CHAPTER-0

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Here we state a few conventions regarding notations and definitions, which will be used throughout; others will be introduced as they become necessary.

0.1. The symbols \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \)

\( \mathbb{N} := \) The set of natural numbers. \\
\( \mathbb{R} := \) The set of real numbers. \\
\( \mathbb{C} := \) The set of complex numbers.

0.2. Limit, supremum and infimum

\( \lim_{k \to \infty} := \) means \( \lim_{k \to \infty} \). \\
\( \sup_{k=1,2,...} := \) means \( \sup_{k=1,2,...} \), unless otherwise stated. \\
\( \inf_{k=1,2,...} := \) means \( \inf_{k=1,2,...} \), unless otherwise stated.

0.3. Summation convention

\( \sum_{k} := \) means summation over \( k = 1 \) to \( k = \infty \), unless otherwise stated.

0.4. Sequences

\( x = (x_k) \), the sequence whose kth term is \( x_k \). \\
\( \theta = (0,0,0,...) \), the zero sequence. \\
\( e_k = (0,0,0,...,0,1,0,0,...) \), the sequence whose k-th component is 1 and others are zeros, for all \( k \in \mathbb{N} \).
\( e = (1, 1, 1, \ldots). \)

\( p = (p_k), \) the sequence of strictly positive real.

**0.5. Difference sequences**

For any sequence \( x, \) the difference sequences \( \Delta x \) and \( \Delta^2 x \) are defined by:

\[
\Delta x = \Delta^1 x_n = x_n - x_{n+1} (\Delta^0 x = x),
\]

\[
\Delta^2 x = \Delta^2 x_n = \Delta x_n - \Delta x_{n+1}.
\]

**0.6. Sequence spaces**

\( \omega := \{ x : x_k \in \mathbb{R} (\text{or } \mathbb{C}) \}, \) the space of all sequences, real or complex.

\( \phi := \) The space of finite sequences, i.e. of all sequences terminating in zeros.

\( \ell := \{ x \in \omega : \sum_k |x_k| < \infty \}, \) (see \([44]\))

\( \ell_\infty := \{ x \in \omega : \sup_k |x_k| < \infty \}, \) the space of bounded sequences, (see\([44]\), p. 29).

\( c_0 := \{ x \in \omega : \lim_k x_k = 0 \}, \) the space of null sequences, (see \([44]\), p. 29).

\( c := \{ x \in \omega : \lim_k x_k = l, \text{ for some } l \in \mathbb{C} \}, \) the space of convergent sequences, (see \([44]\), p. 29).

\( \ell_\infty, c_0, c \) are Banach spaces with their usual norm:

\[
\| x \| = \sup_k |x_k|, \text{ (see Maddox \([44]\), p. 104).}
\]
We observe that
\[ \ell_1 \subset \gamma \subset c_0 \subset c = C \subset \ell_\infty, \]
al\nall inclusions being strict, (see [44], p. 15);
\[ \text{ces}_p := \{x \in \omega : \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^p < \infty, \ 1 < p < \infty, \text{ (see Shiue [68])} \} \]
\[ \text{ces}_\infty := \{x \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty, \ (p < \infty) \text{ (see Shiue [68])} \} \]
The following subspaces of \( \omega \) were first introduced and discussed by Simons [69], and Iyer [19]
\[ \ell(p) := \{x \in \omega : \sum_k |x_k|^p < \infty\}, \ [69]. \]
\[ \ell_\infty(p) := \{x \in \omega : \sup_k |x_k|^p < \infty\}, \ [69]. \]
\[ d := \{ x \in \omega : \sup_k |x_k|^{1/k} < \infty \}, \quad [19] \]

(which is the special case of \( \ell_\infty(p) \), where \( p_k = 1/k \))

\[ c(p) := \{ x \in \omega : \lim_k |x_k - l|^{p_k} = 0, \quad \text{for some } l \in \mathbb{C} \}, \quad [42] \]

\[ c_0(p) := \{ x \in \omega : \lim_k |x_k|^{p_k} = 0 \}, \quad [42] \]

\[ M_\omega(p) := \bigcap_{N>2} \{ x \in \omega : a_k N^{1/p_k < \infty} \}, \quad \text{(see Lascarides and Maddox [34])} \]

All the spaces defined above are complete in their topologies. In general \( \ell_\omega(p) \), \( c_0(p) \), \( c(p) \) and \( \ell(p) \) are not normed spaces (see [13]). If \( p_k = p \) for all \( k \), then \( \ell_\omega(p) = \ell_\infty \), \( c_0(p) = c_0 \), \( c(p) = c \), and \( \ell(p) = \ell_p \).

