CHAPTER I
PRELIMINARIES

The primary aim of this section is to recall some basic notations and definitions which shall be used throughout the dissertation.

1.1. NOTATIONS

\( \mathbb{N} := \) The set of all natural numbers.

\( \mathbb{R} := \) The set of all real numbers.

\( \mathbb{C} := \) The set of all complex numbers.

\( \lim \) : means \( \lim_{k \to \infty} \)

\( \sup \) : means \( \lim_{k \geq 1} \)

\( \inf \) : means \( \inf_{k \geq 1} \), unless otherwise stated

\( \sum \) : means summation over \( k = 1 \) to \( k = \infty \), unless otherwise stated.

\( x := (x_k) \), the sequence whose \( k^{th} \) term is \( x_k \)

\( \theta := (0, 0, 0, ...) \), the zero sequence.

\( e_k := (0, 0, ..., 1, 0, 0, ...) \), the sequence whose \( k^{th} \) component is 1 and others are zeroes, for all \( k \in \mathbb{N} \)

\( e := (1, 1, 1, 1, ....) \)

\( p := (p_k) \), the sequence of strictly positive reals.

\( w := \{ x = (x_k) : x_k \in \mathbb{R} \) (or \( \mathbb{C} \)\} \), the space of all sequences, real or complex.

\( l := \{ x \in w : \sum_k |x_k| < \infty \} \)
\( l_\infty := \{ x \in w : \sup_k |x_k| < \infty \} \), the space of bounded sequences.

\( c_0 := \{ x \in w : \lim_k |x_k| = 0 \} \), the space of null sequences.

\( c := \{ x \in w : \lim_k x_k = l, \text{ for some } l \in C \} \), the space of convergent sequences.

**Remark 1.1.1.** \( l_\infty, c_0, \) and \( c \) are Banach spaces with the usual norm \( \|x\| = \sup_k |x_k| \)

\( l_1 := \{ a = (a_k) : \sum_k |x_k| < \infty \} \), the space of absolutely convergent series.

\( l_p := \{ x \in w : \sum_k |x_k|^p < \infty \}, \text{ where } 0 < p < \infty \)

\( w_p := \{ x \in w : \lim\frac{1}{n} \sum_k |x_k - l|^p = 0; \text{ for some } l \in C \} \)

**Case-I** If \( 1 \leq p < \infty \), the spaces \( l_p \) and \( w_p \) are Banach spaces normed by

\[ \|x\| = \left( \sum_k |x_k|^p \right)^{\frac{1}{p}} \]

and

\[ \|x\| = \sup \left( \frac{1}{n} \sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}}, \]

respectively.

**Case-II** If \( 0 < p < 1 \), then \( l_p \) and \( w_p \) are complete \( p \)-normed spaces, \( p \)-normed by

\[ \|x\| = \sum_k |x_k|^p \]

and

\[ \|x\| = \frac{1}{n} \sum_k |x_k|^p, \]

respectively.

The following subspaces of \( w \) were first introduced and discussed by Simons [51], and Maddox [36].
Let $p = (p_k)$ be bounded. Then $C_0(p)$ is a linear metric space paranormed by:

$$g_1(x) = \sup_k |x_k|^{p_k},$$

where $M = \max(1, \sup_k p_k)$. $l_\infty(p)$ and $C(p)$ are paranormed by $g_1(x)$, defined above if and only if $\inf_k p_k > 0$. $l(p)$ and $w(p)$ are paranormed by:

$$g_2(x) = \left(\sum_k |x_k|^{p_k}\right)^{\frac{1}{p_k}},$$

respectively.

**Remark 1.1.2.** If $p_k = p$, for all $k$, then $l_\infty(p) = l_\infty$, $C_0(p) = C_0$, $C(p) = C$, $l(p) = l$ and $w(p) = w_p$.

