CHAPTER - 5

INVARIANT MEANS AND MATRIX TRANSFORMATIONS
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5.1. Definitions and notations

Here we use the following definitions and notations in addition to those mentioned in Chapters 0, 1 and 3.

The Köthe-Toeplitz dual of the spaces, $\ell(p)$, $\ell_{\infty}(p)$ and $c_{0}(p)$ are given by:

$$ (\ell(p))^\alpha := \{a \in \omega : \sum_{k} |a_k|^{q_k} N^{-q_k/p_k} < \infty \}; $$

$$ (\ell_{\infty}(p))^\alpha := \bigcap_{N>1} \{a \in \omega : \sum_{k} |a_k| N^{-1/p_k} < \infty \}; $$

$$ (c_{0}(p))^\alpha := \bigcup_{N>1} \{a \in \omega : \sum_{k} |a_k| N^{-1/p_k} < \infty \}; $$

where $\frac{1}{p_k} + \frac{1}{q_k} = 1$ and $N>1$ is an integer.

The space $\ell(p)$ was extended to $\ell(p,s)$ for $s \geq 0$ by Bulut and Cakar [10] who determined the Köthe-Toeplitz dual of $\ell(p,s)$ defined by:

$$ (\ell(p,s))^\alpha := \{a \in \omega : \sum_{k} |a_k|^{q_k} N^{-q_k/p_k} < \infty \}. $$

Metin Basarir [7] extended the spaces $c_{0}(p)$, $\ell_{\infty}(p)$ and $c(p)$ to $\ell_{\infty}(p,s)$, $c_{0}(p,s)$ and $c(p,s)$ respectively for $s \geq 0$. Using the same kind of arguments as in Maddox [43], he [7] showed that the necessary and sufficient condition for the above sequence spaces to be linear is $p \in \ell_{\infty}$ and $c_{0}(p,s)$ is paranormed by:

$$ g(x) = \sup_{k} (k^{-s/M} |x_k|^{p_k/M}), $$
where \( M = \max(1, H) \), \( H = \sup_k p_k \).

Also \( \ell_\infty(p,s) \) and \( c(p,s) \) are paranormed by \( g(x) \) if, and only if, \( \inf p_k > 0 \).

All the spaces defined above are complete in their topologies.

Let \( U \) be the space of all sequences \( u \in \omega \), such that \( u_k \neq 0 \) \((k=1, 2, \ldots)\). For \( u \in U \), we define the sequence \( u^{-1} \) by

\[
(u_k)^{-1} = \frac{1}{u_k}, \quad k=1,2,\ldots,
\]

and for \( E \subseteq \omega \), \( u \in U \), we put

\[
(5.1.1) \quad E_u = \{ x \in \omega : xu \in E \}
\]

Obviously

\[
(5.1.2) \quad (E_u)^\alpha = (E^\alpha)_u^{-1}
\]

If \( u=(u_k) \) is defined by

\[
(5.1.3) \quad u_k = \frac{1}{k^{s/p_k}}, \quad s \geq 0; \quad k=1,2,\ldots
\]

Then, from (5.1.1), we have

\[
\ell(p,s) = (\ell(p))_u^\alpha, \quad c_0(p,s) = (c_0(p))_u^\alpha, \quad \ell_\infty(p,s) = (\ell_\infty(p))_u^\alpha,
\]

and by (5.1.2)

\[
(5.1.4) \quad (\ell(p,s))^\alpha = (\ell(p))_{u^{-1}}^\alpha = \{ a \in \omega : \sum_k a_k \{ k^{p_k/q_k} \}_{N=q_k/p_k<\infty} \},
\]

\[
(5.1.5) \quad (c_0(p,s))^\alpha = (c_0(p))_{u^{-1}}^\alpha = \bigcup_{N>1} \{ a \in \omega : \sum_k a_k \{ k^{s/p_k} N^{-1/p_k<\infty} \} \},
\]
Throughout this chapter we shall use the notation \( a(n,k) \) to denote the element \( a_{nk} \) of the matrix \( A \) and we write for all integers \( m,n \geq 1 \),

\[
t_{mn}(Ax) = \frac{(Ax_n + TAx_n + \ldots + T^mAx_n)}{(m+1)} = \sum_k t(n,k,m) x_k,
\]

where \( t(n,k,m) = \frac{1}{m+1} \sum_{j=0}^{m} a(\sigma^j(n), k) \).