\( Q \) is the set of \( p \) for which a number \( N=N(p)>1 \) exists such that

\[ \sum N^{-1/p_k} < \infty, \quad \text{(see Maddox [42]).} \]

Let \( q = (q_k) \) be any bounded sequence. Then the space \( \Gamma(q) \) is defined by :

\[ \Gamma(q) := \{ x \in \omega : \lim_k |x_k|^{q_k} \to 0, \quad \text{as } k \to \infty \}, \quad \text{the space of entire functions. (see Nanda, Srivastava and Nayak [59]).} \]

\( \Gamma(q) \) is a linear metric space under the metric topology generated by the paranorm

\[ g(f) = \sup_k |k! x_k|^{q_k/M}, \]

where \( M = \max (1, \sup q_k) \).

### 0.7. Paranorm and total paranorm

A paranorm on a linear topological space \( X \) is a function...
g : X → IR which satisfies the following axioms:

For any x, y, x₀ ∈ X and λ, λ₀ ∈ C,

(i) g(0) = 0,
(ii) g(x) = g(-x),
(iii) g(x + y) ≤ g(x) + g(y) (subadditivity), and
(iv) the scalar multiplication is continuous, that is,

\[ \lambda \mapsto \lambda_0, x \mapsto x_0 \text{ imply } \lambda x \mapsto \lambda_0 x_0; \]
in other words,

\[ |\lambda - \lambda_0| \to 0, g(x - x_0) \to 0 \text{ imply } g(\lambda x - \lambda_0 x_0) \to 0. \]

A paranormed space is a linear space X with a paranorm g and it is written as (X, g), (see Maddox [44], p. 92).

Any function g which satisfies all the conditions (i)-(iv) together with the condition

(v) g(x) = 0 if and only if x = 0,

is called a total paranorm on X, and the pair (X, g) is called a total paranormed space, (see Maddox [44], p. 92).

If X is a linear space and g is any given function having the properties (i)-(v), then it follows that d* defined by d*(x, y) = g(x - y) is such that (X, d*) is a linear metric space, (see [44]).

Let (X, g) be a paranormed space. A sequence \( (b_k) \) of elements of X is called a basis for X if and only if, for each \( x \in X \), there exists
a unique sequence \( (\lambda_k) \) of scalars such that

\[
x = \sum_{k=1}^{\infty} \lambda_k b_k,
\]

that is,

\[
g( x - \sum_{k=1}^{n} \lambda_k b_k ) \to 0, \text{ as } n \to \infty.
\]

This idea of a basis was introduced by J. Schauder [66] and is often called a Schauder basis (see Maddox [44], p. 98).

**0.08. Continuous and Köthe-Toeplitz duals**

If \( X \) is a space of sequences \( x \in \omega \), then we denote the continuous dual of \( X \) by \( X^* \), that is, the set of all continuous linear functionals on \( X \), (see [44], p. 113).

We denote the absolute Köthe-Toeplitz dual (or \( \alpha \)-dual) and generalized Köthe-Toeplitz dual (or \( \beta \)-dual) by \( X^\alpha \) and \( X^\beta \) respectively, defined by :

\[
X^\alpha := \{ a \in \omega : \sum_{k} |a_k x_k| < \infty, \text{ for all } x \in X \}
\]

and

\[
X^\beta := \{ a \in \omega : \sum_{k} a_k x_k \text{ converges, for all } x \in X \}, \text{ (see [43] and [48]).}
\]

**0.09. Class of matrices**

Let \( X \) and \( Y \) be two non-empty subsets of the space \( \omega \).

