A CLASS OF MATRIX TRANSFORMATIONS

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Abstract

The main purpose of the present note is to characterize matrices of classes \( M_{\infty}(\rho, \lambda) \), where \( \lambda \) denotes one of the sequence spaces \( ms, cs \) and \( (c_0)s \). Also we investigate some matrix transformations in case \( \rho \in \mathbb{Q} \).

1. Introduction

Let \( X \) and \( Y \) be two non-empty subsets of the space \( S \) of complex sequences. Let \( A = (a_{n,k}) \) \( (n, k = 1, 2, \ldots) \) be an infinite matrix of complex numbers. We write \( A \colon X \to Y \) if \( A_n(x) = \sum_k a_{n,k}x_k \) converges for each \( n \). (Throughout \( \sum \) denotes summation over \( k \) from 1 to \( \infty \).) If \( x = (x_k) \in X \Rightarrow Ax = (A_n(x)) \in Y \), we say that \( A \) defines a matrix transformation from \( X \) into \( Y \) and we denote it by \( A : X \to Y \). By \((X, Y)\) we mean the class of matrices \( A \) such that \( A : X \to Y \). If \( \rho \) is real and \( \rho > 0 \), we define (see Maddox [3] and Simons [6]),

\[
I_{\infty}(\rho) = \left\{ x : \sup_k |x_k|^{\rho_k} < \infty \right\}
\]

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Let $E$ be a subset of $S$. Then $E^*$ denotes the generalized K"othe-Toeplitz dual of $E$, i.e.
\[ E^* = \left\{ a : \sum a_k x_k \text{ converges for every } x \in E \right\}. \]

If $p_k > 0$, then $l_\infty^*(p) = M_\infty(p)$ (See Theorem 2 [2]), where
\[ M_\infty(p) = \bigcap_{N=2}^{\infty} \left\{ a : \sum \left| a_k \right| N^{-1/p_k} < \infty \right\}. \]

And $M_\infty(p) = l_1$ if and only if $\inf p_k > 0$.

Now we define (See Steiglitz and Tietz [7]),
\[ ms = \left\{ x : \left( \sum_{i=1}^{n} x_i \right) \in l_\infty \right\}. \]
\[ cs = \left\{ x : \left( \sum_{i=1}^{n} x_i \right) \in c \right\}. \]
\[ (c_0)s = \left\{ x : \left( \sum_{i=1}^{n} x_i \right) \in c_0 \right\}. \]

$Q$ is the set of $p$ for which a number $N = N(p) > 1$ exists such that $\sum N^{-1/p_k} < \infty$.
(See Maddox [4]).

2. Some Results

In this paper we shall be using the idea and some notations given by Nanda in [5]. For all integers $n \geq 1$, we write
\[ l_n(Ax) = \sum_{i=1}^{n} A_i(x) = \sum b_{nk} x_k. \]
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where

\[ b_{nk} = \sum_{i=1}^{n} a_{ik}. \]

Now we prove:

**Theorem 1.** Let \( p_k > 0 \) for every \( k \). Then \( A \in \left( M_\infty(p), ms \right) \) if and only if there exists an integer \( B > 1 \) such that

\[ D = \sup_{n, k} \left| b_{nk} \right| B^{-1/p_k} < \infty. \] (i)

**Proof.** Suppose that the condition holds and \( x \in M_\infty(p) \). Then \( \sum_k x_k |B^{1/p_k} < \infty \) for \( B > 1 \). Since \( D < \infty \), we have \( \left| b_{nk} \right| B^{-1/p_k} \leq D \) for every \( n, k \) and therefore \( \left| b_{nk} \right| \leq MB^{1/p_k} \) where \( M = \max (1, D) \). We have

\[
\left| t_n(Ax) \right| = \left| \sum_k b_{nk} x_k \right|
\leq \sum_k \left| b_{nk} \right| \left| x_k \right|
\leq \sum_k MB^{1/p_k} \left| x_k \right|
= M \sum_k \left| x_k \right| B^{1/p_k} < \infty.
\]

Hence \( A \in \left( M_\infty(p), ms \right) \).

Conversely, suppose that \( A \in \left( M_\infty(p), ms \right) \) and \( \sup_{n, k} \left| b_{nk} \right| B^{-1/p_k} = \infty \) for every integer \( B > 1 \). Then there exist increasing sequences \( \{k(B)\}, \{n(B)\} \) such that

\[ \left| b_{nk} \right| B^{-1/p_k} > B^2 \left( k = k(B), n = n(B), B = 2, 3, \ldots \right) \]
Define $x = (x_k)$ by $x_k = 0$ for $k \neq k(B)$, $x_k = \text{sgn}(b_{n(B), k}) b_{n(B)}^{(2 + 1/p_k)}$ for $k = k(B), (B = 2, 3, \ldots)$. Then for any integer $T > 1$ and every $B \geq T$, we have

$$
\sum_k |x_k| B^{-1/p_k} \leq \sum B^{-2} < \infty
$$

Finally, we have for each $B \geq 2$,

$$
\left| t_{n(B)} A(x) \right| = \left| \sum_k b_{n(B), k} x_k \right| > B
$$

Thus the subsequence $\left( t_{n(B)} A(x) \right)$ of $\left( t_{n(Ax)} \right)$ is unbounded which contradicts the fact that $A \in \langle M_{\infty}(p), ms \rangle$. This completes the proof of the theorem.

**Corollary 2.** Let $p_k > 0$ for every $k$. Then $A \in \langle M_{\infty}(p), c_0 \rangle$ if and only if

(i) The condition (i) of Theorem 1 holds.

(ii) $\lim_{n \to \infty} b_{nk} = \alpha_k$ for each fixed $k$.

**Corollary 3.** Let $p_k > 0$ for every $k$. Then $A \in \langle M_{\infty}(p), (c_0)_s \rangle$ if and only if

(i) The condition (i) of Theorem 1 holds.

(ii) $(b_{nk})_{n=1}^\infty \in (c_0)_s$ for each fixed $k$.

**Remark.** We note that $M_{\infty}(p) \subseteq c_0(p)$. If $p \in Q$, then $c_0(p) \subseteq M_{\infty}(p)$.

Applying Theorem 13 in (Lascarides [1]), we have $c_0(p) = M_{\infty}(p)$ if and only if $p \in Q$.

