CHAPTER - 6
DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS
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SEQUENCE OF ORLICZ FUNCTIONS

6.1. Definitions and notations

As usual we use the set $E$, $F \subset \omega$ to denote the spaces $c_0$, $c$ or $\ell_\infty$, and

$$\Delta x_k = x_k - x_{k+1},$$

$$\Delta^2 x_k = \Delta(\Delta x_k) = \Delta x_k - \Delta x_{k+1}.$$

Kizmaz [30] has introduced and studied the difference sequence spaces:

$$E(\Delta) := \{ x \in \omega : \Delta x \in E \},$$

Recently, Mikail Et [50] has defined the space

$$E(\Delta^2) := \{ x \in \omega : \Delta^2 x \in E \}$$

Qamaruddin [62] has studied the following difference sequence spaces of Orlicz type with the aid of certain convex function $M$, for some $\rho > 0$.

$$E(\Delta, M) := \left\{ x = (x_k) : M^{\left[ \frac{\Delta x_k}{\rho} \right]} \in E, \text{ for some } \rho > 0 \right\}$$
We define the new difference Orlicz spaces in the manner of Mikail Et [50] as:

\[ E(\Delta, M, p) = \left\{ x = (x_k) : M \left[ \frac{|\Delta^2 x_k|}{p} \right] \in E(p), \text{ for some } p > 0 \right\} \]

\[ E(\Delta^2, M) = \left\{ x = (x_k) : M \left[ \frac{|\Delta^2 x_k|}{p} \right] \in E(p), \text{ for some } p > 0 \right\} \]

\[ E(\Delta^2, M, p) = \left\{ x = (x_k) : M \left[ \frac{|\Delta^2 x_k|}{p} \right] \in E(p), \text{ for some } p > 0 \right\} \]

With the help of sequence of Orlicz functions \( M = \{M_k\} \) and for some \( p > 0 \), we also define:

\[ F(M, \Delta^2) = \left\{ x = (x_k) : M_k \left[ \frac{|\Delta^2 x_k|}{p} \right] \in F \right\} \]

Thus, we have

\[ c_0(\Delta^2, M) = \left\{ x = (x_k) : \lim_{k \to \infty} M \left[ \frac{|\Delta^2 x_k|}{p} \right] = 0 \right\} \]

\[ c(\Delta^2, M) = \left\{ x = (x_k) : \lim_{k \to \infty} M \left[ \frac{|\Delta^2 x_k|}{p} \right] = 0, \text{ and } I \in C \right\} \]

\[ \ell_\infty(\Delta^2, M) = \left\{ x = (x_k) : \sup_{k \geq 0} M \left[ \frac{|\Delta^2 x_k|}{p} \right] < \infty \right\} \]

\[ c_0(M, \Delta^2) = \left\{ x = (x_k) : \lim_{k \to \infty} M_k \left[ \frac{|\Delta^2 x_k|}{p} \right] = 0 \right\} \]

\[ \ell_\infty(M, \Delta^2) = \left\{ x = (x_k) : \sup_{k \geq 0} M_k \left[ \frac{|\Delta^2 x_k|}{p} \right] < \infty \right\} \]
We also have

\[ c_0(\Delta^2, M, p) := \left\{ x=(x_k) : \lim_{k \to \infty} \left[ M \left[ \frac{|\Delta x_k|}{\rho} \right]^p \right] = 0 \right\}; \]

\[ c(\Delta^2, M, p) := \left\{ x=(x_k) : \lim_{k \to \infty} \left[ M \left[ \frac{|\Delta x_k|}{\rho} \right]^p \right] = 0, \quad \| x \|_c < \infty \right\}; \]

\[ \ell_\infty(\Delta^2, M, p) := \left\{ x=(x_k) : \sup_{k \in \mathbb{N}} \left[ M \left[ \frac{|\Delta x_k|}{\rho} \right]^p \right] < \infty \right\}; \]

When \( p_k \) is a constant, for all \( k \),

\[ c_0(\Delta^2, M, p) = c_0(\Delta^2, M); \]

\[ c(\Delta^2, M, p) = c(\Delta^2, M); \]

\[ \ell_\infty(\Delta^2, M, p) = \ell_\infty(\Delta^2, M). \]

6.2. Introduction

We know the following theorems due to Qamaruddin [62].