### 1.2. DIFFERENCE SEQUENCE SPACES

The idea of difference sequence spaces was first introduced by Kizmaz [27] in the year 1981 and was generalised by Et. and Colak [10] in the year 1995. Kizmaz [27] defined the sequence spaces:

$$l_\infty(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in l_\infty\},$$
\[ c(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in c \}, \]

and

\[ c_0(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in c_0 \}, \]

where \( \Delta x = (x_k - x_{k+1}) \). These are Banach spaces with the norm

\[ ||x||_\Delta = |x_1| + ||\Delta x||_\infty. \]

After then Colak and Et [8] defined the sequence spaces:

\[ l_\infty(\Delta^m) = \{ x = (x_k) \in w : (\Delta^m x_k) \in l_\infty \}, \]

\[ c(\Delta^m) = \{ x = (x_k) \in w : (\Delta^m x_k) \in c \}, \]

and

\[ c_0(\Delta^m) = \{ x = (x_k) \in w : (\Delta^m x_k) \in c_0 \}, \]

where \( m \in \mathbb{N} \) and

\[ \Delta^0 x = (x_k), \quad \Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}), \]

\[ \Delta x = (x_k - x_{k+1}), \quad \Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}. \]

and showed that these spaces are Banach spaces with the norm

\[ ||x||_\Delta = \sum_{i=1}^{m} |x_i| + ||\Delta^m x||_\infty. \]

These spaces were generalised by Et and Basarir [15].

Esi and Isik [13] defined the sequence spaces:

\[ l_\infty(\Delta^m, s, p) = \{ x = (x_k) \in w : \sup_k k^{-s} |\Delta^m x_k|^p \leq \infty, \ s \geq 0 \}, \]

\[ c(\Delta^m, s, p) = \{ x = (x_k) \in w : k^{-s} |\Delta^m x_k - L|^p \to 0, (k \to \infty), \ s \geq 0, \text{ for some } L \}, \]

\[ c_0(\Delta^m, s, p) = \{ x = (x_k) \in w : k^{-s} |\Delta^m x_k|^p \to 0, (k \to \infty), \ s \geq 0 \}, \]
where \( v = (v_k) \) is any fixed sequence of non zero complex numbers and \( m \in \mathbb{N} \) is fixed and

\[
\begin{align*}
\Delta^0_v x_k &= (v_k x_k), \\
\Delta^m_v x_k &= (\Delta^{m-1}_v x_k - \Delta^{m-1}_v x_{k+1}), \\
\Delta_v x_k &= (v_k x_k - v_{k+1} x_{k+1}), \\
\Delta^m_v x_k &= \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}.
\end{align*}
\]

**Remark 1.2.1.** If \( s = 0, m = 1, v = (1, 1, 1, \cdots) \) and \( p_k = 1 \) for all \( k \in \mathbb{N} \), we have \( l_\infty(\Delta), c(\Delta) \) and \( c_0(\Delta) \), which were defined by Kizmaz [27].

If \( s = 0 \) and \( p_k = 1 \) for all \( k \in \mathbb{N} \), we have the following sequence spaces which were defined by Et and Esi [12].

\[
\begin{align*}
l_\infty(\Delta^m_v) &= \{ x = (x_k) \in w : (\Delta^m_v x_k) \in l_\infty \}, \\
c(\Delta^m_v) &= \{ x = (x_k) \in w : (\Delta^m_v x_k) \in c \}, \\
c_0(\Delta^m_v) &= \{ x = (x_k) \in w : (\Delta^m_v x_k) \in c_0 \}.
\end{align*}
\]

Difference sequence spaces have also been studied by Bektas and Çolak [2], Et. [11], Khan [22, 23, 24, 25, 26], Ahmad H.A.Bataineh [1], Bilgin [4], Malkowsky [37] and many others.