**Corollary 4.** Let $p = (p_k) \in Q$. Then $A \in \langle M_{\infty}(p), ms \rangle$ if and only if
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\begin{equation}
(1) \quad C = \sup_{n,k} |b_{nk}|^{p_k} < \infty.
\end{equation}

**Proof.** It is enough to prove that, in case \( p \in Q \) condition (i)' and condition (i) of Theorem 1 are equivalent. If condition (i) of Theorem 1 holds, then for every integer \( B > 1 \)

\[
|b_{nk}|^{p_k} \leq D^{p_k} B \leq B \max\left(1, D^H\right) < \infty, \quad \text{for every } n, k,
\]

where \( H = \sup_{k} p_k \). Therefore (i) holds. On the other hand if \( p \in Q \), then

\[
\sum_{k} B_0^{-1/p_k} < \infty.
\]

Now there exists an integer \( B \) such that \( B \geq \max \{1, B_0 C\} \). So we have

\[
|b_{nk}| \leq C^{1/p_k} \leq B_0^{1/p_k} B_0^{-1/p_k} \quad \text{for every } n, k.
\]

Thus (i) in Theorem 1 holds.

**Corollary 5.** Let \( p = (p_k) \in Q \). Then \( A \in \left(M_{ac}(p),(c_0)_s\right) \) if and only if together with the condition of the Corollary 4, the following also hold

\[
(u) \quad (b_{nk})_{n=1}^{\infty} \in (c_0)_s \quad \text{for each fixed } k.
\]

**References**


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In this paper we have determined necessary and sufficient conditions for an infinite matrix 
\( A = (a_{nk}) \) to transform \( l_p(s) \), \( c_0(p, s) \) and \( l_\infty(p, s) \) into \( \Gamma(q) \). Where \( \Gamma(q) \) is the generalized 
\( \chi \)-space introduced by Kamthan \[2\].

**KEY WORDS AND PHRASES**: Matrix transformations, Köthe-Töplitz dual, sequence 
spaces.

**INTRODUCTION**: If \( X \) and \( Y \) are two sequence spaces, let \( (X, Y) \) denote the class of all 
matrices \( A = (a_{nk}) \), \( n, k = 1, 2, \ldots \) such that \( (A_n(x)) \in Y \) whenever \( x = (x_k) \in X \), where \( A_n(x) \) is 
the \( A \)-transform of \( x = (x_k) \) given by

\[
A_n(x) = \sum_k a_{nk} x_k, \quad n = 1, 2, \ldots
\]

(Throughout \( \Sigma \) denotes summation over \( k \) from \( k = 1 \) to \( k = \infty \)).

The main purpose of this paper is to characterize the matrices of class \( (l_p(s), \Gamma(q)) \), 
\( (c_0(p, s), \Gamma(q)) \) and \( (l_\infty(p, s), \Gamma(q)) \). For \( p = (p_k) > 0 \) and \( \sup p_k < \infty \) and \( q_k \) be any bounded 
sequence then we define the required sequence spaces (See \[1\], \[3\], \[4\]) as follows:

\[
l(p, s) = \left\{ x = (x_k) : \sum_k k^{-s} |x_k|^p < \infty \right\}
\]

\[
c_0(p, s) = \left\{ x = (x_k) : k^{-s} |x_k|^p \to 0 \quad \text{as} \quad k \to \infty \right\}
\]

\[
l_\infty(p, s) = \left\{ x = (x_k) : \sup_k k^{-s} |x_k|^p < \infty \right\}
\]

where \( s \) is a non-negative real number. And

\[
\Gamma(q) = \left\{ x = (x_k) : |k ! x_k|^{q_k} \to 0 \quad \text{as} \quad k \to \infty \right\}
\]

\( \Gamma(q) \) is a linear metric space under the metric topology generated by the paranorm 
\( g(f) = \sup_k |k ! x_k|^{q_k/M} \). For the properties of \( \Gamma(q) \), one can refer Nanda, Srivastav and Nayak \[4\].

In \[1\], Bulut and Cakar showed that \( l(p, s) \) is a paranormed space with 
\( g(x) = \left( \sum_k k^{-s} |x_k|^{p_k} \right)^{1/M} \) where \( M = \max (1, \sup p_k) \).

If we take \( p_k = p \) for all \( k \), then we have

\[
l_p(s) = \left\{ x = (x_k) : \sum_k k^{-s} |x_k|^p < \infty, \quad s \geq 0 \right\}.
\]

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Obviously, the sequence space $L_p = \left\{ x = (x_k) : \sum_k |x_k|^p < \infty \right\}$ is a special case of $L_p(s)$ which corresponds to $s = 0$. And $L_p(s) \supset L_p$.

Recently, Metin Basarir [3] showed that the space $c_0(p, s)$ is a paranormed space with $g'(x) = \sup (k^{-\gamma/M} |x_k|^p_k / M)$. Also $l_\infty(p, s)$ is a paranormed space by $g'(x)$ if and only if $\inf p_k > 0$. All the spaces defined above are complete in their topologies.

Let $X^*$ denote the Kothe-Toeplitz dual of $X$ such that

$$X^* = \left\{ a = (a_k) : \sum_k |a_k x_k| < \infty \text{ for all } (x_k) \in X \right\}.$$ 

Now, we quote some known results as the following theorems which will be used in section 2 for establishing our results.

**Lemma 1.** (See [1]).

$$L_p^*(s) = \left\{ a = (a_k) : \sum_k |a_k|^p k^{4(q-1)} < \infty, s \geq 0 \right\}.$$

**Lemma 2.** (See [3]).

$$c_0(p, s) = \bigcup_{M > 1} \left\{ a = (a_k) : \sum_k |a_k| k^{M^{-1/p_k} k / p_k} < \infty \right\}.$$

**Lemma 3.** (See [3]).

$$l_\infty(p, s) = \bigcap_{M > 1} \left\{ a = (a_k) : \sum_k |a_k| M^{1/p_k} k / p_k < \infty \right\}.$$

**Matrix Transformations:**

**Theorem 1.** $A \in (L_p(s), \Gamma(q))$ if and only if

$$\left( n ! \sum_k |a_{nk}|^q k^{4(q-1)} \right)^{1/q} \to 0 \text{ as } n \to \infty \text{ uniformly in } k. \quad \text{(1.1)}$$

where $p > 1$ and $p^{-1} + q^{-1} = 1$.

**Proof:** SUFFICIENCY: Since $(x_k) \in L_p(s)$, then there exist a finite $M \geq 1$ such that

$$\sum_k |x_k|^p k^{-3} \leq M \quad \text{(1.2)}$$

Let (1.1) hold good. Then given an $\varepsilon > 0$, there exist a $N = N(\varepsilon)$ independent of $k$ such that

$$\left( n ! \sum_k |a_{nk}|^q k^{4(q-1)} \right)^{1/q} < \varepsilon/M \text{ for } n \geq N \quad \text{(1.3)}$$

Hence using (1.2) and (1.3), we get

$$\left( n ! A_n(s) \right)^{1/q} \leq \left( n ! \sum_k |a_{nk}| k^{4(q-1)} \right)^{1/q} \leq \frac{\varepsilon}{M}.$$
This shows that \( A_n(x) \in \Gamma(q) \).

