag{5.5} \]

Same reason as in Theorem 5.4.4, there exists \( c' \in [0, c] \) such that \( F \) has a unique fixed point in \( \mathcal{P}_{\phi, a_1}^{[T_0, T_0 + c]} \). This fixed point will extend the mild solution \( x \) to \((-\infty, T_0 + c'] \) which contradicts the definition of \( T_0 \).