### 1.3. DEFINITIONS

**Definition 1.3.1.** A paranorm is a function \( g : X \to \mathbb{R} \) which satisfies the following axioms: for any \( x, y, x_0 \in X \), and \( \lambda, \lambda_0 \in \mathbb{C} \),

(i) \( g(\theta) = 0; \)

(ii) \( g(x) = g(-x); \)

(iii) \( g(x + y) \leq g(x) + g(y) \)

(iv) the scalar multiplication is continuous, that is \( \lambda \to \lambda_0, x \to x_0 \) imply \( \lambda x \to \lambda_0 x_0 \).
In other words,
\[ |\lambda - \lambda_0| \to 0, \quad g(x - x_0) \to 0 \quad \text{imply} \quad g(\lambda x - \lambda_0 x_0) \to 0. \]

A paranormed space is a linear space \( X \) with a paranorm \( g \) and it is written as \((X, g)\). Any function \( g \) which satisfies all the condition (i)-(iv) together with the condition

(v) \( g(x) = 0 \) if and only if \( x = \theta \),

is called a total paranorm on \( X \) and the pair \((X, g)\) is called total paranormed space, (see Maddox [35]).

Example 1.3.1. \( l_p \) is totally paranormed for any \( p = (p_k) \in l_{\infty} \).

Definition 1.3.2. A seminorm is a function \( \nu : X \to \mathbb{R} \), defined on linear space \( X \) such that for all \( x, y \in X \).

(i) \( \nu(x) = 0 \) if \( x = \theta \);

(ii) \( \nu(\alpha x) = |\alpha|\nu(x) \), for all scalars \( \alpha \);

(iii) \( \nu(x + y) \leq \nu(x) + \nu(y) \)

The property expressed by (ii) is called absolute homogeneity of \( \nu \) and that expressed by (iii) is called subadditivity of \( \nu \). Thus, a seminorm is a real subadditive and absolutely homogeneous function on \( X \).

Moreover, it follows from (ii) and (iii) that
\[ 0 = \nu(\theta) = \nu(x + (-x)) \leq \nu(x) + \nu(-x) = 2\nu(x). \]

Whence a seminorm is necessarily non-negative. Also, a seminorm \( \nu \) is convex on \( X \), since if \( \lambda + \mu = 1, \lambda \geq 0, \mu \geq 0 \) and \( x + y \in X \), then
\[ \nu(\lambda x + \mu y) \leq |\lambda|\nu(x) + |\mu|\nu(y) = \lambda \nu(x) + \mu \nu(y) \]

Example 1.3.2. For each \( x \in \mathbb{C} \), \( \nu(x) = |x| \) defines a seminorm on \( \mathbb{C} \).
Example 1.3.3. $\nu_1(x) = \sup |x_n|$ and $\nu_2(x) = |\lim x_n|$ defines seminorms $\nu_1$ and $\nu_2$ on the linear space $c$ of all convergent subsequences $x = (x_n)$.

Remark 1.3.1. Every seminormed space is a paranormed space but not conversely.

Example 1.3.4. Consider the paranorm $g$ on $l_p$, where $g(x) = \sum |x_k|^p_k$ and $p_k = \frac{1}{k}$, for all $k \in \mathbb{N}$. Let $x=(0,1,0,0,\ldots)$, then $g(2x) \leq 2g(x)$. Thus $g$ is not a seminorm.

If $X$ is a space of complex sequences $x = (x_k)$, we denote the continuous dual of $X$ by $X'$, that is the set of all continuous linear functionals on $X$.

For a Banach space $X$, $X'$ is the dual Banach space of continuous linear functionals on $X$ with

$$
\|f\| = \sup\{|f(x)| : \|x\| \leq 1\};
$$

$f \in X'$ if and only if $\|f\| < \infty$;

$$
|f(x)| \leq \|x\|\|f\|.
$$

Definition 1.3.3. A continuous function $M : \mathbb{R} \to \mathbb{R}$ is called convex if

$$
M\left(\frac{u+v}{2}\right) \leq \frac{M(u) + M(v)}{2},
$$

for all $u, v \in \mathbb{R}$.