**Theorem A.** \( \ell_\infty(\Delta, M) \) is a Banach space with the norm

\[ \| x \|_\Delta := \inf \left\{ \rho > 0 : \sup_{k \in \mathbb{N}} M \left[ \frac{|\Delta x_k|}{\rho} \right] \leq 1 \right\}. \]

**Theorem B.** \( \ell_\infty(\Delta, M, p) \) is a complete paranormed space with

\[ G(x) := \inf \left\{ \rho^{p_n/H} : \sup_{k \in \mathbb{N}} \left[ M \left[ \frac{|\Delta x_k|}{\rho} \right]^p \right] \leq 1 \right\}. \]
where $H = \max \{1, \sup_{k \geq 0} p_k\}$.

**Theorem C.** Let $M$ be an Orlicz function which satisfies $\Delta_2$-condition. Then

(i) $c_0(\Delta) \subseteq c_0(\Delta, M)$;

(ii) $c(\Delta) \subseteq c(\Delta, M)$;

(iii) $\ell_\infty(\Delta) \subseteq \ell_\infty(\Delta, M)$;

**Theorem D.** Let $0 < p_k \leq q_k < \infty$, for each $k$. Then $c_0(\Delta, M, p) \subseteq c_0(\Delta, M, q)$.

**Theorem E.**

(i) Let $0 < \inf p_k \leq 1$. Then $c_0(\Delta, M, p) \subseteq c_0(\Delta, M)$

(ii) Let $1 \leq p_k \leq \sup p_k < \infty$. Then $c_0(\Delta, M) \subseteq c_0(\Delta, M, p)$

**Theorem F.** Let $\mathcal{M} = \{M_k\}$ be a sequence of Orlicz functions. Then the following statements are equivalent:

(i) $\ell_\infty(\Delta) \subseteq \ell_\infty(M, \Delta)$;

(ii) $c_0(\Delta) \subseteq \ell_\infty(M, \Delta)$;

(iii) $\sup_k M_k [t/p] < \infty$ for $t, p > 0$
Theorem G. The following statements are equivalent for a sequence of Orlicz functions $\mathcal{M}=(M_k)$

(i) $c_0(\mathcal{M},\Delta) \subseteq c_0(\Delta)$;

(ii) $c(\mathcal{M},\Delta) \subseteq \ell_\infty(\Delta)$;

(iii) $\inf_k M_k \frac{t}{\rho} > 0 \quad (t, \rho > 0)$.

Theorem H. The inclusion $\ell_\infty(\mathcal{M},\Delta) \subseteq c_0(\Delta)$ holds if, and only if,

$$\lim_k M_k \frac{t}{\rho} = \infty \quad (t, \rho > 0).$$

Theorem I. The inclusion $\ell_\infty(\Delta) \subseteq c_0(\mathcal{M},\Delta)$ holds if, and only if,

$$\lim_k M_k \frac{t}{\rho} = 0 \quad (t, \rho > 0).$$

In this chapter we have applied the second order difference sequences, i.e. $\Delta^2 x$, to Orlicz function spaces and proved theorems analogous to the above results of Qamaruddin.

6.3. Main results

We prove theorems concerning the topological properties of the new spaces defined in Section 6.2 and also give relations among themselves.
Theorems 6.3.1. $\ell_\infty(\Delta^2, M)$ is a Banach space with the norm:

$$\|x\|_{\Delta^2} := \inf \left\{ \rho > 0 : \sup_{k \geq 0} M \left[ \frac{|\Delta^2 x_k|}{\rho} \right] \leq 1 \right\}.$$ 

Theorem 6.3.2. $\ell_\infty(\Delta^2, M, p)$ is a complete paranormed space with

$$G(x) := \inf \left\{ \rho^{p_n/H} : \left[ \sup_{k \geq 0} M \left[ \frac{|\Delta^2 x_k|}{\rho} \right]^p \right]^{1/H} \leq 1 \right\}$$

where $H = \max \{ \sup_{k \geq 0} p_k \}$.