**NECESSITY:** If (1.1) does not hold good, then there exist a subsequence of values of \( n \) such that

\[
\left( n ! \sum_k \left| a_{nk} \right|^{p \frac{q}{q - 1}} \right)^{\frac{q}{q - 1}} \geq \varepsilon \tag{1.4}
\]

Since the matrix \((a_{nk})\) is applicable to each member of \( l_p(s) \), \((a_{nk}) \in l_p^s(s)\).

Hence by Lemma 1, we get

\[
\sum_k \left| a_{nk} \right|^{p \frac{q}{q - 1}} \text{ is convergent} \tag{1.5}
\]

when \( x_k = 1 \) and \( x_j = 0 \) for \( j \neq k \), \((x_k) \in l_p(s)\) so that

\[
A_n(x) = (a_{nk}) \in \Gamma(q).
\]

Hence

\[
\left( n ! \left| a_{nk} \right|^{p \frac{q}{q - 1}} \right)^{\frac{q}{q - 1}} \leq A_k^v \text{ for all } n \text{ and each fixed } k. \tag{1.6}
\]

So that \((n!a_{nk}k^{q(q-1)})^{\frac{q}{q-1}} \leq A_k^v\) for each fixed \( k \) and for all \( n \).

Now we shall construct a sequence \((x_k) \in l_p(s)\) and show that \((A_n(x)) \notin \Gamma(q)\) using (1.4), (1.5) and (1.6). Then that will suffice to prove the necessity of the condition.

Now by (1.4), choose \( n = n_1 \) and \( k = r_1 \) such that

\[
\left( n_1 ! \sum_{k=1}^{r_1} \left| a_{nk} \right|^{p \frac{q}{q - 1}} \right)^{\frac{q}{q - 1}} > 1 \tag{1.7}
\]

Having fixed \( n_1 \), by (1.5) choose \( k_1 > r_1 \) such that

\[
\left( n_1 ! \sum_{k=r_1+1}^{k_1} \left| a_{nk} \right|^{p \frac{q}{q - 1}} \right)^{\frac{q}{q - 1}} < \varepsilon \tag{1.8}
\]

for \( \varepsilon > 0 \), the series being convergent.

Taking \( x_k = \left| a_{nk} \right|^{q \frac{q}{q - 1} - 1} \text{sgn}(a_{nk}) \) for \( 1 \leq k \leq k_1 \)

\[
\left| n_1 ! A_n(x) \right|^{q} \geq (n_1 !) \left( \sum_{k=1}^{k_1} a_{nk} x_k \right)^{q} \frac{q^n}{q - 1} - \left( \sum_{k=k_1+1}^{k_1} a_{nk} x_k \right)^{q} \frac{q^n}{q - 1} \geq \left( n_1 ! \sum_{k=1}^{k_1} \left| a_{nk} \right|^{q \frac{q}{q - 1}} \right)^{\frac{q}{q - 1}} \tag{1.9}
\]
From (1.6) we have for all \( n, \)
\[
\left( \frac{k}{n} \sum_{k=1}^{q_n} |a_{nk}|^q k^{q(q-1)} \right)^{\frac{p}{q}} \geq \left( \frac{k}{n} \sum_{k=1}^{q_n} |a_{nk}|^q k^{q(q-1)} \right)^{\frac{p}{q}} \leq A_1^q + A_2^q + \ldots + A_k^q \leq C_k, \quad \text{where} \quad C_k = A_1^q + A_2^q + \ldots + A_k^q \ldots (1.10)
\]

Now by (1.4) choose \( n_2 > n_1 \) and \( r_2 > k_1 \) such that
\[
\left( \frac{n_2}{k_1} \sum_{k=1}^{r_2} |a_{nk}|^q k^{q(q-1)} \right)^{\frac{p}{q}} > 2 + C_{k_1} \ldots (1.13)
\]

Having fixed \( n_2, \) by (1.5) choose \( k_2 > r_2 \) such that
\[
\left( \frac{n_2}{k_1} \sum_{k=1}^{k_2} |a_{nk}|^q k^{q(q-1)} \right)^{\frac{p}{q}} < \varepsilon \ldots (1.12)
\]

Taking \( x_k = |a_{nk}|^q k^{q(q-1)} \) \( \text{sgn} \) \( a_{nk} \) for \( k_1 < k \leq k_2, \) we have
\[
(n_2 ! A_n(x))^q > \left( n_2 ! \sum_{k=1}^{k_2} a_{nk} x_k \right)^q - \left( n_2 ! \sum_{k=1}^{k_1} a_{nk} x_k \right)^q - \left( n_2 ! \sum_{k=1}^{k_2} a_{nk} x_k \right)^q
\]

\[
> \left( n_2 ! \sum_{k=1}^{k_2} |a_{nk}|^q k^{q(q-1)} \right)^q - \left( n_2 ! \sum_{k=1}^{k_1} |a_{nk}|^q k^{q(q-1)} \right)^q - \left( n_2 ! \sum_{k=1}^{k_2} |a_{nk}|^q k^{q(q-1)} \right)^q
\]

\[
> 2 + C_{k_1} - C_{k_1} - \left( n_2 ! \sum_{k=1}^{k_2} |a_{nk}|^q k^{q(q-1)} \right)^q - \left( n_2 ! \sum_{k=1}^{k_2} |x_k|^p k^{q(q-1)} \right)^q
\]

\[
> 2 - \varepsilon, \quad \text{by} \ (1.10), (1.11) \text{and} (1.12).
\]

Proceeding like this by (1.4), we can choose \( n_m > n_{m-1} \) and \( r_m > k_{m-1} \)
such that
\[
\left( n_m \left[ \sum_{k=k_m+1}^{n_m} \left| a_{n_k} \right| q^k (q-1) \right] \right)^{\frac{1}{q}} > m + (m-1)C_{k_1} + (m-2)C_{k_2} + \ldots + C_{k_{m-1}} \ldots (1.13)
\]

Having fixed \( n_m \) by (1.5) choose \( k_m > r_m - 1 \) such that
\[
\left( n_m \left[ \sum_{k=k_m+1}^{n_m} \left| a_{n_k} \right| q^k (q-1) \right] \right)^{\frac{1}{q}} < \varepsilon \ldots (1.14)
\]

Taking \( x_k = \left| a_{n_k} \right| q^{-1} k^q (q-1) \text{sgn} (a_{n_k}) \) for \( k_{m-1} < k \leq k_m \)

we can show as above that \( (n_m ! A_n(x))^{\frac{1}{q}} > m - \varepsilon. \) Since \( \varepsilon \) is arbitrary \( (n_m ! A_n(x))^{\frac{1}{q}} \to \infty \) as \( m \to \infty \). Hence \( (A_n(x)) \notin \Gamma(q) \) so that the condition (1.1) is necessary.