Let \( A = (a_{nk}) \), \( (n, k = 1,2,3,\ldots) \) be an infinite matrix with elements of real or complex numbers. We write

\[
A_n(x) = \sum_{k} a_{nk} x_k.
\]
Then $Ax = (A_n(x))$ is called the A-transform of $x$.

$$\lim_{n} Ax = \lim_{n \to \infty} A_n(x),$$

whenever it exists.

If $x \in X$ implies $Ax \in Y$, we say that $A$ defines a (matrix) transformation from $X$ into $Y$, denoted by $A : X \to Y$. By $(X,Y)$ we mean the class of matrices $A$ such that $A : X \to Y$. By $(X,Y;P)$ we mean the subset of $(X,Y)$ for which limits or sums are preserved. We also write $(X,Y)_{\text{reg}}$ for $(X,Y;P)$.

If $A$ is regular we write $A \in (X,Y)_{\text{reg}}$.

0.10. **Some inequalities**

The following inequalities will be useful.

0.10.1. The triangle inequality, (see [44] and [18]).

For any $a, b \in \mathbb{C}$, $|a + b| \leq |a| + |b|$.

0.10.2. Holder's inequality, (see [44], p. 20; see also [18]).

Let $p > 1$, $1/p + 1/q = 1$, $a_1, ..., a_n \geq 0$ and $b_1, ..., b_n \geq 0$. Then

$$\sum_{k=1}^{n} a_k b_k \leq \left( \sum_{k=1}^{n} a_k^p \right)^{1/p} \left( \sum_{k=1}^{n} b_k^q \right)^{1/q}.$$

Also,

$$\sum_{k=1}^{n} a_k b_k \leq \left( \sum_{k=1}^{n} a_k \right) \max b_k$$

0.10.3. Minkowski's inequality, (see [44]; see also [18]).

Let $p \geq 1$, $a_1, ..., a_n \geq 0$ and $b_1, ..., b_n \geq 0$. Then
\[
\left( \sum_{k=1}^{n} (a_k + b_k)^p \right)^{1/p} \leq \left( \sum_{k=1}^{n} a_k^p \right)^{1/p} + \left( \sum_{k=1}^{n} b_k^p \right)^{1/p},
\]
where sums run from \( k = 1 \) to \( k = n \).

0.10.4. Let \( 0 < p \leq 1 \), \( a_1, \ldots, a_n \geq 0 \) and \( b_1, \ldots, b_n \geq 0 \). Then
\[
\sum_{k=1}^{n} (a_k + b_k)^p \leq \sum_{k=1}^{n} a_k^p + \sum_{k=1}^{n} b_k^p, \text{ (see [44] and [18])}
\]

Inequalities 0.10.1, 0.10.3 and 0.10.4 yield the following frequently used results valid for complex \( a_k, b_k \): (see Maddox [44], pp. 20-22 and Hardy, Littlewood and Polya [18]).

\[
\left( \sum_{k=1}^{n} |a_k + b_k|^p \right)^{1/p} \leq \left( \sum_{k=1}^{n} |a_k|^p \right)^{1/p} + \left( \sum_{k=1}^{n} |b_k|^p \right)^{1/p} (p \geq 1),
\]

\[
\sum_{k=1}^{n} |a_k + b_k|^p \leq \sum_{k=1}^{n} |a_k|^p + \sum_{k=1}^{n} |b_k|^p (0 < p \leq 1).
\]

0.10.5. For any \( E > 0 \) and any two complex numbers \( a, b \),
(see Maddox [43] and Maddox and Willey [47]).

(i) \( |ab| \leq E (|a|^q E^{-q} + |b|^p) \), \( (p > 1, 1/p + 1/q = 1) \)

(ii) \( |a|^p - |b|^p \leq |a+b|^p \leq |a|^p + |b|^p (0 < p \leq 1) \).

(iii) \( |ab| \leq |a|^p + |b|^q \) \( (1 < p < \infty, \ p^{-1} + q^{-1} = 1) \), and

(iv) \( |\lambda|^p \leq \max (1, |\lambda|) (0 < p \leq 1) \),

where \( \lambda \) is a scalar, and for \( p = (p_k) \), a strictly positive real sequence, such that \( H = \sup p_k < \infty \),

\[
|\lambda|^p_k \leq \max (1, |\lambda|^H).
\]