Theorem 6.3.3. Let $M$ be an Orlicz function which satisfies $\Delta^2$-condition. Then

(6.3.1) $c_0(\Delta^2) \subset c_0(\Delta^2, M)$;

(6.3.2) $c(\Delta^2) \subset c(\Delta^2, M)$;

(6.3.3) $\ell_\infty(\Delta^2) \subset \ell_\infty(\Delta^2, M)$.

Theorem 6.3.4. Let $0 < p_k \leq q_k < \infty$, for each $k$. Then

$c_0(\Delta^2, M, p) \subset c_0(\Delta^2, M, q)$

Theorem 6.3.5.

(6.3.4) Let $0 < \inf p_k \leq p_k \leq 1$. Then $c_0(\Delta^2, M, p) \subset c_0(\Delta^2 M)$,

(6.3.5) Let $1 \leq p_k \leq \sup p_k < \infty$. Then $c_0(\Delta^2, M) \subset c_0(\Delta^2, M, p)$. 
Theorem 6.3.6. Let \( M=(M_k) \) be a sequence of Orlicz functions. Then the following statements are equivalent:

(6.3.6) \( \ell_\infty(\Delta^2) \subseteq \ell_\infty(M, \Delta^2) \);

(6.3.7) \( c_0(\Delta^2) \subseteq \ell_\infty(M, \Delta^2) \);

(6.3.8) \( \sup_{k} M_k(t/p) < \infty, \ (t, p > 0) \).

Theorem 6.3.7. The following statements are equivalent for a sequence of Orlicz functions \( M=(M_k) \)

(6.3.9) \( c_0(M, \Delta^2) \subseteq c_0(\Delta^2) \);

(6.3.10) \( c(M, \Delta^2) \subseteq \ell_\infty(\Delta^2) \);

(6.3.11) \( \inf_{k} M_k(t/p) > 0, \ (t, p > 0) \).

Theorem 6.3.8. The inclusion \( \ell_\infty(M, \Delta^2) \subseteq c_0(\Delta^2) \) holds if, and only if,

(6.3.12) \( \lim_{k} M_k(t/p) = \infty, \ (t, p > 0) \).

Theorem 6.3.9. The inclusion \( \ell_\infty(\Delta^2) \subseteq c_0(M, \Delta^2) \) holds if, and only if,

(6.3.13) \( \lim_{k} M_k(t/p) = 0, \ (t, p > 0) \).
6.4. Proof of Theorems:

Proof of Theorem 6.3.1.

Proof: The proof is similar to the proof of Theorem 6.3.2 given below:

Proof of Theorem 6.3.2.

Proof: Let \((x^i_k)\) be a Cauchy sequence in \(\ell_\infty(\Delta^2, M, p)\). Let \(r\) and \(x_0\) be fixed. Then for each \(\frac{\varepsilon}{rx_0} > 0\), there exists a positive integer \(N\) such that \(G(x^i_k - x^{j}_k) < \frac{\varepsilon}{rx_0}\), for all \(i, j \geq N\). Using the definition of paranorm, we get

\[
\left\{ \sup_{k \geq 0} \left[ M \left[ \frac{|x^i_k - x^{j}_k|}{G(x^i_k - x^{j}_k)} \right]^p \right]^{1/H} \right\} \leq 1, \text{ for all } i, j \geq N.
\]

Thus

\[
\sup_{k \geq 0} \left[ M \left[ \frac{|x^i_k - x^{j}_k|}{G(x^i_k - x^{j}_k)} \right]^p \right] \leq 1, \text{ for all } i, j \geq N.
\]

It follows that

\[
M \left[ \frac{|x^i_k - x^{j}_k|}{G(x^i_k - x^{j}_k)} \right] \leq 1, \text{ for each } k \geq 0, \text{ and for all } i, j \geq N.
\]

For \(r > 0\) with \(M \left[ \frac{rx_0}{2} \right] \geq 1\), we have
This implies that 
\[ |\Delta^2 x_i - \Delta^2 x_j| \leq \frac{rx_o}{2} \leq \varepsilon. \]

Hence \((\Delta^2 x_i)\) is a Cauchy sequence in \(\mathbb{R}\).

