SUMMARY

The thesis consists of nine chapters. Chapter 0 has been devoted to recall some conventions regarding notations and definitions used throughout the thesis. Chapter I is introductory and contains a résumé of hitherto known results which have direct interconnection with our investigations.

Let \( z \) be any sequence and let \( Y \) be any subset of \( w \). Then we shall write

\[
\mathcal{z}^{-1}.Y := \{ x \in w : zx = (z_kx_k) \in Y \}.
\]

For any subset \( X \) of \( w \), the \( \alpha, \beta \)-duals of \( X \) are defined by:

\[
X^\alpha = \bigcap_{x \in X} (x^{-1} \cdot \ell_1) \quad \text{and} \quad X^\beta = \bigcap_{x \in X} (x^{-1} \cdot cs).
\]

We define the linear operators \( \Delta, \Delta^2, \Delta^{-2} : w \to w \) by

\[
\Delta x = (\Delta x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty,
\]

\[
\Delta^2 x = (\Delta^2 x_k)_{k=1}^\infty = (\Delta x_k - \Delta x_{k+1})_{k=1}^\infty,
\]

and

\[
\Delta^{-2} x = (\Delta^{-2} x_k)_{k=1}^\infty = \left( \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} x_i \right)_{k=1}^\infty.
\]

Let \( U \) be the set of all sequences \( u = (u_k) \) such that \( u_k \neq 0 \) \((k = 1, 2, \ldots)\). We define the sets

\[
E(u; \Delta^2) = (u^{-1}.E)(\Delta^2) := \{ x \in w : (u_k \Delta^2 x_k)_{k=1}^\infty = 1 \in E \};
\]

and

\[
E(p; u, \Delta^2) = (u^{-1}.E(p))(\Delta^2) := \{ x \in w : (u_k \Delta^2 x_k)_{k=1}^\infty = 1 \in E(p) \}.
\]

We define the operator \( S : E(p; u, \Delta^2) \to E(p; u, \Delta^2) \) by \( x \to Sx = (0, 0, x_3, x_4, \ldots) \).

In Chapter II we prove the following theorems.
Theorem 2.1. Let \( u \in U \). Then \([SE(p; u, \Delta^2)]^\alpha = M_\alpha(p)\), where for a constant \( M > 1 \),
\[
M_\alpha(p) := (\Delta^{-2}(M^{-1/p}/|u|))^{-1} \cdot \ell_1
\]
\[
:= \left\{ a \in w : \sum_{k=1}^{\infty} a_k \left| \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} M^{-1/p_i}/|u_i| \right| < \infty \right\}.
\]

Theorem 2.2. Let \( u \in U \). We write,
\[
M_\beta(p) := \left\{ a \in (\Delta^{-2}(B^{1/p}/|u|))^{-1} \text{ cs : } R \in B^{-1/p}u \cdot \ell_1 \right\}
\]
\[
:= \left\{ a \in w : \sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} B^{1/p_i}/|u_i| \right. \text{ converges}
\]
\[
\left. \text{ and } \sum_{k=1}^{\infty} R_k \left| \frac{B^{-1/p_k}}{|u_k|} \right| < \infty \right\}.
\]
for real \( B > 0 \), where, for all \( a \in \text{cs} \),
\[
\gamma_k = \sum_{v=k+1}^{\infty} a_v \quad \text{and} \quad R_k = \sum_{m=k+1}^{\infty} \gamma_m.
\]

Then
(a) \([SE(p; u, \Delta^2)]^\beta = M_\beta(p)\), for \( E = \ell_\infty \), or \( c \).
(b) If \( 1/u \in \ell_1 \), then \([SE(p; u, \Delta^2)]^\beta = M_\beta(p)\), for \( E = c_0 \).
(c) If \( 1/u \in \ell_\infty \setminus \ell_1 \), then \([SE(p; u, \Delta^2)]^\beta \neq M_\beta(p)\), for \( E = c_0 \).

Theorem 2.3. \( A \in (E, F(p; u, \Delta^2)) \) if and only if
(i) \( \sum_k |a_{nk}| < \infty \), for each \( n \);
(ii) \( C \in (E, F(p)) \),
where \( C = (c_{nk}) = ((\Delta a_{nk} - \Delta a_{n-1,k})u_k) \).