If in addition, the two sides of above are not equal for $u \neq v$, then we call $M$ to be strictly convex. Also see [28].

Definition 1.3.4. A continuous function $M : \mathbb{R} \to \mathbb{R}$ is said to be uniformly convex if for any $\varepsilon > 0$ and any $u_0 > 0$ there exists $\delta > 0$ such that

$$
M\left(\frac{u+v}{2}\right) \leq (1 - \delta)\frac{M(u) + M(v)}{2},
$$

satisfying $|u - v| \geq \varepsilon \max\{|u|, |v|\} \geq eu_0$.

Remark 1.3.2. If $M$ is convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all $\lambda$ with $0 < \lambda < 1$.

1.4. SOME INEQUALITIES.

The following inequalities will be used throughout this dissertation.
(1.4.1.) For any $a, b \in \mathbb{C}$,

$$|a + b| \leq |a| + |b|,$$

is called as the Triangle inequality.

(1.4.2.) For all $u, v \in \mathbb{C}$

$$u \cdot v \leq M(u) + N(v),$$

is called as the Young's inequality.

(1.4.3.) If $\rho_M(u) < \infty$ then

$$M\left(\frac{1}{\mu G} \int_G u(t) dt\right) \leq \frac{1}{\mu G} \int_G M(u(t)) dt,$$

is called as the Jensen's inequality.

(1.4.4.) Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup p_k = G$ and let $D = \max(1, 2^{G-1})$. For $a_k, b_k \in \mathbb{C}$, the set of complex numbers, for all $k \in \mathbb{N}$, we have (from Maddox [32]),

$$|a_k + b_k|^p_k \leq D\left(|a_k|^p_k + |b_k|^p_k\right).$$

1.5. SEQUENCE OF ORLICZ FUNCTIONS.

**Definition 1.5.1.** An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which satisfies the following properties:

(i) $M$ is continuous, non-decreasing and convex

(ii) $M(0) = 0$, $M(x) > 0$ for $x > 0$ and

(iii) $M(x) \to \infty$, as $x \to \infty$.

If convexity of $M$ is replaced by $M(x + y) \leq M(x) + M(y)$, then it is called a Modulus function, defined and discussed by Ruckle [46], Maddox [36] and many others.
Definition 1.5.2. The integral form of an Orlicz function is given as

\[ M(x) = \int_0^{|x|} p(t) \, dt. \]

where \( p \) is the right derivative of \( M \) and satisfies

(i) \( p \) is right continuous and non decreasing.

(ii) \( p(t) > 0 \) whenever \( t > 0 \) and \( p(0) = 0 \)

(iii) \( \lim_{t \to \infty} p(t) = \infty. \)

Definition 1.5.3. Let \( p \) be the right derivative of \( M \), then

\[ q(s) = \sup \{ t : p(t) \leq s \} \]

is called as the right inverse function of \( p \). Let \( M \) be an Orlicz function and \( p \) be the right derivative of \( M \) and \( q \) be the right inverse function of \( p \). Then

\[ N(v) = \int_0^{|v|} q(s) \, ds \]

is called as the complementary function of \( M \).

Definition 1.5.4. If to every positive integer \( n \), there is assigned a number \( a_n \), then the collection \( (a_1, a_2, ..., a_n, a_{n+1}, ...) \) is said to be a sequence, denoted as \( (a_n) \). A Sequence of Orlicz functions is a similar collection of Orlicz functions \( M_k \) where \( k = 1, 2, 3, ... \) and is denoted by \( (M_k) \).