**THEOREM 2.** \( A \in \left( c_{0}(p,s), \Gamma(q) \right) \) if and only if
\[
\left( n ! \left[ \sum_{k=1}^{n} \left| a_{n_k} \right| M^{-1/p_1} k^{q/p_1} \right] \right)^{\frac{1}{q}} \to 0 \text{ as } n \to \infty
\]

for every positive integer \( M > 1 \).

**PROOF.** By adopting a method similar to the theorem 1 and taking for all \( n, \)
\[
(x_k) \in c_{0}(p,s)
\]

as
\[
x_k = \begin{cases} 
\text{sgn}(a_{n_k}) (M+1)^{-1/p_1} k^{q/p_1} & \text{for } 1 \leq k \leq k_1 \\
\text{sgn}(a_{n_k}) (M+1)^{-1/p_1} k^{q/p_1} & \text{for } k_1 \leq k \leq k_2
\end{cases}
\]

and using Lemma 2 we can establish the result.

**THEOREM 3.** \( A \in \left( L_\infty(p,s), \Gamma(q) \right) \) if and only if
\[
\left( n ! \left[ \sum_{k=1}^{n} \left| a_{n_k} \right| k^{q/p_1} M^{1/p_1} \right] \right)^{\frac{1}{q}} \to 0 \text{ as } n \to \infty
\]

for every positive integer \( M > 1 \).

**PROOF.** By taking for all \( n, (x_k) \in L_\infty(p,s) \) as \( x_k = \text{sgn}(a_{n_k}) M^{1/p_1} k^{q/p_1} \) for \( 1 \leq k \leq k_i; \ i = 1, 2, 3, \ldots \) and using lemma 3 we can establish the result.

**NOTE:** When we take \( s = 0 \) in theorems 2 and 3 we get the results of Somasundram [5] and if we take \( q_n = \frac{1}{n} \) and \( s = 0 \) in theorems 1, 2 and 3 we get the results of Sirajudeen [6] as Corollaries.

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MATRICES TRANSFORMATIONS AND ALMOST BOUNDEDNESS

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The main purpose of this paper is to determine the matrices of the classes (l(p), m(r)), (l(p, s), b (r)), (c0(p, s), m(r)) and (c0(p, s), b (r)). Additionally some new result have been obtained as special cases.

1. Introduction, Definitions and Notations

Let \( A = (a_{nk}) \) be an infinite matrix of complex numbers \( a_{nk} (n, k = 1, 2, \ldots) \) and \( X, Y \) be two nonempty subsets of the space \( S \) of all complex sequences. We say that the matrix \( A \) defines a transformation from \( X \) into \( Y \), if for every sequence \( x = (x_k) \in X \) the sequence \( Ax = (A^k x) \) exists and is in \( Y \), where \( A^k x = \sum a_{nk} x_k \).

Here and onwards the sum without limits is always taken from \( k = 1 \) to \( k = \infty \). By \( (X, Y) \), we denote the class of all such matrices.

In this paper let \( m \) and \( bs \) denote the spaces of bounded sequences and bounded series. Let \( D^i \) be the composition of the shift operator \( D \), defined on \( S \) by \( (Dx)_n = x_{n+1} \), with itself \( i \) times. By extending the concept of boundedness to that of almost boundedness, S. Nanda introduced the space \( \hat{m} \) of almost bounded sequences in [9], as

\[
\hat{m} = \left\{ x : \sup_{j, n} |t_j (x)_n| < \infty \right\},
\]

where \( t_j (x) = \frac{1}{j+1} \sum_{i=0}^{j} (D^i x)_n \).

Now we define (see Maddox [5] and Nanda [9]),

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where $p = (p_k)$ denotes a sequence of strictly positive numbers. It is natural to expect that the space $bs$ can be extended to $bs(p)$ and $bs^*(p)$ just as $m$ is extended to $m(p)$ and $m^*(p)$ respectively. Let us reconsider the bounded sequence $p = (p_k)$ of strictly positive numbers and write $P_x$ to denote the sequence of partial sums of an infinite series $\sum x_k$ i.e. $(P_x)_n = \sum_{k=0}^n x_k$. Then we describe respectively, the generalized linear spaces $bs(p)$ and $bs^*(p)$ of bounded and almost bounded series (see [2]), as

$$bs(p) = \{x : P_x \in m(p)\} \quad \text{and} \quad \hat{bs}^*(p) = \{x : P_x \in \hat{m}(p)\}.$$

In the case $p_k = p > 0$ for all $k$, we have $m(p) = m$, $\hat{m}(p) = \hat{m}$, $bs(p) = bs$ and $bs^*(p) = bs^*$. Also in the case $p = (\frac{1}{k})$ we have $\hat{m}(p) = \hat{d}$ where

$$\hat{d} = \{x : \sup_{j,n} |t_{jn}(x)|^{1/k} < \infty\}$$

which is an extension of the space

$$d = \{x : \sup_k |x_k|^{1/k} < \infty\}$$

introduced by Iyer [3]. Again in the same case we obtain the linear spaces $ds$ and $\hat{ds}$ from $bs(p)$ and $bs^*(p)$, respectively.

In this paper, our main purpose is to determine the matrices of the classes $(l(p,s),\hat{m}(r)), (l(p,s),\hat{bs}(r)), (c_0(p,s),\hat{m}(r))$ and $(c_0(p,s),\hat{bs}(r))$. In addition, some new results are obtained as special cases. Here the space $l(p,s)$ is defined by Bulut and Cakar [1] for $p = (p_k)$ with $p_k > 0$, as

$$l(p,s) = \left\{x : \sum_{k=1}^{\infty} k^{-s} |x_k|^{p_k} < \infty, s \geq 0\right\}$$

which generalizes the sequence space $l(p)$. They showed that $l(p,s)$ is a linear
MATRIX TRANSFORMATIONS AND ALMOST ...

space paranormed by \( g(x) = \left( \sum_k k^{-s} |x_k|^p \right)^{1/p} \) where \( M = \max \{ 1, H \} \), and \( H = \sup_k p_k < \infty \).