Theorem 2.4. \( A \in (F(p; u, \Delta^2), E) \) if and only if
(i) \((a_{n1}) \) and \((a_{n2}) \in F(p; u, \Delta^2) \);
(ii) \( \sum_{k=1}^{\infty} a_{nk} \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} B^{1/p_i}/|u_i|^{-1} < \infty, (B > 1) \), for each \( n \);
(iii) $\sum_{k=1}^{\infty} |R_{nk}| B^{-1/p_k} |u|^{-1} < \infty, (B > 1)$, for each $n$;

(iv) $R u^{-1} \in (F(p), E)$,

where $R = (R_{nk}) = (\sum_{m=k+1}^{\infty} \gamma_{nm})$ and $\gamma_{nm} = \sum_{i=m+1}^{\infty} a_{ni}$.

**Theorem 2.5.** $A \in (E(p; u, \Delta^2), F(p; u, \Delta^2))$ if and only if

(i) $\sum_{k} a_{nk} \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} B^{1/p_i} |u_i|^{-1} < \infty, (B > 1)$, for each $n$;

(ii) $\sum_{k} |R_{nk}/u_k| B^{-1/p_k} < \infty, (B > 1)$, for each $n$;

(iii) $R u^{-1} \in (E(p), F(p))$.

(iv) $(c_{n1})$ and $(c_{n2}) \in E(p; u, \Delta^2)$,

where $R_{nk} = (R_{nk}), C = (c_{nk})$, with

$$R_{nk} = \sum_{m=k+1}^{\infty} \sum_{i=m+1}^{\infty} c_{ni} \quad \text{and} \quad c_{nk} = (\Delta^2 a_{nk}) u_k = (\Delta a_{nk} - \Delta a_{n-1,k}) u_k.$$

In Chapter III we prove some theorems relating to $\sigma$-convergence. We recall the following:

$c^\sigma := \{x \in \ell_\infty : \lim_{m \to \infty} t^\sigma_{mn}(x) = L, \text{ uniformly in } n, L = \sigma\text{-lim } x\}$,

where

$$t^\sigma_{mn}(x) = (x_n + T x_n + \cdots + T^m x_n)/(m+1) = \frac{1}{m+1} \sum_{j=0}^{m} x_{\sigma^j(n)}.$$

$\hat{c} := \{x \in \ell_\infty : \lim_{m \to \infty} \hat{t}_{mn}(x) = L', \text{ uniformly in } n, L' = B\text{-lim } x\}$,

where

$$\hat{t}_{mn}(x) = (x_n + x_{n+1} + \cdots + x_{n+m})/(m+1) = \frac{1}{m+1} \sum_{j=0}^{m} x_{n+j}.$$

We write $\hat{C} = \hat{c} A$ and $C^\sigma = c^\sigma A$, as iteration products of two spaces. We define the spaces associated with the summability method $C^\sigma$ as :

$$\ell(C^\sigma, p) := \{x \in \ell_\infty : \sum_{m} |t^\sigma_{mn}(Ax)|^p < \infty, \text{ uniformly in } n\},$$
and
\[ \ell(C^\sigma, p) := \{ x \in \ell_\infty : \sup_n \sum_m t_m^\sigma(Ax) |p^m < \infty \}, \]

where
\[ t_m^\sigma(Ax) = (Ax + T(Ax) + \cdots + T^m(Ax))/(m + 1) \]
\[ = \frac{1}{m+1} \sum_{k=1}^\infty \sum_{j=0}^\infty a(\sigma^j(n), k)x_k \]
\[ = \sum_k a(n, k, m)x_k \]

and
\[ \alpha(n, k, m) = \sum_{j=0}^m a(\sigma^j(n), k)/(m + 1). \]

The spaces:
\[ \ell(A, p) := \{ x \in \ell_\infty : \sum_n |A_m(x)| |p^m < \infty \}, \]

and
\[ \ell(A, p)_\infty := \{ x \in \ell_\infty : \sup_m |A_m(x)| |p^m < \infty \} \]

are known.

We prove:

**Theorem 3.1.** \( \ell(C^\sigma, p) \subset \ell(C^\sigma, p). \)

**Theorem 3.2.** \( \ell(C^\sigma, p) \subset \ell(A)_\infty, \) where \( \ell(A)_\infty \) is the special case of \( \ell(A, p)_\infty \) for \( p = 1 \) for all \( m. \)

**Theorem 3.3.** \( \ell(C^\sigma, p) \) is a linear topological space paranormed by:

\[ (3.3.1) \quad h(x) = \sup_n \left( \sum_m | \sum_k a(n, k, n)x_k |p^m \right)^{1/M}, \]

where
\[ M = \max(1, \sup p_m). \]