1.6. MODULAR SPACES AND ORLICZ SPACES.

Let \( X \) be a real vector space. A functional \( \rho : X \to [0, \infty] \) is called a modular if

(i) \( \rho(x) = 0 \) if and only if \( x = \theta \);

(ii) \( \rho(\alpha x) = \rho(x) \) for all scalar \( \alpha \) with \( |\alpha| = 1 \);
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

We always denote by $(G, \Sigma, \mu)$, the lebesgue measure space in a Euclidean space with $0 < \mu G < \infty$ and by $M$ and $N$ a pair of Orlicz functions complementary to each other. Moreover, for a measurable function $u$ on $G$, we introduce its modular by

$$\rho_M(u) = \int_G M(u(t))dt.$$ 

Then the Orlicz space $L_M$ and its subspace $E_M$ are defined as follows:

$$L_M = \{u : \rho_M(\lambda u) < \infty \text{ for some } \lambda > 0\},$$

$$E_M = \{u : \rho_M(\lambda u) < \infty \text{ for all } \lambda > 0\}.$$ 

If $\rho$ is a modular in $X$, we define

$$X_\rho = \{x \in X : \lim_{\lambda \to 0^+} \rho(\lambda x) = 0\}$$

and

$$X_\rho^* = \{x \in X : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$ 

It is clear that $X_\rho \subseteq X_\rho^*$. If $\rho$ is a convex modular, for $x \in X_\rho$ we define

$$||x|| = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\}. \quad (1.6.1)$$

It is known that if $\rho$ is a convex modular on $X$, then $X_\rho = X_\rho^*$ and $||.||$ is a norm on $X_\rho$ for which it is a Banach space. The norm $||.||$ defined as in equation (1.6.1) is called as the Luxemburg norm. The following known results give some relationships between the modular $\rho$ and the Luxemburg norm $||.||$ on $X_\rho$.

**Theorem 1.6.1.** (see [34], theorem 1.4) Let $\rho$ be a convex modular on $X$ and let $x \in X_\rho$ and $(x_n)$ be a sequence in $X_\rho$. Then $||x_n - x|| \to 0$ as $n \to \infty$ if and only if $\rho(\lambda(x_n - x)) \to 0$ as $n \to \infty$, for every $\lambda > 0$.

**Theorem 1.6.2.** (see [34], theorem 1.4) Let $\rho$ be a continuous convex modular on $X$. Then

(i) $||x|| < 1$ if and only if $\rho(x) < 1$,
(ii) \( \|x\| \leq 1 \) if and only if \( \rho(x) \leq 1 \),

(iii) \( \|x\| = 1 \) if and only if \( \rho(x) = 1 \).

Note that an Orlicz function satisfies the inequality
\[
M(\lambda x) \leq \lambda M(x) \quad \text{for all } \lambda \text{ with } 0 < \lambda < 1.
\]

Lindenstrauss and Tzafriri [30] used the idea of Orlicz sequence space;
\[
l_M := \left\{ x \in w : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty, \text{ for some } \rho > 0 \right\}
\]
which is Banach space with the norm the norm
\[
\|x\|_M = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1 \right\}.
\]

For more details on Orlicz sequence spaces we refer to [3], [30], [33], [42] and [43].

Remark 1.6.1. If \( M \) is convex function and \( M(0) = 0 \), then \( M(\lambda x) \leq \lambda M(x) \) for all \( \lambda \) with \( 0 < \lambda < 1 \).

We say that an Orlicz function \( M \) satisfies the \( \Delta_2 \) condition (see Krasnoselskii et al. [28]) or \( M \in \Delta_2 \) for short) if there exist constant \( k \geq 2 \) and \( u_0 > 0 \) such that
\[
M(2u) \leq KM(u) \quad \text{whenever } |u| \leq u_0.
\]

Proposition 1.6.1. Let \( M \) be an Orlicz function and \( x \in l_M \).

(i) If \( \|x\| \leq 1 \), then \( \varrho \leq \|x\| \).

(ii) If \( \|x\| > 1 \), then \( \varrho > \|x\| \).

(iii) If \( M \in \delta_2 \), then \( \|x\| = 1 \implies \varrho_M(x) = 1 \).

Proof. The proof is trivial.