Therefore, for each \(\varepsilon (0 < \varepsilon < 1)\), there exists a positive integer \(N\) such that 
\[ |\Delta^2 x_i - \Delta^2 x_j| < \varepsilon, \quad \text{for all } i, j \geq N. \]

Using the continuity of \(M\), we can find that
\[
\left\{ \sup_{k\geq N} \left[ M \left[ \frac{|\Delta^2 x_i^j - \Delta^2 x_i^j|}{\rho} \right] \right]^{p_k} \right\}^{1/H} \leq 1.
\]

Thus
\[
\left\{ \sup_{k\geq N} \left[ M \left[ \frac{|\Delta^2 x_i^j - \Delta^2 x_i^j|}{\rho} \right] \right]^{p_k} \right\}^{1/H} \leq 1.
\]

Taking infimum of such \(\rho\)'s we get
\[
\inf \left\{ \rho^{p_n/H} : \left\{ \sup_{k\geq N} \left[ M \left[ \frac{|\Delta^2 x_i^j - \Delta^2 x_i^j|}{\rho} \right] \right]^{p_k} \right\}^{1/H} \leq 1 \right\} < \varepsilon,
\]
for all \(i \geq N\) and \(j \to \infty\). Since \((x^i) \in \ell_\infty(\Delta^2, M, p)\) and \(M\) is continuous, it follows that \(x \in \ell_\infty(\Delta^2, M, p)\).

This completes the proof of the theorem.
Proof of Theorem 6.3.3.

Proof of (6.3.3): Let \( x \in \ell_\infty(\Delta^2) \), this implies that \( |\Delta^2 x_k| \leq N \), for all \( k \), so that

\[
M \left[ \frac{|\Delta^2 x_k|}{\rho} \right] \leq M \left[ \frac{N}{\rho} \right] \leq K / M(N),
\]

by \( \Delta_2 \) - condition.

Hence \( \sup_{k \geq 0} M \left[ \frac{|\Delta^2 x_k|}{\rho} \right] < \infty \). This shows that \( \ell_\infty(\Delta^2) \subseteq (\Delta^2, M) \).

Proof of conditions (6.3.1) and (6.3.2) follow in similar way.

Proof of Theorem 6.3.4.

Proof. Let \( x \in c_0(\Delta^2, M, p) \). Then, there exists some \( \rho > 0 \) such that

\[
\lim_{k \to \infty} \left[ M \left[ \frac{|\Delta^2 x_k|}{\rho} \right]^p \right]^q_k = 0.
\]

This implies that

\[
M \left[ \frac{|\Delta^2 x_k|}{\rho} \right] \leq 1,
\]

for sufficiently large \( k \), since \( M \) is non-decreasing.

Hence, we get

\[
\lim_{k \to \infty} \left[ M \left[ \frac{|\Delta^2 x_k|}{\rho} \right]^q_k \right] \leq \lim_{k \to \infty} \left[ M \left[ \frac{|\Delta^2 x_k|}{\rho} \right]^p_k \right] = 0,
\]
Proof of Theorem 6.3.5.

Proof of (6.3.4): Let \( x \in c_0(\Delta^2 M, p) \), that is,

\[
\lim_{k \to \infty} \left[ M \left| \frac{\Delta^2 x_k}{\rho} \right| \right]^{p_k} = 0.
\]

Since \( 0 < \inf p_k \leq p_k \leq 1 \), it follows that

\[
\lim_{k \to \infty} \left[ M \left| \frac{\Delta^2 x_k}{\rho} \right| \right] \leq \lim_{k \to \infty} \left[ M \left| \frac{\Delta^2 x_k}{\rho} \right| \right]^{p_k} = 0,
\]

and hence \( x \in c_0(\Delta^2, M) \).