### 5.2. Introduction

Schaefer [67] introduced the concept of \( \sigma \)-conservative, \( \sigma \)-regular and \( \sigma \)-coercive matrices and characterized the classes of matrices \( (c, \mathcal{C}) \), \( (c, \mathcal{C})_{reg} \) and \( (\ell_\infty, \mathcal{C}) \). He proved the following theorems:

**Theorem A.** \( A \in (c, \mathcal{C}) \) i.e. \( A \) is \( \sigma \)-conservative if, and only if,

\[
(5.2.1) \quad ||A|| = \sup_n \sum_k |a_{nk}| < + \infty,
\]

\[
(5.2.2) \quad a_k = \{a_{nk}\}_{n=1}^\infty \in \mathcal{C}, \quad \text{for each } k, \text{ and}
\]

\[
(5.2.3) \quad a = \{\sum_k a_{nk}\}_{n=1}^\infty \in \mathcal{C}.
\]

In this case, the \( \sigma \)-limit of \( Ax \) is

\[
(\lim x) [u - \sum_k u_k] + \sum_k x_k u_k, \quad \text{for every } x \in c,
\]

where

\[
u_k = \sigma \text{-lim } a(k) \text{ and } u = \sigma \text{-lim } a.
\]
Theorem B. \( A \in (c, \mathfrak{c})_{\text{reg}} \), i.e. \( A \) is \( \sigma \)-regular if, and only if, conditions (5.2.1) and (5.2.2) with \( u_k = 0 \), for each \( k \) and (5.2.3) with \( u = 1 \), hold.

Theorem C. \( A \in (\ell_\infty, \mathfrak{c}) \), i.e. \( A \) is \( \sigma \)-coercive if, and only if,

\[
\| A \| < \infty,
\]

\[
a_{(k)} \in \mathfrak{c}, \text{ for each } k \text{ and}
\]

\[
\lim_{m} \sum_{k} \left| \sum_{j=0}^{m} [a^j(n), k - u_k] \right| / (1+m) = 0,
\]

uniformly in \( n \), where \( u_k = \sigma \)-lim \( a_{(k)} \).

In this case, the \( \sigma \)-limit of \( Ax \) is \( \sum_{k} u_k x_k \), for every \( x = (x_k) \in \ell_\infty \).

Generalizing the above results of Schaefer, in 1979, Ahmad and Mursaleen [3] characterized the classes of matrices, \( (c(p), \mathfrak{c})_{\text{reg}} \) and \( (\ell_\infty(p), \mathfrak{c}) \) and gave the following theorems:

Theorem D. Let \( p_k > 0 \) for each \( k \). Then \( A \in (\ell_\infty(p), \mathfrak{c}) \) if, and only if,

\[
\sup_{m} \sum_{k} |t(n,k,m)| < \infty,
\]

\[
a_k \in \mathfrak{c}, \text{ for each } k \text{ and}
\]

\[
\lim_{m} \sum_{k} |t(n,k,m) - u_k| N^{1/p_k} = 0, \text{ uniformly in } n.
\]

In this case, the \( \sigma \)-limit of \( Ax \) is \( \sum_{k} u_k x_k \) for every \( x \in \ell_\infty(p) \).

Theorem E. Let \( p \in \ell_\infty \). Then \( A \in (c(p), \mathfrak{c}) \) if, and only if, there exists an integer \( B > 1 \), such that, for each \( n \),
(5.2.10) \( C_n = \sup_m \sum_k |t(n,k,m)| B^{-1/p_k} < \infty \).

(5.2.11) \( a_k \in \mathbb{C} \) for each \( k \), and

(5.2.12) \( a \in \mathbb{C} \).

In this case, the \( \sigma \)-limit of \( Ax \) is \( \lim(x) [u - \sum_k u_k] + \sum_k u_k x_k \), for every \( x \in \mathbb{C}(p) \), where \( u = \sigma \)-lim \( a \) and \( u_k = \sigma \)-lim \( a_{(k)} \).