Quite recently, Metin Basarir [8] has defined the sequence space \( c_0(p, s) \) as follows:

\[
c_0(p, s) = \{ x : k^{-s} |x_k|^p \to 0 \text{ as } k \to \infty \}
\]

where \( s \) is a non-negative real number. It is easy to see that \( c_0(p, s) \) is a space paranormed by \( g(x) \). All the spaces defined above are complete in their topologies.

If \( X \) is a nonempty subset of \( S \), then we shall denote the generalized Köthe-Toeplitz dual of \( X \) by \( X^* \), i.e.

\[
X^* = \left\{ \sum_k a_k x_k \text{ converges for all } x \in X \right\}.
\]

The generalized Köthe-Toeplitz duals of \( c_0(p, s) \) and \( l(p, s) \), which are employed in the proof of our theorems, were determined as

\[
c_0^+(p, s) = \bigcup_{N > 1} \left\{ a : \sum_k k^{-N} |x_k|^p < \infty \right\}
\]

and

\[
l^*(p, s) = \begin{cases} E(p, s), & \text{if } 1 < p_k < \infty \\ V(p, s), & \text{if } 0 < p_k \leq 1, \end{cases}
\]

where \( E(p, s) = \{ a : \sum_k k^{d(q_k^{-1})} N^{-d/q_k} |x_k|^p < \infty \text{ for some integer } N > 1 \} \), and \( V(p, s) = \{ a : \sup_k k^{d/q_k} |a_k|^p < \infty, s > 0 \} \) (see Theorem 1 of [8], and Theorem 1 of [1]).

We state the following inequality (see [7]) which will be used later. For any two complex numbers \( a, b \) and \( p > 1 \)

\[
|ab| \leq a^p + b^p, \quad \text{where } p^{-1} + q^{-1} = 1.
\]

(*)
Matrix Transformations on \( l(p, s), c_0(p, s), \) and \( b^s(r) \)

Throughout the paper we shall assume the sequence \( p = (p_k), r = (r_n) \) of strictly positive numbers to be bounded and write for a sequence \( x = (x_k), \)

\[
j_m(Ax) = \frac{1}{j+1} \sum_{i=0}^{j} (D^j Ax)_m = \sum_{k} a(n, k, j) x_k;
\]

where

\[
a(n, k, j) = \frac{1}{j+1} \sum_{i=0}^{j} a_{n+i, k}.
\]

In the special case \( j = 0, \) \( j_m(Ax) \) reduces to \( (A_n(x)). \) For brevity we shall, in the future, also write \( a(n, k) = \sum_{j=0}^{n} a_{n,j}. \)

We now prove:

**Theorem 1.** \( A \in (l(p, s), m(r)) \) if and only if there exists an integer \( N > 1 \) such that

1. For \( 0 < p_k \leq 1, \)
   \[
   B_N = \sup_{n,k,j} (a(n, k, j)|N^{-\frac{1}{k}} k^j|) < \infty;
   \]
2. For \( 1 < p_k < \infty, \)
   \[
   C_N = \sup_{n,j} \sum_{k} |a(n, k, j)|N^{-q_k} k^j (q_j - 1) < \infty,
   \]

where \( p_k^{-1} + q_k^{-1} = 1. \)

**Proof.** Necessity. Let \( A \in (l(p, s), m(r)) \) and \( x \in l(p, s). \) Then \( j_m(Ax) \) exists and is in \( m(r) \) for all \( x \in l(p, s). \) Consider the sequence \( x \in l(p, s) \) defined by \( x_k = k^j N^{-\frac{1}{k}} sgn a(n, k, j) e^k \) for \( N > 1 \) and \( 0 < p_k \leq 1, \) where \( e^k \) is the sequence having \( 1 \) in its \( k-th \) place and zero elsewhere. Now, the necessity of (1) is trivial by the hypothesis, since \( j_m(Ax) = 1 a(n, k, j)|N^{-\frac{1}{k}} k^j|. \) To prove the necessity of (2), we define the sequence of continuous linear functionals \( f_n \) on \( l(p, s) \) by \( f_n(x) = \sup_{j} j_m(Ax)| \) such that \( \sup_{n} f_n(x) < \infty, \) since \( (a(n, k, j)) \in l^* (p, s) \) for
every \( n,j \). Therefore by Banach-Steinhaus Theorem (see [6], pp. 114-115) there is a constant \( K \) such that

\[ f_n(x) \leq K \| x \| \text{ for all } n, \]  

(3)

and for every \( x \in l(p,s) \). The desired result follows from (3) by putting the sequence \( x \in l(p,s) \) defined by

\[ x_k = 1 a(n,k,j) t^{n-1} N^{-q_f} s g n (a(n,k,j)) k_{r_u} - 1 \].

for \( N > 1 \) and \( 1 < p_k \).

Sufficiency. Let \( B_N, C_N < \infty, 0 < p_k \leq 1 \) and \( x \in l(p,s) \). Now

\[ \sum_{k} k^{-s} |x_k|^{p_k} = c, \ M = \max (1, \sup 1/p_k), \ L = \max (1, \sup r_n), \text{ say.} \]

Then we get

\[ \sup_{n,j} |f_n(Ax)|^{p_L} \leq \sup_{n,j} \left( \sum_k |a(n,k,m)| \| x_k \|^{|a|} \right)^{p_L} \]

\[ \leq N^{ML} B_N \left( \sum_{k} k^{-s} |x_k|^{p_k} \right)^{ML} \]

\[ = (Nc)^{ML} B_N < \infty. \]

This proves the sufficiency of (1).

Since for \( p_k > 1 \)

\[ |a(n,k,j) x_k| \leq N (|a(n,k,j)| t^{n-1} N^{-Lq_f} k_{r_u} k^{(q_f-1)}) + k^{-s} |x_k|^{p_k} \]

holds by (*), we then have

\[ |f_n(Ax)|^{p_L} \leq N \left\{ c + \sum_k |a(n,k,j)| t^{n-1} N^{-Lq_f} k_{r_u} k^{(q_f-1)} \right\}. \]

Taking supremum in this last inequality over \( n,j \) we now deduce by (2) that \( Ax \in \ell m (r) \) and this completes the proof.

In the special case \( j = 0, \) and \( s = 0 \) we get the characterization for the class \( l(p), m(r) \) due to Maddox and Willey [7]. An immediate consequence of Theorem
Corollary 1. Let $A \in (l(p,s), \hat{b})$ if and only if (1) and (2) hold with $r_n = \frac{1}{n}$ for all $n$.

We now give a lemma which establishes a linear isomorphism between the spaces $b^*_s(r)$ and $\hat{m}(r)$, and is required in proving Theorem 2 below.