Proof of (6.3.5): Let \( p_k \geq 1 \), for each \( k \) and \( \sup p_k < \infty \). Let \( x \in c_0(\Delta^2, M) \). Then, for each \( \varepsilon (0 < \varepsilon < 1) \), there exists a positive integer \( N \) such that

\[
\lim_{k \to \infty} \left[ M \left| \frac{\Delta^2 x_k}{\rho} \right| \right] \leq \varepsilon < 1,
\]

Since \( 1 \leq p_k \leq \sup p_k < \infty \), we have

\[
\lim_{k \to \infty} \left[ M \left| \frac{\Delta^2 x_k}{\rho} \right| \right]^{p_k} \leq \lim_{k \to \infty} \left[ M \left| \frac{\Delta^2 x_k}{\rho} \right| \right]^{p_k} \leq \varepsilon < 1.
\]

Therefore \( x \in c_0(\Delta^2, M, p) \).

This completes the proof of the theorem.
Proof of Theorem 6.3.6.

Proof : (6.3.6) \Rightarrow (6.3.7) is obvious, since \( c_0(\Delta^2) \subseteq \ell_\infty(\Delta^2) \).

For the proof of the implication (6.3.7) \Rightarrow (6.3.8), let \( c_0(\Delta^2) \subseteq \ell_\infty(M, \Delta^2) \) and suppose that (6.3.8) is not satisfied. Then \( \sup_k M_k [t/\rho] = \infty \), for all \( t, \rho > 0 \) and therefore, there is an index sequence \( (k_i) \) such that

\[
\sum_{n=1}^{i-1} \sum_{m=1}^{n-1} \frac{1}{m} > i, \quad (i = 1, 2, 3, \ldots).
\]

Define \( x = (x_{k_i}) \) with

\[
x_{k_i} = \begin{cases} \sum_{n=1}^{i-1} \sum_{m=1}^{n-1} \frac{1}{m} & \text{for } k_i = k_i \ (i = 1, 2, 3, \ldots), \\ 0 & \text{otherwise}. \end{cases}
\]

Then \( x \in c_0(\Delta^2) \). But by (6.3.14) \( x \notin \ell_\infty(M, \Delta^2) \) which contradicts (6.3.7). Hence (6.3.8) must hold. For (6.3.8) \Rightarrow (6.3.6), let (6.3.8) be satisfied and \( x \in \ell_\infty(\Delta^2) \). If we suppose that \( x \notin \ell_\infty(M, \Delta^2) \), then

\[
\sup_k M_k \left[ \frac{|\Delta^2 x_k|}{\rho} \right] = \infty, \quad \text{for } \Delta^2 x \in \ell_\infty \ \text{(i.e. } x \in \ell_\infty(\Delta^2) \text{)}.
\]

Let \( t = |\Delta^2 x| \). Then, \( \sup_k M_k [t/\rho] = \infty \), which contradicts (6.3.8). Hence \( \ell_\infty(\Delta^2) \subseteq \ell_\infty(M, \Delta^2) \).

This completes the proof of the theorem.
Proof of Theorem 6.3.7.

Proof of (6.3.9) $\Rightarrow$ (6.3.11): Let $c_0(M, \Delta^2) \subseteq \ell_\infty(\Delta^2)$ and suppose that (6.3.11) does not hold. Then

\[(6.3.15) \quad \inf \, M_k \{t/\rho\} = 0 \quad (t, \rho > 0).\]

We can choose an index sequence $(k_i)$ such that

\[M_{k_i} \left[ \frac{1^3}{\rho} \right] < \frac{1}{i} \quad (i = 1, 2, \ldots).\]

Define the sequence $x = (x_k)$ by

\[x_k = \begin{cases} 1^3 & \text{for } k = k_i \,(i = 1, 2, \ldots), \\ 0 & \text{otherwise}. \end{cases}\]

Then, by (6.3.15), $x \in c_0(M, \Delta^2)$. But $x \notin \ell_\infty(\Delta^2)$ which contradicts (6.3.10). Hence (6.3.11) must hold. For (6.3.11) $\Rightarrow$ (6.3.9), let (6.3.11) holds and $x \in c_0(M, \Delta^2)$ then,

\[(6.3.16) \quad \lim_k M_k \left[ \frac{\Delta^2 x_k}{\rho} \right] = 0.\]