**Theorem F.** \( A \in (\mathbb{C}(p), \mathbb{C}) \) if, and only if, conditions (5.2.7), (5.2.8) with \( u_k = 0 \) for each \( k \), and (5.2.9) with \( u = 1 \), hold.

Our aim in this chapter is to characterize the classes of matrices \( (\ell_\infty(p,s), \mathbb{C}), (\mathbb{C}(p,s), \mathbb{C}), (c_0(p,s), \mathbb{C}) \) which are the generalizations of the above mentioned results of [54] and Ahmad and Mursaleen [3].

**5.3. Main results**

We prove the following theorems.

**Theorem 5.3.1.** \( A \in (\ell_\infty(p,s), \mathbb{C}) \) if, and only if,

(5.3.1) \( \sup_m \sum_k |t(n,k,m)| N^{1/p_k} k^{s/p_k} < \infty \),

uniformly in \( n \) for every integer \( N > 1 \),

(5.3.2) \( a_k = \{a_{nk}\}_{n=1}^\infty \in \mathbb{C} \), for each \( k \),

i.e. \( \lim_m t(n,k,m) = u_k \) uniformly in \( n \) for each \( k \).

In this case the \( \sigma \)-limit of \( Ax \) is

\[
\lim_m \sum_{k=1}^\infty t(n,k,m) = \sum_{k=1}^\infty u_k x_k.
\]
Theorem 5.3.2. Let \( p \in \ell_\infty \). Then \( A \in (c(p,s), \mathcal{C}) \) if, and only if, there exists an absolute constant \( B > 1 \), such that,

\[
\begin{align*}
D &= \sup_m \left( \sum_k |t(n,k,m)| B^{-1/p_k} k^{s/p_k} \right) < \infty, \text{ uniformly in } n. \\
\lim_{m \to \infty} t(n,k,m) &= u_k, \text{ for each } k, \text{ uniformly in } n. \\
\lim_{m \to \infty} \sum_k t(n,k,m) &= u, \text{ uniformly in } n.
\end{align*}
\]

5.4. Proof of theorems

Proof of Theorem 5.3.1.

Necessity. For necessity of (5.3.1), suppose that \( A \in (\ell_\infty(p,s), \mathcal{C}) \) but there is an integer \( N > 1 \), such that, (5.3.1) is false. Then the matrix \( B = (t(n,k,m) N^{1/p_k} k^{s/p_k}) \) (see Shaefer [67]), so there is \( x \in \ell_\infty \) with \( \| x \| = 1 \), such that \( \{ t_{m,n} (Bx) \} \) \( \in \mathcal{C} \). Hence, although \( y \in (N^{1/p_k} k^{s/p_k} x_k) \in \ell_\infty(p,s) \), the sequence \( \{ t_{m,n} (Ay) \} \notin \mathcal{C} \). This contradicts the fact that \( A \in (\ell_\infty(p,s), \mathcal{C}) \) and completes the proof.

Necessity of (5.3.2) is obtained by taking \( x = e_k \in \ell_\infty(p,s) \), where \( e_k = (0,0,0,1,0,\ldots) \) such that 1 appears at the kth place.

Sufficiency. Suppose that the conditions hold. Take an integer \( N > 1 \), such that,

\[
N > \max \left\{ 1, \sup_k (k^{-s} |x_k|^{p_k}) \right\}.
\]

Then, we have

\[
\left| \sum_k (t(n,k,m) - u_k) x_k \right| > \sum_k |t(n,k,m) - u_k| N^{1/p_k} k^{s/p_k}.
\]
Hence, by (5.3.1) and (5.3.2) we have

$$\lim_{m \to \infty} \sum_{k} t(n,k,m) x_k = \sum_{k} u_k x_k.$$  

This completes the proof of the theorem.