\( \ell(C^\sigma, p) \) is paranormed by (3.3.1) if \( \inf p_m > 0. \)

**Theorem 3.4.** Let \( 0 < p_m \leq q_m. \) Then \( \ell(C^\sigma, p) \subset \ell(C^\sigma, q). \)
Theorem 3.5. Let $b_{nk} > 0$. If

$$
\sup_n \sum_k |a_{nk}| \left( b_{nk} \right)^{1/p} < \infty
$$

and

$$
\sup_m \sum_k \alpha(n, k, m) \left( b_{nk} \right)^{-1/q} < \infty
$$

where $p^{-1} + q^{-1} = 1$. Then $A \in \ell_p(C^\sigma)$, $\tilde{\ell}_p(C^\sigma)$ being the special case of $\ell(C^\sigma, p)$, for $p_m = p$, for all $m$.

Theorem 3.6. Let $1 \leq p < \infty$. Then $A \in \ell_p(C^\sigma)$ if and only if

$$
\sum_m (\sum_k |\alpha(n, k, m)|)^p < \infty,
$$

uniformly in $n$, where $\ell_p(C^\sigma)$ is the special case of $\ell(C^\sigma, p)$, for $p_m = p$, for all $m$.

In Chapter IV we define some new sequence spaces of $\sigma$-bounded variation as given below.

Let us write, for $m \geq 1$,

$$
\sigma_{mn}^\sigma(x) = t_{mn}^\sigma(x) - t_{m-1,n}^\sigma(x).
$$

Then, we have

$$
\phi_{mn}^\sigma(x) = \frac{1}{m(m+1)} \sum_j \sum_{i=1}^\infty j \left[ x_{\sigma^i(n)} - x_{\sigma^{i-1}(n)} \right],
$$

and if, for a given infinite series $\sum z_n$, denoted by $z$,

$$
x_n = z_0 + z_1 + z_2 + \cdots + z_n.
$$

then, we also write

$$
\phi_{mn}^\sigma \equiv \sigma_{mn}^\sigma(z) = \frac{1}{m(m+1)} \sum_j \sum_{i=1}^\infty h_j \left[ \sum_{i=d_j} z_i \right],
$$

with $d_j = \sigma^{j-1}(n) + 1$ and $h_j = \sigma^j(n)$.

We define:
C(p,3) := \{ z^d : \sup_{m,n} | C^m n_s > 0 \};

\mathcal{C}(p,s) := \{ z \in C : \sum_{m=1}^{\infty} m^{-s} | \phi_m s | < \infty \}

uniformly in n, s \geq 0};

and

\tilde{\mathcal{C}}(p,s) := \{ z \in \mathcal{C}(p,s) : \sup_{n} \sum_{m=1}^{\infty} m^{-s} | \phi_m s | < \infty \}.

We establish the following theorems:

**Theorem 4.1.** \( \mathcal{C}(p,s) \subset \tilde{\mathcal{C}}(p,s) \).

**Theorem 4.2.** \( \mathcal{C}(p,s) \) is a complete linear topological space paranormed by:

\[ G^*(z) = \sup_{m} \left( \sum_{m=1}^{\infty} m^{-s} | \phi_m s | \right)^{1/M}, \]

where \( M = \max(1, \sup p_m) \).

**Theorem 4.3.** Suppose \( p \) is bounded away from 0. Then

(i) \( \tilde{\mathcal{C}}(p,s) \) is a complete linear topological space paranormed by the function \( G^* \) defined by (4.2.1);

(ii) \( \mathcal{C}(p,s) \) is a closed subspace of \( \tilde{\mathcal{C}}(p,s) \);

(iii) if, for all \( m; p_m \leq q_m \), then \( \mathcal{C}(p,s) \subset \mathcal{C}(q,s) \); and \( \tilde{\mathcal{C}}(p,s) \subset \tilde{\mathcal{C}}(q,s) \).

**Theorem 4.4.** If \( \inf_{p_m} > 0 \), then \( \mathcal{C}_\infty(p,s) \) is a complete linear topological space over the complex field \( \mathcal{C} \), paranormed by:

\[ g(z) = \sup_{m,n} \left\{ m^{-s/M} | \phi_m n_s(z) | p_m \right\}^{1/M}, \]

where \( M = \max(1, \sup_{m} p_m) \).

**Theorem 4.5.** Let \( 0 < p_m \leq q_m \), then \( \mathcal{C}_\infty(p,s) \) is a closed subspace of \( \mathcal{C}_\infty(p,s) \).
Theorem 4.6. $\mathcal{L}_\infty^0(p, s)$ is 1-convex.