Suppose that $x \notin c_0(\Delta^2)$. Then, for some number $\varepsilon_0 > 0$ and index $k_0$, we have $|\Delta^2 x_k| > \varepsilon_0$, $(k \geq k_0)$. Therefore,

\[M_k \left[ \frac{\varepsilon_0}{\rho} \right] \leq M_k \left[ \frac{\Delta^2 x_k}{\rho} \right] \quad (k \geq k_0),\]

and consequently by (6.3.16), we get

\[\lim_k M_k \left[ \frac{\varepsilon_0}{\rho} \right] = 0\]
which contradicts (6.3.11). Hence \( c_0(M, \Delta^2) \subseteq c_0(\Delta^2) \).

This completes the proof of the theorem.

**Proof of Theorem 6.3.8.**

**Proof.** Let \( \ell_\infty(M, \Delta^2) \subseteq c_0(\Delta^2) \). Suppose that (6.3.12) is not satisfied. Then, there is a number \( t_0 > 0 \) and an index sequence \((k_i)\), such that

\[
M_{k_i} \left( \frac{t_0 \rho}{\rho} \right) \leq N < \infty.
\]

Define the sequence \( x = (x_k) \) by

\[
x_k = \begin{cases} 
t_0 i^2 & \text{for } k = k_i (i = 1, 2, \ldots), \\ 0 & \text{otherwise}.\end{cases}
\]

Then \( x \in \ell_\infty(M, \Delta^2) \), by (6.3.17). But \( x \not\in c_0(\Delta^2) \), so that (6.3.12) must hold for \( \ell_\infty(M, \Delta^2) \subseteq c_0(\Delta^2) \).

Conversely, let (6.3.12) be satisfied. If \( x \in \ell_\infty(M, \Delta^2) \), then

\[
M_k \left( \frac{|\Delta^2 x_k|}{\rho} \right) \leq N < \infty, \quad (k = 1, 2, 3, \ldots).
\]

Suppose that \( x \not\in c_0(\Delta^2) \). Then, for some number \( \varepsilon_0 > 0 \) and index \( k_0 \), \( |\Delta^2 x_k| \geq \varepsilon_0 \) (\( k \geq k_0 \)). Therefore,

\[
M_k \left( \frac{\varepsilon_0}{\rho} \right) \leq M_k \left( \frac{|\Delta^2 x_k|}{\rho} \right) \leq N, \quad (k \geq k_0),
\]
which contradicts (6.3.12). Hence $x \in c_0(\Delta^2)$, i.e. $\ell_\infty(M,\Delta^2) \subseteq c_0(\Delta^2)$.

This completes the proof of the theorem.

**Proof of Theorem 6.3.9.**

**Proof.** Let $\ell_\infty(\Delta^2) \subseteq c_0(M,\Delta^2)$. Suppose that (6.3.13) does not hold. Then, for some $t_0 > 0$,

$$\lim_{k} M_k \left[ t_0 / \rho \right] = l \neq 0.$$  \hfill (6.3.18)

Define $x = (x_k)$ by $x_k = - t_0 k$, \hspace{1em} ($k = 1, 2, 3 \ldots$).

Then $x \notin c_0(M,\Delta^2)$, by (6.3.18). Hence (6.3.13) must hold.

Conversely, suppose that (6.3.13) holds and $x \in \ell_\infty(\Delta^2)$. Then $|\Delta^2 x_k| 
 \leq N < \infty$, \hspace{1em} ($k = 1, 2, \ldots$). Therefore,

$$M_k \left[ \frac{|\Delta^2 x_k|}{\rho} \right] \leq \lim_{k} M_k \left[ N / \rho \right], \hspace{1em} (k = 1, 2, \ldots),$$

and

$$\lim_{k} M_k \left[ \frac{|\Delta^2 x_k|}{\rho} \right] \leq \lim_{k} M_k \left[ N / \rho \right] = 0,$$

by (6.3.13). Hence $x \in c_0(M,\Delta^2)$.

This completes the proof of the theorem.