**Lemma [2].** The space $b^*_s(r)$ is linearly isomorphic to the space $\hat{m}(r)$.

It may be noted here that the Lemma holds even if $b^*_s(r)$ and $\hat{m}(r)$ are replaced by $bs(r)$ and $m(r)$ respectively.

**Theorem 2.** Let $B \in (l(p,s), b^*_s(r))$ if and only if there exists an integer $N > 1$ such that

(i) for $0 < p_k < 1$,

$$\sup_{n,j} \left. \left( \frac{1}{j+1} \sum_{i=0}^{j} b(n+i,k) x^N b^*_s(r) k^n \right) \right| < \infty,$$

(ii) for $1 < p_k < \infty$,

$$\sup_{n,j} \left. \left( \sum_{k=1}^{j} \frac{1}{j+1} \sum_{i=0}^{j} b(n+i,k) x^N b^*_s(r) k^{(q_k-1)} \right) \right| < \infty,$$

where $p_k + q_k^{-1} = 1$.

**Proof.** Let $B \in (l(p,s), b^*_s(r))$ and $x \in l(p,s)$. Now consider the following equality obtained from the $n,j$-th partial sums of $(Bx)_n$, as $j \to \infty$.

$$\sum_{i=0}^{n} \sum_{k} b_{ik} x_k = \sum_{k} b(n,k) x_k, \text{ for all } n.$$ 

It is then obvious by lemma that $Bx \in b^*_s(r)$ if and only if $Ax \in \hat{m}(r)$, where $A = (a_{nk})$ with $a_{nk} = b(n,k)$ for all $n,k$. This completes the proof.

**Remark.** It is comprehensible by examining the proof of Theorem 2 that the conditions (i) and (ii) of Theorem 2 are respectively equivalent to the conditions (i) and (ii) of Theorem 1 whenever the relation $a_{nk} = b(n,k)$ exists between $A = (a_{nk})$ and $B = (b_{nk})$ although they are contained in different matrix classes.
By Theorem 2, we have

**COROLLARY 2.** \( B \in (l(p, s), ds) \) if and only if (4) and (5) hold with \( r_n = \frac{1}{n} \), for all \( n \).

Again as a result of Theorem 2, we can give

**THEOREM 3.** \( B \in (l(p, s), bs(r)) \) if and only if the conditions (4) and (5) hold with \( j = 0 \) for all \( n, k \).

By Theorem 3, we have

**COROLLARY.** \( B \in (l(p, s), ds) \) if and only if the conditions (4) and (5) of Theorem 3 hold with \( r_n = \frac{1}{n} \) for all \( n \).

We want to conclude the paper by giving the necessary and sufficient conditions on an infinite matrix which transforms \( c_0(p, s) \) into \( m(r), bs(r) \) and \( bs(r) \). Since it is now feasible to prove them, on the same lines as that of previous theorems, so we omit their proofs.

**THEOREM 4.** \( A \in (c_0(p, s), m(r)) \) if and only if there exists an integer \( N > 1 \) such that

\[
\sup_{n, j} \left( \sum_k |a(n, k, j)| n^{-\frac{1}{2k}r} \right) < \infty. \tag{6}
\]

In the special case \( j = 0 \) and \( s = 0 \), we get the class \( (c_0(p), m(r)) \) due to Luh [4]. Again as a result of Theorem 4, we can give

**THEOREM 5.** \( A \in (c_0(p, s), bs(r)) \) if and only if there exists an integer \( N > 1 \) such that

\[
\sup_{n, j} \left( \sum_k \frac{1}{j+1} \sum_{i=0}^j |a(n+i, k)| n^{-\frac{1}{2k}r} \right) < \infty. \tag{7}
\]

**THEOREM 6.** \( A \in (c_0(p, s), (bs(r))) \) if and only if there exists an integer \( N > 1 \) such that

\[
\sup_n \left( \sum_k |a(n, k)| n^{-\frac{1}{2k}r} \right) < \infty. \tag{8}
\]

Reducing the conditions (6), (7) and (8) to the case \( r = \left( \frac{1}{n} \right) \), we may respec-
tively obtain the characterizations of the classes \((c_0(\ell, s), \delta), (c_0(\ell, s), \delta^c)\) and \((c_0(\ell, s), \delta^c)\).

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The main purpose of this paper is to characterize the class of matrices 
\( (L_p, Vg), (c_p, Vg) \) and \( (c, Vg) \). Here the spaces \( L_p, c_p, c_p, \) and \( c \) have been defined by Metin Basarir [7] while the space \( Vg \) is the set of bounded sequences all of whose \( \sigma \)-means are equal, due to P. Schaefer [9].

**INTRODUCTION:** Let \( \sigma \) be mapping of the set of positive integers into itself. A continuous linear functional \( \phi \) on \( c \), the space of real bounded sequences, is said to be an invariant mean or a \( \sigma \)-mean if and only if (i) \( \phi (x) \neq 0 \) when the sequence \( x = (x_n) \) has \( x_n \neq 0 \), for all \( n \) (ii) \( \phi (e) = 1 \), where \( e = (1, 1, 1, \ldots) \), and (iii) \( \phi ((x_{\sigma}(n))) = \phi (x) \), for all \( x \in c \).

The mappings \( \sigma \) are one to one and such that \( \sigma^m(n) \neq n \) for all positive integers \( n \) and \( m \), when \( \sigma^m \) denotes the \( m \)th iterate of the mapping \( \sigma \) at \( n \). For such mappings, every \( \sigma \)-mean extends the limit functional on the space \( c \) of real convergent sequences, in the sense that \( \phi (x) = \lim x \) for all \( x \in c \) [8]. Consequently, \( c \subset Vg \) where \( Vg \) is the set of bounded sequences all of whose \( \sigma \)-means are equal. When \( \sigma (n) = n + 1 \), the \( \sigma \)-means are classical Banach limits on \( L_p \) and \( vg \) is the set of almost convergent sequences [9]. If \( x = (x_n) \), write

\[
T x = (T x_n) = (x_{\sigma(n)}),
\]

then

\[
Vg = \left\{ x \in L_p : \lim_{n \to \infty} \frac{t_{m,n}(x)}{(m+1)} \right\},
\]

where

\[
t_{m,n}(x) = (x_n + T x_n + \ldots + T^n x_n)/(m+1).
\]

Several problems have been done by the different authors (see [5, 6], [10]) concerning the space \( Vg \).

Now let \( X \) and \( Y \) be two subsets of the space \( W \) of all complex sequences and \( A = (a_{nk}) \) be an infinite matrix of complex numbers \( a_{nk} \) converges for each \( n \) (Here and afterwards the sum without limits is taken from \( k = 1 \) to \( k = \infty \)). If \( x \in X \Rightarrow Ax \in Y \), we say that \( A \) is a \((X, Y)\)-matrix.