**Proof of Theorem 5.3.2.**

**Necessity.** Let $A \in (c(p,s), c)$ and put $t_{mn}(Ax) = \sigma_{mn}(x)$. Since $(c(p,s), c) \subseteq (c_0(p,s), c)$ and $\{\sigma_{mn}(x)\}_m$ is a sequence of continuous linear functionals on $c_0(p,s)$ (see [7], Theorem 1), such that $\lim_{n} \sigma_{mn}(x)$ exists for each $m$. Therefore, by uniform boundedness principle for $0 < \delta < 1$, there exists a sphere $S_\delta[0] \subseteq c_0(p,s)$ and a constant $K$ such that $\sigma_{m,n}(x) \leq K$, for each $n$ and $x \in S_\delta[0]$. Define for each $r$.  

$$\gamma^r_n = \begin{cases} 
\delta^{M/p_k} \text{sgn } t(n,k,m) k^{s/p_k} & 0 \leq k \leq r, \\
0 & r < k.
\end{cases}$$  

where $M = \max \{1, \sup_k p_k\}$.

Now $\gamma^r_n \in S_\delta[0]$ and $\sum_{k=1}^{r} |t(n,k,m)| B^{-1/p_k} k^{s/p_k} \leq K$, for each $n$, and $r$ where $B = \delta^{-M}$. Therefore, (5.3.3) holds and conditions (5.3.4) and (5.3.5) trivially hold.

**Sufficiency.** Suppose that conditions (5.3.3) - (5.3.5) hold and $x \in c(p,s)$. Then, there exists $l$ such that,

$$k^{-s} |x_k - l|^{p_k} \to 0.$$  

Hence, for $0 < \varepsilon < 1$ there exists $k_0$ such that, for all $k > k_0$. 


and therefore, for \( k > k_0 \),

\[
B^{1/p_k} \left| x_k - l \right| < B^{M/p_k} \left| x_k - l \right| < \frac{\varepsilon^{M/p_k}}{(2D+1)} < \frac{\varepsilon}{(2D+1)} .
\]

By (5.3.3) and (5.3.4) we have

\[
\sum_k |t(n,k,m) - u_k| B^{-1/p_k} k^{-s/p_k} < 2D,
\]

Hence

\[
\sum_{k > k_0} |(t(n,k,m) - u_k) (x_k - l)| < \varepsilon .
\]

Also

\[
\sum_{k \leq k_0} |(t(n,k,m) - u_k) (x_k - l)| \to 0, \text{ as } m \to \infty .
\]

Therefore, we have

\[
\lim_{m \to \infty} \sum_k t(n,k,m) x_k = lu + \sum_k u_k(x_k - l),
\]

and this completes the proof of the theorem.

5.5. Corollaries:

**Corollary 5.5.1.** \( A \in (\ell_{\infty}(s), \ell_{\infty}(\delta)) \) if, and only if,

\[
(5.5.1) \quad \sup_m \sum_k k^s |t(n,k,m)| < \infty, \text{ uniformly in } n;
\]

\[
(5.5.2) \quad \lim_{m \to \infty} t(n,k,m) = u_k, \text{ uniformly in } n, \text{ for each } k.
\]
**Remark:** If we take \( s = 0 \) in Corollary 5.5.1, then it reduces to Theorem B of Schaefer [67].

**Corollary 5.5.2.** Let \( p \in \ell_\infty \). Then \( A \in (c(p,s), c(P)) \) if, and only if,

(5.5.3) condition (5.3.3) of Theorem 5.3.2 holds,

(5.5.4) \( \lim_{m \to \infty} t(n,k,m) = 0 \), for each \( k \), uniformly in \( n \),

(5.5.5) \( \lim_{m \to \infty} \sum_k t(n,k,m) = 1 \), uniformly in \( n \).

**Proposition:** \( (\ell_\infty(p,s), c) \cap (c(p,s), c(P)) = \phi. \)

**Corollary 5.5.3.** Let \( p \in \ell_\infty \). Then \( A \in (c(p,s), c) \) if, and only if, the conditions (5.3.3) and (5.3.4) of Theorem 5.3.2 hold.

**Remark:** If we take \( p_k = 1 \) for all \( k \) and \( s = 0 \), then Theorem 5.3.2 reduces to Theorem A of Schaefer [67].