Theorem 4.7. Let $p \in \ell_\infty$. Then $A \in (c_0(p), \mathcal{L}_\infty^0(p, s))$ if and only if there is an absolute constant $B > 1$, such that

$$D = \sup_{m,n} \left[ m^{-s/p_m} \sum_k |\alpha(n, k, m) | B^{-1/p_k} \right]^{p_m} < \infty.$$  

Theorem 4.8. $A \in (\ell_\infty(p), \mathcal{L}_\infty^0(s))$ if and only if for every integer $N > 1$,

$$\sup_{m,n} \left\{ m^{-s} \sum_k |\alpha(n, k, m) | N^{-1/p_k} \right\} < \infty,$$

where $\mathcal{L}_\infty^0(s)$ is the special case of $\mathcal{L}_\infty^0(p, s)$, for $p_m = p$, for all $m$.

Theorem 4.9. $A \in (\ell(p), \mathcal{L}_\infty^0(s))$ if and only if there exists an integer $N > 1$, such that

$$\sup_{m,n} \left\{ m^{-s} \sum_k |\alpha(n, k, m) | q_k N^{-1/q_k} \right\} < \infty,$$

for $1 < p_k < \infty, 1/p_k + 1/q_k = 1$; and

$$\sup_{m,n} \left\{ m^{-s} \sum_k |\alpha(n, k, m) | p_k \right\} < \infty,$$

for $0 < p_k \leq 1$.

In Chapter V we study the topological properties and mapping theorems for our new spaces $\mathcal{L}_\infty^0(s)$ and $\mathcal{L}_\infty^0(p, s)$ and prove some theorems.

For any sequence $x = (x_n) \in \mathcal{w}$ and for any given sequence $B = (B_i)$ of infinite matrices with $B_i = (b_{nk}(i))$, we write

$$t^B_n(x) = (B_i x)_n = \sum_k b_{nk}(i)x_k$$

and let $x_n$ be the $n$th partial sum of a given series $\sum z_n$ (denoted by $z$), so that

$$x_n = z_0 + z_1 + \cdots + z_n$$

and

$$x_n - x_{n-1} = z_n.$$
Then, for \( n \geq 0 \), we have

\[
\psi_{in}(z) = t_{in}^G(x) - t_{i,n-1}^G(x)
\]

\[
= \sum_{k=0}^{\infty} \left\{ b_{nk}(i) - b_{n-1,k}(i) \right\} x_k
\]

\[
= \sum_{k=0}^{\infty} \Delta b_{nk}(i) x_k
\]

\[
= \sum_{v=0}^{\infty} \left\{ \sum_{k=v}^{\infty} \Delta b_{nk}(i) \right\} z_v,
\]

\[
= \sum_{v=0}^{\infty} g(n,v,i) z_v,
\]

where

\[
g(n,v,i) = \sum_{k=v}^{\infty} \Delta b_{nk}(i),
\]

and

\[
\Delta b_{nk}(i) = \{ b_{nk}(i) - b_{n-1,k}(i) \}.
\]

Let \( p = (p_n) \) be a sequence of strictly positive real numbers with \( \sup p_n < \infty \). Then we define

\[
\mathcal{L}_\infty^G(p,s) := \left\{ z \in \mathbb{C} : \sup_{n,\mathbf{i},n} n^{-s} |\psi_{in}(z)| p_n < \infty, s \geq 0 \right\}.
\]

We write

\[
\psi_{in}(Az) = \sum_{j=0}^{\infty} \sum_{v=0}^{\infty} \sum_{k=v}^{\infty} \Delta b_{nk}(i) a_{vj} z_j
\]

\[
= \sum_{j=0}^{\infty} \sum_{v=0}^{\infty} g(n,v,i) a_{vj} z_j
\]

\[
= \sum_{j=0}^{\infty} g^*(n,j,i) z_j,
\]

where

\[
g^*(n,j,i) = \sum_{v=0}^{\infty} g(n,v,i) a_{vj},
\]

provided that the infinite sums involved exist.

We prove the following theorems.
Theorem 5.1. If \( \inf p_n > 0 \), then \( \mathcal{L}_\infty^g(p, s) \) is a complete linear topological space over the complex field \( \mathbb{C} \), paranormed by:

\[
g(z) = \sup_{i, n} n^{-s/M} |\psi_{in}(z)|^{p_n/M},
\]

for all \( z \in \mathcal{L}_\infty^g(p, s) \), where \( M = \max(1, \sup_n p_n) \). In particular, \( \mathcal{L}_\infty^g(s) \) is a Banach space normed by

\[
\| z \| = \sup_{i, n} n^{-s} |\psi_{in}(z)|,
\]

where \( \mathcal{L}_\infty^g(s) \) is the special case of \( \mathcal{L}_\infty^g(p, s) \), for \( p_n = p \), for all \( n \).

Theorem 5.2. Let \( 0 < p_n \leq q_n \). Then \( \mathcal{L}_\infty^g(q, s) \) is a closed subspace of \( \mathcal{L}_\infty^g(p, s) \).

Theorem 5.3. \( \mathcal{L}_\infty^g(p, s) \) is 1-convex.

Theorem 5.4. \( A \in (\ell_\infty, \mathcal{L}_\infty^g(s)) \) if and only if

\[
\sup_{i, n} n^{-s} \sum_{j=0}^{\infty} |g^*(n, j, i)| < \infty.
\]

Theorem 5.5. \( A \in (\ell_\infty(p), \mathcal{L}_\infty^g(s)) \) if and only if

\[
\sup_{i, n} n^{-s} \sum_{j=0}^{\infty} |g^*(n, j, i)|^{L^{1/p_j}} < \infty,
\]

for every integer \( L > 1 \).

Theorem 5.6. Let \( \inf p_n > 0 \). Then \( A \in (\ell_\infty, \mathcal{L}_\infty^g(p, s)) \) if and only if

\[
\sup_{i, n} n^{-s} \left( \sum_{j=0}^{\infty} |g^*(n, j, i)|^{p_n} \right)^{p_n} < \infty.
\]

Theorem 5.7. \( A \in (\ell(p), \mathcal{L}_\infty^g(s)) \) if and only if

(i) there exists an integer \( K > 1 \) such that

\[
\sup_{i, n} n^{-s} \sum_{j=0}^{\infty} |g^*(n, j, i)|^{q_k} K^{-q_k} < \infty, \quad (1 < p_k < \infty, \: p_k^{-1} + q_k^{-1} = 1),
\]
Theorem 5.8. \( A \in (c_0(p), L^\infty(p,s)) \) if and only if there exists an integer \( K > 1 \), such that
\[
\sup_{i,n} n^{-s} \left( \sum_k \left| g^*(n,j,i) \left| K^{-1/p_k} \right. \right)^{p_n} < \infty.
\]

In Chapter VI we define some new spaces and prove some theorems concerning spaces of \( B \)-bounded variation and spaces related to them.

We define the spaces:
\[
L^B(p,s) := \left\{ z \in c^B : \sum_n n^{-s} \left| \psi_{in}(z) \right|^{p_n} < \infty, \text{ uniformly in } i, \text{ and for } s \geq 0 \right\}
\]
\[
\hat{L}^B(p,s) := \left\{ z \in L^B(p,s) : \sup_i \sum_n n^{-s} \left| \psi_{in}(z) \right|^{p_n} < \infty \right\}
\]

Theorem 6.1. If \( p = (p_n) \) is bounded, then \( L^B(p,s) \) is a complete linear topological space over the complex field \( \mathbb{C} \), paranormed by \( g \) defined by
\[
g(z) = \sum_n n^{-s/M} \left| \psi_{in}(z) \right|^{p_n/M},
\]
for all \( z \in L^B(p,s) \), where \( M = \max(1,\sup p_n) \). In particular, \( L^B_p(s) \) is a Banach space normed by
\[
\| z \| = \sum_n n^{-s} \left| \psi_{in}(z) \right|^p,
\]
where \( L^B_p(s) \) is the special case of \( L^B(p,s) \), for \( p_n = p \), for all \( n \).

Theorem 6.2. \( L^B(p,s) \subseteq L^\infty(p,s) \).

Theorem 6.3. Let \( B = (B_i) \) be a sequence of infinite matrices with \( B_i = (b_{nk}(i)) \), such that \( \sum_k |b_{nk}(i)| < \infty \), for all \( i, n = 0,1,2,\ldots \). Then
\[
L^B(p,s) \subseteq \hat{L}^B(p,s).
\]

Theorem 6.4. If \( \inf p_n > 0 \), then \( L^B(p,s) \) is locally bounded.
Theorem 6.5. If \( \inf p_n > 0 \), then \( \mathcal{L}^G(p, s) \) is 1-convex. If \( \inf p_n \leq 1 \), then \( \mathcal{L}^G(p, s) \) is not 1-convex.

Theorem 6.6. Let \( 1 \leq p < \infty \). Then \( A \in (\ell_1, \mathcal{L}^G_p(s)) \) if and only if

\[
(6.6.1) \quad \sup_k \sum_n n^{-s} | \triangle g^*(n, k, i) |^p < \infty,
\]

uniformly in \( i \).

Theorem 6.7. \( A \in (\ell_1, \mathcal{L}^G(p, s)) \) if and only if (6.6.1) with \( p = 1 \) holds and

\[
\sum_n n^{-s} \triangle g^*(n, k, i) = 1,
\]

for all \( i, k \).

Theorem 6.8. \( z \in \mathcal{L}^G(p, s) \) implies that there exists a constant \( Q > 0 \) such that

\[
\sum_n n^{-s} | \psi_n(z) |^p n \leq Q,
\]

for all \( i \).

Theorem 6.9. Let \( 1 \leq p < \infty \). Then \( A \in (\ell_\infty, \mathcal{L}^G_p(s)) \) if and only if

\[
\sum_n n^{-s} \left( \sum_k | \triangle g^*(n, k, i) | \right)^p < \infty,
\]

uniformly in \( i \).

Theorem 6.10. Let \( 1 \leq p < \infty \). Then \( A \in (\ell_\infty, (p), \mathcal{L}^G_p(s)) \) if and only if, for every integer \( L > 1 \),

\[
\sum_n n^{-s} \left( \sum_k | \triangle g^*(n, k, i) | \right)^p < \infty,
\]

uniformly in \( i \).

Theorem 6.11. Let \( 1 < p_k < \infty \). Then \( A \in (\ell_\infty(p), \mathcal{L}^G(p, s)) \) if and only if, for every integer \( L > 1 \),

\[
\sum_n n^{-s} \left( \sum_k | \triangle g^*(n, k, i) | \right)^{p_n} < \infty,
\]
uniformly in $i$.

In Chapter VII we recall that a sequence $x \in \ell_\infty$ is said to be $B$-summable if
\[ t_{in}^B (x) \equiv (B; x) n = \sum_k b_{nk}(i)x_k \]
converges as $n \to \infty$, uniformly in $i = 0, 1, 2, \ldots$.

We define the difference sequence spaces generated by the summability $B$, by:
\[
(B, p; u, \Delta) := \{ x \in \ell_\infty : |t_{in}^B(u \Delta x - le)|^n \to 0, \text{ as } n \to \infty, \\
\text{uniformly in } i, \text{ for some } l \in \mathbb{C} \},
\]
\[
(B, p; u, \Delta)_0 := \{ x \in \ell_\infty : |t_{in}^B(u \Delta x)|^n \to 0, \text{ as } n \to \infty, \\
\text{uniformly in } i \},
\]
\[
(B, p; u, \Delta)_\infty := \{ x \in \ell_\infty : \sup_{i,n} |t_{in}^B(u \Delta x)|^n < \infty \}.
\]

We prove the following theorems:

**Theorem 7.1.** $(B, p; u, \Delta) \subset (B, p; u, \Delta)_\infty$ if and only if
\[
(7.1.1) \quad \sup_{i,n} |\sum_k b_{nk}(i)|^n \to 0,
\]
holds.

**Theorem 7.2.** $(B, p; u, \Delta), (B, p; u, \Delta)_0$ and $(B, p; u, \Delta)_\infty$ are linear spaces over the complex field $\mathbb{C}$.

**Theorem 7.3.** $(B, p; u, \Delta)_0$ is a linear topological space paranormed by $g$ defined by:
\[
(7.3.1) \quad g(x) = \sup_{0 \leq i < \infty, r \leq n < \infty} \left| \sum_k b_{nk}(i)u_k \Delta x_k \right|^n,
\]
for a whole number $r \geq 0$. $(B, p; u, \Delta)_\infty$ is paranormed by $g$ if $\inf p_n > 0$. If $(7.1.1)$ holds, then $(B, p; u, \Delta)$ is paranormed with the same paranorm $g$. 
Theorem 7.4. \((B, p; u, \Delta)_0\) and \((B, p; u, \Delta)_\infty\) (if \(p_n > 0\)) are complete with respect to the topology generated by the paranorm \(g^*\) defined by
\[
g^*(x) = \sup_{0 \leq i < \infty, \, r \leq n < \infty} \left| \sum_k b_{nk}(i) u_k \Delta x_k \right|^{p_n},
\]
\((r\) is the same as in (7.3.1)). If
\[
\left| \sum_k b_{nk}(i) \right|^{p_n} \to 0, \quad \text{as} \quad n \to \infty.
\]
uniformly in \(i\), holds, then \((B, p; u, \Delta)\) is complete with respect to \(g^*\).

Theorem 7.5. \((B, q; u, \Delta)_0 \subset (B, p; u, \Delta)_0\), if
\[
\lim \inf \frac{p_n}{q_n} > 0.
\]

Theorem 7.6. If \(0 < p_n < q_n \leq 1\), then \((B, q; u, \Delta)_\infty\) is a closed subspace of \((B, p; u, \Delta)_\infty\).

Theorem 7.7. \((B, p; u, \Delta)_0\) and \((B, p; u, \Delta)_\infty\) are locally bounded if \(\inf p_n > 0\). If (7.1.1) holds, then \((B, p; u, \Delta)\) has the same property.

Theorem 7.8. \((B, p; u, \Delta)_0\) and \((B, p; u, \Delta)_\infty\) are \(r\)-convex for all \(r\), where \(0 < r < \inf p_n\). Moreover, if \(p_n = p \leq 1\), for all \(n\), then they are \(p\)-convex.

In Chapter VIII we prove some theorems concerning some new sequences defined by Orlicz functions.

Let \(u = (u_k)\) be an arbitrary sequence such that \(u_k \neq 0\) \((k = 1, 2, \ldots)\). We define the space:
\[
\ell_M(p, u) := \left\{ x \in w : \sum_{k=1}^{\infty} \left[ M \left( \frac{|u_k x_k|}{\rho} \right) \right]^{p_k} < \infty, \quad \text{for some} \quad \rho > 0 \right\}.
\]

Let \(p = (p_n), \, q = (q_n)\) and \(\bar{q} = (\bar{q}_n)\) denote the sequences of positive real numbers and the sequence \(\bar{Q} = (\bar{Q}_n)\) is such that
\[
\bar{Q}_n = \bar{q}_1 + \bar{q}_2 + \cdots + \bar{q}_n \neq 0.
\]

For a sequence \(x = (x_n)\), we write
\[
i_n^*(x, \rho) = \frac{1}{\bar{Q}_n} \sum_{k=1}^{n} \bar{q}_k \left[ M \left( \frac{|x_k|}{\rho} \right) \right]^{p_k}, \quad \text{for some} \quad \rho > 0.
\]
Then we define the new spaces:

\[ W(M, q; p, u) := \{ x \in w : \ell_n^*(ux, \rho) \to 0, \text{ as } n \to \infty, \text{ for some } \rho > 0 \text{ and } l \in \mathbb{C} \}, \]

\[ W_0(M, q; p, u) := \{ x \in w : \ell_n^*(ux, \rho) \to 0, \text{ as } n \to \infty, \text{ for some } \rho > 0 \} \]

and

\[ W_\infty(M, q; p, u) := \left\{ x \in w : \sup_n \ell_n^*(ux, \rho) < \infty, \text{ for some } \rho > 0 \right\}. \]

We prove the following theorems:

**Theorem 8.1.** Let \( H = \sup_k p_k \). Then \( \ell_M(p, u) \) is a linear space over the complex field \( \mathbb{C} \).

**Theorem 8.2.** \( \ell_M(p, u) \) is total paranormed space with paranorm defined by:

\[
\text{(8.2.1)} \quad h(x) = \inf_n \left\{ \rho^{p_n/H} : \left( \sum_{k=1}^\infty \left[ M \left( \frac{|u_k x_k|}{\rho} \right) \right]^{p_k} \right)^{1/H} \leq 1, n = 1, 2, \ldots \right\}.
\]

where \( H = \max(1, \sup_k p_k) \).

**Theorem 8.3.** Let \( 1 \leq p_k < \sup_k p_k < 1 \). Then \( \ell_M(p, u) \) is complete paranormed space with paranorm defined by (8.2.1).

**Theorem 8.4.** Let \( 0 < p_k \leq q_k < \infty \), for each \( k \). Then

\[ \ell_M(p, u) \subseteq \ell_M(q, u). \]

**Theorem 8.5.** Let \( p \) be bounded. Then \( W(M, \bar{q}; p, u), W_0(M, \bar{q}; p, u), \) and \( W_\infty(M, \bar{q}; p, u) \) are linear spaces.

**Theorem 8.6.** Let \( H = \sup_k p_k \). Then \( W_0(M, \bar{q}; p, u) \) is a linear topological space paranormed by \( h^* \) defined by

\[
h^*(x) = \inf_n \left\{ \rho^{p_n/H} : \left( \frac{1}{Q_n} \sum_{k=1}^n \bar{q}_k \left( M \left( \frac{|u_k x_k|}{\rho} \right) \right)^{p_k} \right)^{1/H} \leq 1, n = 1, 2, \ldots \right\}.
\]
Theorem 8.7. Let $M$ be an Orlicz function which satisfies $\Delta_2$-condition.

Then $W(\tilde{q}; u) \subseteq W(M, \tilde{q}; u), W_0(\tilde{q}; u) \subseteq W_0(M, \tilde{q}; u)$ and $W_\infty(\tilde{q}; u) \subseteq W_\infty(M, \tilde{q}; u)$.

Theorem 8.8

(i) Let $0 < \inf p_k \leq p_k < 1$. Then $W(M, \tilde{q}; p, u) \subseteq W_\infty(M, \tilde{q}; u)$.

(ii) Let $1 \leq p_k < \sup p_k < \infty$. Then $W(M, \tilde{q}; u) \subseteq W(M, \tilde{q}; p, u)$.

Theorem 8.9. Let $0 < p_k \leq q_k$, and $(q_k/p_k)$ be bounded. Then

$W(M, \tilde{q}; q, u) \subseteq W(M, \tilde{q}; p, u)$.

Finally, in Chapter IX we prove some theorems on summability of trigonometric sequences by sequence of infinite matrices generalizing some known results.

Let $f(x)$ be a periodic function, with period $2\pi$, and integrable $(L)$, that is, integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let the Fourier series of $f(x)$ be given by:

$$
\frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).
$$

Then the series conjugate to it is:

$$
\sum_{k=1}^{\infty} (b_k \cos kx - a_k \sin kx)
$$

and its derived series is:

$$
\sum_{k=1}^{\infty} k(b_k \cos kx - a_k \sin kx).
$$

Let $s_n(x)$ and $s'_n(x)$ denote the partial sums of the series (9.1) and (9.2) respectively. We write

$$
\psi_x(t) = \psi(f, t) = \begin{cases} 
  f(x + t) - f(x - t) & 0 < t \leq \pi, \\
  g(x) & t = 0,
\end{cases}
$$

where $g(x) = \{f(x + 0) - f(x - 0)\}$.
and
\[ h_x(t) = \frac{\psi_x(t)}{4\sin \frac{t}{2}}. \]

Let \( B = (B_i) \), with \( B_i = (b_{nk}(i)) \), be a sequence of infinite matrices. Then, a sequence \( x = (x_n) \in l_\infty \) is said to be \( B \)-(or \( F_B \))-convergent or summable \( B \) to the generalized limit \( Bx \), if
\[
\lim_{n \to \infty} (B_i x)_n = \lim_{n \to \infty} \sum_{k=0}^{\infty} b_{nk}(i)x_k
\]
\[ = \text{Lim } Bx, \text{ uniformly for } i = 0, 1, 2, \ldots. \]

We establish the following theorems.

**Theorem 9.1.** Let \( B = (B_i) \) be a family of matrices, with
\[(9.1.1) \quad N(B_i) < \infty, \quad \text{for each } i.\]

Let \( A = (a_{nk}) \) be a \( B \)-(or \( F_B \))-regular matrix, i.e. \( A \in (c,B)_{\text{reg}} \). Then, for every \( x \in [-\pi, \pi] \) for which \( h_x(t) \in BV[0, \pi] \),
\[
\lim_n \sum_l \sum_k b_{nk}(i) a_{kl} s_l(x) = h_x(0^+),
\]
uniformly in \( i \) if and only if
\[
\lim_n \sum_l \sum_k b_{nk}(i) a_{kl} \sin(l + \frac{1}{2})t = 0.
\]

**Theorem 9.2.** Let \( B = (B_i) \) be a family of matrices with the condition (9.1.1).

Let \( A = (a_{nk}) \) be a \( B \)-regular matrix, i.e. \( A \in (c,B)_{\text{reg}} \). Then, for each \( x \in [0, 2\pi] \), for which \( f(x) \in BV[0, 2\pi] \),
\[
\lim_n \sum_l \sum_k b_{nk}(i) a_{kl} s_l(x) = \pi^{-1} g(x),
\]
uniformly in \( i \) if and only if
\[
\lim_n \sum_l \sum_k b_{nk}(i) a_{kl} \cos kt = 0,
\]
for all \( t \in [0, \pi], \delta > 0, \text{ uniformly in } i. \)