For a sequence \( p = (p_k) \) with \( p_k > 0 \), the following sequence spaces and their Kothe-Toeplitz duals have been studied by various authors (see [3], [4], and [11]).

\[
L_p = \left\{ x : \sum \frac{|x_k|}{p_k} < \infty \right\},
\]

\[
L_\infty(p) = \left\{ x : \sup_k |x_k|^{1/p_k} < \infty \right\},
\]

\[
c_p = \left\{ x : |x_k|^{1/p_k} \rightarrow 0 \right\}.
\]
Let $E$ be a sub set of $W$ then

$$E^* = \{a = (a_k) : \sum a_k x_k \text{ converges, for every } x \in E\},$$

will denote the Koth-Toeplitz dual of $E$.

We know that

$$c^*(p) = \{x : |x_k - l|^p/k \to 0 \text{ for some } l (k \to \infty)\}.$$

$$L^*_n(p) = \bigcap_{N>1} \{a : \sum a_k |N^{-1/p}k < \infty\},$$

$$c^*_n(p) = U \{a : \sum a_k |N^{-1/p}k < \infty\},$$

where $\frac{1}{p_k} + \frac{1}{q_k} = 1$ and $N > 1$ is an integer.

In [1] the space $l(p)$ was extended to $l(p, s)$ for $s \geq 0$ i.e.,

$$l(p, s) = \{x : \sum |x_k|^p k^{-s} < \infty\},$$

and

$$l^*(p, s) = \{a : \sum a_k |q_k k^{(q_k - 1)} N^{-1/p}k < \infty\}.$$

Metin Basarir [7] extended the space $c_o(p)$, $c(p)$ and $l_\infty(p)$ to $c^*_o(p, s)$, $c^*(p, s)$ and $l^*_\infty(p, s)$ respectively for $s \geq 0$ and defines:

$$c^*_o(p, s) = \{x : x_k |p_k k^{-s} \to 0 (k \to \infty)\}$$

$$c^*(p, s) = \{x : x_k - l |p_k k^{-s} \to 0 \text{ for some } l (k \to \infty)\}$$

$$l^*_\infty(p, s) = \{x : \sup_k |x_k|^p k^{-s} < \infty\}.$$

Using the same kind of argument as in Maddox [4], we get that the necessary and sufficient conditions for above sequence spaces to be linear is $p \in l_\infty$ and $c^*_o(p, s)$ is paranormed space by $g(x) = \sup (k/M) |x_k|^p k^{-s}$, where $M = \max (1, H), H = \sup k p_k$. Also $l^*_\infty(p, s)$ and $c^*(p, s)$ are paranormed by $g(x)$ if and only if $\inf p_k > 0$. All the spaces defined above are complete in their topologies.

Now we give an easy setting to find Koth-Toeplitz duals of these spaces.

Let $W$ be the space of all convergent series. For arbitrary $x, y \in W$, we put

$$xy = (x_k y_k). \quad \ldots (1.1)$$

Therefore, for $E \subseteq W$, we can write

$$E^* = \{a = (a_k) : ax \in W\}.$$

Let $U$ be the space of all sequences $u \in W$, such that $u_k \neq 0 (k = 1, 2, \ldots)$. For $u \in U$, we define the sequence $u^{-1}$ by

$$u^{-1}_k = \frac{1}{u_k}, \quad k = 1, 2, \ldots \quad \ldots (1.2)$$
and for $E \subseteq W, u \in U$, we put
\[ E_u = \{ x \in W : xu \in E \}. \]  
\[ \text{(1.3)} \]

Obviously
\[ (Eu)^+ = (E^+)u^{-1}. \]  
\[ \text{(1.4)} \]

If $u = (u_k)$ is defined by
\[ u_k = \frac{1}{k^{y/p_k}}, s \geq 0; \quad k = 1, 2, \ldots \]  
\[ \text{(1.5)} \]

then from (1.3), we have
\[ l(p, s) = (l(p))u^{+}, \]  
\[ c_\alpha (p, s) = (c_\alpha (p))u^{+}, \]  
\[ l_\omega (p, s) = (l_\omega (p))u^{+}, \]  

and by (1.4)
\[ f^\alpha (p, s) = (f^\alpha (p))u^{+} = \{ a : \Sigma |a_k| q_k^{1/p_k} \leq \infty \} \]  
\[ \text{(1.6)} \]
\[ c_\alpha^\alpha (p, s) = (c_\alpha^\alpha (p))u^{+} = \{ a : \Sigma |a_k| k^{1/p_k} \leq \infty \} \]  
\[ \text{(1.7)} \]
\[ f_\omega^\alpha (p, s) = (f_\omega^\alpha (p))u^{+} = \bigcup_{N > 1} \{ a : \Sigma |a_k| k^{1/p_k} \leq \infty \} \]  
\[ \text{(1.8)} \]

The object of this paper is to determine the necessary and sufficient condition of the matrix $A = (a_{nk})$ such that $A \in (X, Y)$, where $X = c(p, s), c_\alpha (p, s)$ and $l_\omega (p, s)$ and $Y = V_\sigma$. Many known and unknown results have been deduced from our results as Corollaries.

**SOME MATRIX TRANSFORMATIONS** : Throughout this paper we shall use the notation $a(n, k)$ to denote the element $a_{nk}$ of the matrix $A$, and we write for all integers $n, m \geq 1$
\[ t_{mn} (Ax) = (Ax_n + TAx_n + \ldots + T^{m} Ax_n)/(m + 1) \]  
\[ = \Sigma (n, k, m) x_k \]  
where $t(n, k, m) = \frac{1}{m + 1} \Sigma a(\sigma^\alpha (n), k)$.

Now we prove:

**THEOREM 1** : $A \in (l_\omega (p, s), v_\sigma)$ if and only if
\begin{enumerate}
  
  
  
  
  
  
  \item \( \sup_{m} \Sigma (n, k, m) |N^{1/p_k} k^{1/p_k} \leq \infty \) uniformly in $n$ for every integer $N > 1$.
  
  \item $a_k = (a_{nk})_{n=1}^\infty \in v_\sigma$ for each $k$.
\end{enumerate}

\[ i.e., \lim_{m} t(n, k, m) = u_k \text{ uniformly in } n \text{ for each } k. \]  
\[ \text{In this case, the } \sigma\text{-limit of } Ax \text{ is} \]  
\[ \lim_{m} \Sigma_{k=1}^{\infty} t(n, k, m) = \Sigma_{k=1}^{\infty} u_k x_k. \]  

**PROOF** : For necessity of (i) suppose that $A \in (l_\omega (p, s), v_\sigma)$ but there is an integer $N > 1$ such that (i) is false. Then the matrix $B = (t(n, k, m) N^{1/p_k} k^{1/p_k}) \in (l_\omega, v_\sigma)$ (see Schaefer [9]).
So there is \( x \in l_\infty \) with \( \|x\| = 1 \) such that \( \{t_{m,n}(Bx)\} \not\in v_\sigma \). Hence, although \( y = (N^{1/p_k}k^{{r/2p_k}}x_k) \in l_\infty(p,s) \), the sequence \( \{t_{m,n}(Ay)\} \not\in v_\sigma \). This contradicts the fact that \( A \in (l_\infty(p,s),v_\sigma) \) and completes the proof.

The necessity of (ii) is obtained by taking \( x = e_k \in l_\infty(p,s) \) where \( e_k = [0,0,0,\ldots,0,1,0,0,\ldots] \) such that 1 appears at the \( k \)th place.

For sufficiency, suppose that the conditions hold. Take an integer \( N > \max_k \left( \sup \|t_k^* \|_{l_\infty(p,s)} \right) \).

\[
\left| \sum_{k} t_k(n,k,m)x_k \right| > \sum_{k} \|t_k(n,k,m) - u_k\| N^{1/p_k} k^{r/p_k}.
\]

Hence by (i) and (ii) we have

\[
\lim_{m \to \infty} \sum_{k} t_k(n,k,m)x_k = u_k x_k.
\]

This completes the proof.

Taking \( p_k \) = 1 for all \( k \), let us define the sequence space \( l_\infty(s) = \{x: \sup_k \|t_k^* x_k\| < \infty\} \).

\begin{corollary}
A \in (c(l_\infty(s),v_\sigma)) if and only if

(i) \( \sup_{k} \|t_k^* x_k\| < \infty \) uniformly in \( n \).

(ii) \( \lim_{m \to \infty} t_k(n,k,m) = u_k \) uniformly in \( n \) for each \( k \).

\end{corollary}

\begin{remark}
If we take \( s = 0 \) in corollary 1, then it reduces to Theorem 2 of Schaefer [9].

\end{remark}

\begin{theorem}
Let \( p \in l_\infty \). Then \( A \in (c(p,s),v_\sigma) \) if and only if

(i) There exist an absolute constant \( B > 1 \) such that

\[
D = \sup_{m} \left( \sum_{k} \|t_k(n,k,m)\| B^{-1/p_k} k^{r/p_k} \right) < \infty, \text{ uniformly in } n.
\]

(ii) \( \lim_{m \to \infty} t_k(n,k,m) = u_k \) for each \( k \), uniformly in \( n \).

(iii) \( \lim_{m \to \infty} \sum_{k} t_k(n,k,m) = u \) uniformly in \( n \).

\end{theorem}

\begin{proof}
Necessity—Let \( A \in (c(p,s),v_\sigma) \).

\begin{proof}
Put \( t_{m,n}(Ax) = \sigma_{m,n}(x) \). Since \( (c(p,s),v_\sigma) \subset (c_0(p,s),v_\sigma) \), \( \{\sigma_{m,n}(x)\} \) \( m \) is sequence of continuous linear functionals on \( c_0(p,s) \), see theorem 1 [7].

Such that \( \lim \sigma_{m,n}(x) \) exist for each \( m \). Therefore by uniform boundedness principle for \( 0 < \delta < 1 \), there exists a sphere \( S_\delta \{0\} \subset c_0(p,s) \) and a constant \( K \) such that \( \sigma_{m,n}(x) \leq K \) for each \( n \) and \( x \in S_\delta \{0\} \). Define for each \( r \)

\[
\gamma_n^r = \begin{cases} 
\delta_n^{M/p_k} \text{sgn } t_k(n,k,m) k^{r/p_k} & 0 \leq k \leq r \\
0, & r < k 
\end{cases}
\]

where \( M = \max(1, \sup_k p_k) \).

\end{proof}

\end{proof}
Now \( \gamma_n \in S_k(0) \) and \( \sum_{k=1}^{r} |t(n, k, m)| B^{-1/p_k} k^{1/p_k} \leq k. \)

for each \( n, r \) where \( B = 5^{-M} \). Therefore (i) holds conditions (ii) and (iii) trivially hold.

**Sufficiency**—Suppose (i)-(iii) hold and \( x \in c(p, s) \).

Then there exists \( l \) such that \( k^{-l} |x_k - l|^p_s \rightarrow 0 \). Hence for \( 0 < \varepsilon < 1 \) there exists \( k_\alpha \) such that for all \( k > k_\alpha \),

\[
| x_k - l |^{p_s} \leq \frac{\varepsilon}{B(2D + 1)} < 1
\]

and therefore for \( k > k_\alpha \)

\[
B^{1/p_k} k^{-l/p_k} |x_k - l| < B^{M/p_k} k^{-l/p_k} |x_k - l|
\]

\[
< \frac{e^{M/p_k}}{(2D + 1)} < \frac{e}{2D + 1}
\]

By (i) and (ii) we have

\[
\sum_k |t(n, k, m) - u_k| B^{-1/p_k} k^{1/p_k} < 2D
\]

Hence \( \sum_{k > k_\alpha} |(t(n, k, m) - u_k)(x_k - l)| < \varepsilon \)

Also \( \sum_{k \leq k_\alpha} |(t(n, k, m) - u_k)(x_k - l)| \rightarrow 0 \) as \( m \rightarrow \infty \).

Therefore we have

\[
\lim_{m \rightarrow \infty} \sum_{k} t(n, k, m)x_k = lu + \sum_{k} u_k(x_k - l).
\]

and this completes the proof.

**COROLLARY 2.** Let \( p \in I^\infty \). Then \( A \in (c(p, s), \varepsilon_\alpha p) \) if and only if

(i) condition (i) of theorem (2), holds

(ii) \( \lim_{m \rightarrow \infty} t(n, k, m) = 0 \) for each \( k \), uniformly in \( n \)

(iii) \( \lim_{m \rightarrow \infty} \sum_k t(n, k, m) = 1 \) uniformly in \( n \)

**PROPOSITION :** \( (I^\infty(p, s), \varepsilon_\alpha) \cap (c(p, s), \varepsilon_\alpha p) = \Phi \)

**COROLLARY 3.** Let \( p \in I^\infty \). Then \( A \in (c(p, s), \varepsilon_\alpha p) \) if and only if the condition (i) and (ii) of theorem 2, hold.

**REMARK :** If we take \( p_k = 1 \forall k \) and \( s = 0 \) then theorem 2 reduces to theorem 1 of Shafer [9].
REFERENCES:


