CHAPTER 1
PRELIMINARIES

1.1. Introduction

The main objective of this introductory chapter is to discuss some basic concepts, preliminary notions and some fundamental results that are required for the development of the subject in the dissertation. We also discuss manifolds admitting different structures, space forms, theory of submanifolds, invariant and anti invariant submanifolds of complex and almost complex manifolds briefly. Although most of these results are readily available in research article and some in standard books e.g., Nomizu and Kobayshi [25], D.E.Blair [9], Yano [48], B.Y.Chen [12], A.Bejancu [3], we have collected them here for ready references and to set up our terminology.

1.2. Structures on $C^\infty$-Manifolds

We can explain the geometry of a differentiable manifold by knowing a Riemannian metric on it. Further refined informations can be had by knowing additional structures on the manifold, for example, almost complex, almost Kaehler, nearly Kaehler and almost contact structures etc., ([21], [22]). In this section, we briefly discuss some of these structures.

By a Riemannian metric $g$ on a manifold $M$, we mean a map $g: p \mapsto g_p$, where $g_p$ is a positive definite inner product on $T_p(M)$. We require this map to be smooth in the sense that the function

$$p \mapsto g_{ij}(p) = g_p \left( \frac{\partial}{\partial x_i}, \left. \frac{\partial}{\partial x_j} \right|_p \right)$$

is smooth for all $i, j$ on any chart $(U, x)$ on $M$. On a paracompact manifold there exists a smooth Riemannian metric $g$.

For a Riemannian manifold $(M, g)$ with a unique connection $\nabla$, the Koszul formula is given by

$$2g(\nabla X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) - g([Z, X], Y). \quad (1.2.1)$$

In what follows, we shall take a differentiable manifold which is connected and paracompact, so that it can always be endowed with a Riemannian metric $g$ and a
An almost complex structure on a real differentiable manifold $\bar{M}$ is a $(1,1)$-tensor field $J$ which is at every point $p \in \bar{M}$, an endomorphism of the tangent space $T_p(M)$ such that $J^2 = -I$, where $I$ is the identity transformation. A manifold with a fixed almost complex structure is called an almost complex manifold. On an almost complex manifold, there always exists a Riemannian metric $g$ invariant by the almost complex structure $J$, satisfying

$$g(JU, JV) = g(U, V)$$

(1.2.2)

for all $U, V \in T(\bar{M})$, where $T(\bar{M})$ is the tangent bundle of $\bar{M}$. By virtue of (1.2.2), $g$ is called a Hermitian metric. An almost complex manifold (resp. a complex manifold) equipped with a Hermitian metric is called an almost Hermitian manifold (resp. a Hermitian manifold).

Analogous to the almost complex structure $J$, there is defined a fundamental $2$-form which plays an important role in the geometry as well as in the mechanics on the manifold [32]. We describe it as follows.

Let $\Phi$ denote the fundamental $2$-form associated with the Hermitian metric $g$ on $\bar{M}$, 

$$\Phi(U, V) = g(U, JV)$$

(1.2.3)

for all vectors $U, V$ in $T(\bar{M})$. Since $g$ is invariant under $J$ so is $\Phi$, 

$$\Phi(JU, JV) = \Phi(U, V)$$

(1.2.4)

**Definition 1.2.1.** A symplectic form of dimension $n$ on a real vector space $V$ is a non-degenerate exterior $2$-form $\Phi$ of rank $n$. If $V$ admits a symplectic form $\Phi$, then we say that $\Phi$ defines a symplectic structure on $V$ or that $(V, \Phi)$ is a symplectic vector space.

A symplectic structure on a manifold $\bar{M}$ is defined by a choice of a differentiable $2$-form $\Phi$ satisfying following two conditions:

1. For all $p \in \bar{M}$, $\Phi_p$ is non-degenerate,
2. $\Phi$ is closed, $d\Phi = 0$

The almost complex structure $J$ is not parallel in general with respect to the Riemannian connection $\nabla$ on $\bar{M}$, defined by the Hermitian metric $g$. In fact, we have the following formula

$$4g(\nabla_U J)V, W) = 6d\Phi(U, JV, JW) - 6d\Phi(U, V, W) + g(N(V, W), JU),$$

(1.2.5)
where $N$ is the Nijenhuis tensor of $J$, defined by
\[
\] (1.2.6)

It is easy to verify that $N$ satisfies
\[
N(JU, V) = N(U, JV) = -JN(U, V).
\] (1.2.7)

It is well known that vanishing of the tensor $N(U, V)$ is the necessary and sufficient condition for an almost complex manifold to be a complex manifold [22].

If we extend the Riemannian connection $\nabla$ to be a derivative on the tensor algebra of $M$, then we have the following formulae
\[
(\tilde{\nabla}_UJ)V = \tilde{\nabla}_UJV - J\tilde{\nabla}_UV,
\] (1.2.8)
\[
(\tilde{\nabla}_U\Phi)(V, W) = g((\tilde{\nabla}_UJ), W).
\] (1.2.9)

We define a Kaehler manifold by using the fundamental 2-form $\Phi$, almost complex structure $J$ and the Riemannian metric $g$ as follows:

**Definition 1.2.2.** A Hermitian metric on an almost complex manifold is called a Kaehler metric if the fundamental 2-form $\Phi$ is closed. A complex manifold equipped with a Kaehler metric is called a Kaehler manifold. In other words, an almost complex manifold $\tilde{M}$ is Kaehler if
\[
(\tilde{\nabla}_UJ)V = 0
\] (1.2.10)
or equivalently,
\[
\tilde{\nabla}_UJV = J\tilde{\nabla}_UV
\]
for all $U, V$ in $T(\tilde{M})$. The connection $\tilde{\nabla}$ on $\tilde{M}$ is said to be a Kaehler connection.

A Hermitian manifold $\tilde{M}$ is said to be nearly Kaehler if
\[
(\tilde{\nabla}_UJ)V + (\tilde{\nabla}_VJ)U = 0
\]
for all $U, V$ in $T(\tilde{M})$ and is almost Kaehler if
\[
(\tilde{\nabla}_UJ)U + (\tilde{\nabla}_{JU}J)U = 0
\]
for all $U$ in $T(\tilde{M})$.

For the relation among these classes, let us denote by $K$, $AK$, $NK$ and $H$ the classes of Kaehler, almost Kaehler, nearly Kaehler and Hermitian manifolds respectively. Then it can be easily seen that
\[ K \subseteq AK \]
\[ \cap_{NK} \quad \text{and} \quad K \subseteq H, \quad K = AK \cap NK. \]

Figure (1.2.1)

**Remark 1.2.1.** It is clear that every Kaehler manifold is nearly Kaehler, but converse need not be true in general.

For the Nijenhuis tensor \( N \) of \( J \) in nearly Kaehler manifold, we have

**Proposition 1.2.1.** Let \( \tilde{M} \) be a nearly Kaehler manifold. Then the Nijenhuis tensor \( N \) of \( J \) is given by

\[
N(U, V) = 4J(\nabla V J)U
\]

for any \( U, V \) in \( T(\tilde{M}) \).

**Definition 1.2.3.** A Kaehler manifold \( \tilde{M} \) of constant holomorphic sectional curvature is called a complex space form. It is denoted by \( \tilde{M}(c) \).

The curvature tensor \( \tilde{R} \) of \( \tilde{M}(c) \) is given by

\[
\tilde{R}(U, V)W = \frac{c}{4} \{g(V, W)U - g(U, W)V + g(JV, W)JU - g(JU, W)JV
+ 2g(U, JV) JW\} \tag{1.2.11}
\]

for any vector fields \( U, V, W \) tangent to \( \tilde{M} \).

We now discuss some examples of Kaehler and nearly Kaehler manifolds.

**Example 1.2.1.** Consider the complex \( n \)-space \( C^n \) with the metric

\[
ds^2 = \sum_{j=1}^{n} dz^j d\bar{z}^j.
\]

The fundamental 2-form \( \Phi \) in this case is given by

\[
\Phi = -i \sum_{j=1}^{n} dz^j \wedge d\bar{z}^j.
\]

Clearly, \( \Phi \) is closed and so the metric defines a Kaehlerian structure on \( C^n \). Thus, \( C^n \) is a complete, flat Kaehlerian manifold.
Example 1.2.2. Let $CP^n$ be the complex projective space with homogeneous coordinates $z^0, z^1, z^2, \ldots, z^n$. The complex quadratic $Q^{n-1}$ is a complex hypersurface of $CP^n$ defined by the equation

$$(z^0)^2 + (z^1)^2 + \ldots \ldots (z^n)^2 = 0.$$ 

Then, $Q^{n-1}$ is a Kaehlerian manifold.

Example 1.2.3. $S^6$ with usual almost complex structure is nearly Kaehler but not Kaehler [46].

Let $\tilde{M}$ be a $(2n + 1)$-dimensional (i.e., odd dimensional) differentiable manifold. A triplet $(\phi, \xi, \eta)$ is said to be an almost contact structure on $\tilde{M}$, where $\phi$ is a $(1,1)$-tensor field on $\tilde{M}$, $\xi$ is a vector field on $\tilde{M}$ and $\eta$ is a 1-form such that

$$\phi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1,$$ \hspace{1cm} (1.2.12)

where $I$ is the identity map on $\tilde{M}$. The vector field $\xi$ is known as the structure vector field and 1-form $\eta$ is the dual of $\xi$. A manifold $\tilde{M}$ equipped with the almost contact structure $(\phi, \xi, \eta)$ is said to be an almost contact manifold. The condition (1.2.12) imply that

$$\phi \xi = 0 \quad \text{and} \quad \eta \circ \phi = 0.$$ \hspace{1cm} (1.2.13)

Now suppose, there is given a Riemannian metric tensor field $g$ on $\tilde{M}$ which satisfies

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V)$$ \hspace{1cm} (1.2.14)

for any vector fields $U, V$ in $T(\tilde{M})$. Then the structure $(\phi, \xi, \eta, g)$ is said to be an almost contact metric structure on $\tilde{M}$. In this case, it is easy to check that

$$g(U, \xi) = \eta(U)$$ \hspace{1cm} (1.2.15)

for any vector field $U$ in $T(\tilde{M})$.

An almost contact metric structure is called a contact metric structure if

$$d\eta = \Phi,$$

where $\Phi$ is a fundamental 2-form defined by

$$\Phi(U, V) = g(U, \phi V).$$

In this case for any vector field $U$ in $T(\tilde{M})$, we have

$$\nabla_U \xi = -\phi U - \phi h U,$$ \hspace{1cm} (1.2.16)
where $h = \frac{1}{2} L_\xi \phi$, $L_\xi \phi$ being the Lie derivative of $\phi$ with respect to $\xi$. The operator $h$ satisfies
\begin{equation}
    g(hU, V) = g(U, hV), \quad \phi \circ h = -h \circ \phi.
\end{equation}

By a Sasakian manifold, we mean a contact metric manifold which is normal i.e.,
\begin{equation}
    S_{\phi} + 2d\eta \otimes \xi = 0,
\end{equation}
where $S_{\phi}$ is the Nijenhuis tensor of $\phi$.

1.3. Submanifolds

If an $n$-dimensional differentiable manifold $M$ admits an immersion
\begin{equation}
    f : M \hookrightarrow \bar{M}
\end{equation}
into an $m$-dimensional differentiable manifold $\bar{M}$, then $M$ is said to be a submanifold of $\bar{M}$. Obviously $n \leq m$.

If $\bar{M}$ is a Riemannian manifold with a Riemannian metric $g$, then $M$ also admits a Riemannian metric induced from $\bar{M}$ which is denoted by the same symbol $g$. The immersion $f$ is said to be an isometric immersion if the differentiable map
\begin{equation}
    f_* : T(M) \hookrightarrow T(\bar{M})
\end{equation}
preserves the Riemannian metric i.e., for $U, V \in T(M)$,
\begin{equation}
    g(f_* U, f_* V) = g(U, V).
\end{equation}

For every point $p \in M$, the tangent space $T_{f(p)}(\bar{M})$ of $\bar{M}$ admits the following decomposition
\begin{equation}
    T_{f(p)}(\bar{M}) = T_p(M) \oplus T^\perp_p (M),
\end{equation}
where $T_p(M)$ is the tangent space of $M$ at $p$ and $T^\perp_p (M)$ is the orthogonal compliment of $T_p(M)$ in $T_{f(p)}(\bar{M})$ consisting of all vectors normal to $M$.

The Riemannian connection $\nabla$ of $\bar{M}$ induces canonically the connection $\nabla$ and $\nabla^\perp$ on $T(M)$ and $T^\perp (M)$ respectively, governed by the Gauss and Wiengarten formulae, viz,
\begin{align}
    \nabla_U V &= \nabla_U V + h(U, V), \tag{1.3.2} \\
    \nabla_U N &= -A_N U + \nabla^\perp_U N \tag{1.3.3}
\end{align}
for any tangent vector fields $U, V$ on $M$ and $N \in T^\perp (M)$. $h$ and $A_N$ are called second fundamental form and shape operator respectively and are related by
\begin{equation}
    g(h(U, V), N) = g(A_N U, V). \tag{1.3.4}
\end{equation}
For the second fundamental form \( h \), we define the covariant differentiation \( \nabla h \) with respect to the connection in \( T(M) \oplus T^\perp(M) \) by

\[
(\nabla_U h)(V, W) = \nabla_U^h h(V, W) - h(\nabla_U V, W) - h(V, \nabla_U W)
\]  
(1.3.5)

for any vector fields \( U, V \) and \( W \) tangent to \( M \).

Using Gauss and Wiengarten formul\( \bar{a} \), we obtain the following celebrated equations due to Gauss, Coddazi and Ricci [12].

\[
\tilde{R}(U, V; W, Z) = R(U, V, W, Z) + g(h(U, Z), h(V, W)) - g(h(U, W), h(V, Z)),
\]  
(1.3.6)

\[
(\tilde{\nabla}_U h)(V, W) - (\tilde{\nabla}_V h)(U, W),
\]  
(1.3.7)

\[
\tilde{R}(U, V; N_1, N_2) = R(U, V, N_1, N_2) - g([A_{N_1}, A_{N_2}]U, V)
\]  
(1.3.8)

for any vector fields \( U, V, W, Z \) tangent to \( M \) and \( N_1, N_2 \) are vector fields normal to \( M \). In (1.3.7), \( (\tilde{R}(U, V)W)^\perp \) denotes the normal component of \( \tilde{R}(U, V)W \) and \( R^\perp \) is the curvature tensor of the normal connection \( \mathcal{D} \).

1.4. Some Special Submanifolds

Looking into the Guass formula, we can easily classify the submanifolds, putting conditions on \( h \) as follows:

**Definition 1.4.1** [12]. A submanifold \( M \) of a Riemannian manifold \( \bar{M} \) is said to be a totally geodesic submanifold if the second fundamental form \( h \) is identically zero on \( M \) i.e., \( h \equiv 0 \).

**Definition 1.4.2** [12]. A submanifold \( M \) of a Riemannian manifold \( \bar{M} \) is said to be a totally umbilical submanifold of \( \bar{M} \) if its second fundamental form \( h \) satisfies

\[
h(U, V) = g(U, V)H,
\]

where \( H = \frac{1}{n}(\text{trace of } h) \) is called the mean curvature vector and the squared norm of second fundamental form \( h \) is defined as

\[
\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).
\]  
(1.4.1)

**Definition 1.4.3** [12]. A submanifold \( M \) is called minimal if the mean curvature vector vanishes identically i.e., \( H=0 \).
Remark 1.4.1. A minimal totally umbilical submanifold is totally geodesic submanifold.

Definition 1.4.4. The vector sub-bundle $\mu$ of the normal bundle $T^\perp(M)$ is said to be parallel (in the normal bundle) if

$$\nabla_U N \in \mu$$

for any $U \in T(M)$ and any local cross-section $N$ in $\mu$.

On an almost Hermitian manifold $\tilde{M}$,

$$g(JU, JV) = g(U, V)$$

for any vector fields $U, V$ in $T(\tilde{M})$. In other words

$$g(JU, U) = 0$$

i.e., $JU \perp U$ for each tangent vector field $U$ on $\tilde{M}$.

Hence, for a submanifold $M$ of $\tilde{M}$ if $U \in T_p(M)$, $JU$ may or may not belong to $T_p(M)$. Thus, action of the almost complex structure $J$ on the tangent vectors of the submanifold of the almost Hermitian manifold gives rise to its classification into invariant and anti-invariant submanifolds. These are defined as follows:

Definition 1.4.5 [41]. A submanifold of $M$ an almost Hermitian manifold $\tilde{M}$ is said to be invariant (or holomorphic) if

$$J(T_p(M)) \subset T_p(M)$$

for all $p \in M$.

Definition 1.4.6 [48]. A submanifold $M$ of an almost Hermitian manifold $\tilde{M}$ is said to be anti-invariant (or totally real) if

$$J(T_p(M)) \subseteq T^\perp_p(M)$$

for all $p \in M$.

Remark 1.4.2. $M$ is a holomorphic submanifold of $\tilde{M}$ if for any nonzero vector $U$ tangent to $M$ at any point $p \in M$, the angle between $JU$ and the tangent space $T_p(M)$ is equal to zero, whereas $M$ is totally real if and only if for any non-zero tangent vector $U$ in $M$ at any point $p \in M$, the angle between $JU$ and $T_p(M)$ is equal $\pi/2$.

In 1978, A.Bejancu ([1], [2]) considered a new class of submanifolds of an almost Hermitian manifold of which the above classes namely invariant and totally
real submanifolds are particular cases and named this class of submanifolds as CR-submanifolds that is, a CR-submanifold provides a single setting to study the invariant and anti-invariant submanifolds of an almost Hermitian manifold.

Let $\tilde{M}$ be an almost Hermitian manifold with an almost complex structure $J$ and Hermitian metric $g$ and $M$ be a Riemannian submanifold immersed in $\tilde{M}$. At each point $p \in M$, let $\mathcal{D}_p$ be the maximal holomorphic subspace of the tangent space $T_p(M)$ i.e.,

$$\mathcal{D}_p = T_p(M) \cap JT_p(M).$$

If the dimension of $\mathcal{D}_p$ is same for all $p \in M$, we get a holomorphic distribution $\mathcal{D}$ on $M$.

**Definition 1.4.7.** A Riemannian submanifold $M$ is said to be a CR-submanifold of an almost Hermitian manifold $\tilde{M}$ if there exists a holomorphic distribution $\mathcal{D}$ on $M$ such that its orthogonal complementary distribution $\mathcal{D}^\perp$ is totally real i.e.,

$$JD^\perp \subseteq T^\perp(M)$$

for all $p \in M$.

Clearly every real hypersurface $M$ of an almost Hermitian manifold is a CR-submanifold, if $\dim(M) > 1$.

**Remark 1.4.3.** It is clear from the above definition that the dimension of $\mathcal{D}$ is always even and $JD^\perp$ is a sub-bundle of $T^\perp(M)$, the normal bundle splits as

$$T^\perp(M) = JD^\perp \oplus \mu,$$

where $\mu$ is the compliment of $JD^\perp$ in $T^\perp(M)$ and $\mu$ is invariant under $J$.

**Note.** Throughout the dissertation $M$ denotes a submanifold of the ambient space $\tilde{M}$, unless mentioned otherwise.

**Definition 1.4.8.** A CR-submanifold $M$ is called anti-holomorphic submanifold if

$$JD^\perp_p = T^\perp_p(M)$$

for all $p \in M$.

**Definition 1.4.9.** A CR-submanifold $M$ is said to be proper if neither $\mathcal{D}$ nor $\mathcal{D}^\perp = 0$. Obviously if $\mathcal{D} = 0$, then $M$ is totally real submanifold and if $\mathcal{D}^\perp = 0$, then $M$ is holomorphic submanifold.
Definition 1.4.10. A CR-submanifold $M$ is called a CR-product if it is locally a Riemannian product of a holomorphic submanifold $N_T$ and a totally real submanifold $N_L$.

From the above definition, it is obvious that on a CR-product submanifold, the leaves of $\mathcal{D}$ and $\mathcal{D}^\perp$ are totally geodesic in $M$ and vice-versa.

We know that the leaves of a distribution $\mathcal{D}$ on a manifold $M$ are totally geodesic in $M$ if and only if
\[ \nabla_X Y \in \mathcal{D} \]
for all $X, Y \in \mathcal{D}$.

Thus, in the setting of CR-submanifold of an almost Hermitian manifold, the leaves of $\mathcal{D}$ are totally geodesic in $M$ if and only if
\[ \nabla_X Y \in \mathcal{D} \]
for all $X, Y \in \mathcal{D}$ which is equivalent to the condition
\[ \nabla_X W \in \mathcal{D}^\perp \]
for all $X \in \mathcal{D}$ and $W \in \mathcal{D}^\perp$.

Similarly, for the totally geodesicness of the leaves of $\mathcal{D}^\perp$, the conditions
\[ \nabla_Z W \in \mathcal{D}^\perp, \]
\[ \nabla_Z X \in \mathcal{D} \]
for all $X$ in $\mathcal{D}$ and $Z, W$ in $\mathcal{D}^\perp$, are equivalent.

Now, we discuss the condition of the totally geodesicness of the leaves of $\mathcal{D}$.

Lemma 1.4.1 [14]. The leaves of the holomorphic distribution $\mathcal{D}$ on a CR-submanifold $M$ of a Kaehler manifold $\tilde{M}$ are totally geodesic in $M$ if and only if
\[ g(h(\mathcal{D}, \mathcal{D}), J\mathcal{D}^\perp) = 0. \]
(1.4.6)

For the integrability of the distribution $\mathcal{D}^\perp$, we need the following lemma

Lemma 1.4.2 [14]. Let $M$ be a CR-submanifold of a Kaehler manifold $\tilde{M}$. Then
\[ g(\nabla_U Z, X) = g(JA_{ZU} U, X), \]
(1.4.7)
\[ A_{ZW} = A_{ZW} \]
(1.4.8)
for all $X$ in $\mathcal{D}$ and $Z, W$ in $\mathcal{D}^\perp$.

From the definition 1.4.10, a $CR$-submanifold is a $CR$-product if and only if the leaves of $\mathcal{D}$ and $\mathcal{D}^\perp$ are totally geodesic in $M$. Hence by combining (1.4.2) and (1.4.5), we conclude that a $CR$-submanifold of an almost Hermitian manifold is a $CR$-product if and only if

$$\nabla_U X \in \mathcal{D}$$

(1.4.9)

for all $U \in T(M)$.

Similarly, by combining (1.4.3) and (1.4.4), it is concluded that a $CR$-submanifold is a $CR$-product if and only if

$$\nabla_U Z \in \mathcal{D}^\perp.$$  \hspace{1cm} (1.4.10)

Conditions (1.4.9) and (1.4.10) are equivalent because

$$g(\nabla_U X, Z) = 0 \Leftrightarrow g(X, \nabla_U Z) = 0.$$

Next we have,

**Theorem 1.4.1** [14]. A $CR$-submanifold of a Kaehler manifold is a $CR$-product if and only if

$$A_{\mathcal{J} \mathcal{D}^\perp \mathcal{D}} = 0.$$  \hspace{1cm} (1.4.11)

The generalization of Riemannian products namely warped product is defined as follows:

**Definition 1.4.11** [15]. Let $(B, g_B)$ and $(F, g_F)$ be two Riemannian manifolds with Riemannian metrics $g_B$ and $g_F$ respectively and $f$ be a positive differentiable function on $B$. The warped product of $B$ and $F$ is the Riemannian manifold

$$B \times_f F = (B \times F, g),$$

where $g = g_B + f^2 g_F$.

More explicitly, if $U$ is tangent to $M = B \times_f F$ at $(p, q)$, then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p) \|d\pi_2 U\|,$$

where $\pi_i$ ($i = 1, 2$) are the canonical projections of $B \times F$ onto $B$ and $F$ respectively and the function $f$ is known as the warping function.

**Definition 1.4.12** [41]. A doubly warped product $(M, g)$ is a product manifold of the form $M = _f B \times_b F$ with the metric $g = f^2 g_B \oplus b^2 g_F$, where $b : B \rightarrow (0, \infty)$ and $f : F \rightarrow (0, \infty)$ are smooth maps and $g_B$, $g_F$ are the metrics on the Riemannian
manifolds $B$ and $F$ respectively. If either $b \equiv 1$ or $f \equiv 1$ but not both, then we get a (single) warped product. If both $b \equiv 1$ and $f \equiv 1$, then we have a product manifold. If neither $b$ nor $f$ is constant, then we have a non trivial doubly warped product.

If $X \in \chi(B)$ and $Z \in \chi(F)$, then the Levi-Civita connection is

$$\nabla_X Z = (Z \ln f) X + X (ln b) Z.$$  \hfill (1.4.12)

Bishop and O’Neill [6] obtained the following lemma which provides some basic formulae on warped product manifolds.

**Lemma 1.4.3** [6]. Let $M = B \times_f F$ be a warped product manifold. If $X, Y \in T(B)$ and $V, W \in T(F)$, then

(i) $\nabla_X Y \in T(B),$

(ii) $\nabla_X V = \nabla_Y X = \left( \frac{Xf}{f} \right) V,$

(iii) $\text{nor}(\nabla_Y W) = - \left( \frac{g(V, W)}{f} \right) \nabla f,$

where $\text{nor}(\nabla_Y W)$ is the component of $\nabla_Y W$ in $T(B)$ and $\nabla f$ is the gradient vector field of the warping function $f$ and is defined as

$$g(\nabla f, U) = U f$$  \hfill (1.4.13)

for all $U \in T(M)$.

From (ii) of above lemma, we can see that

$$\nabla_U V = \nabla_Y U = (U \ln f)V$$  \hfill (1.4.14)

for any vector field $U$ tangent to $B$ and $V$ tangent to $F$.

For any vector field $U$ tangent to $M$, we can decompose $JU$ as

$$JU = PU + FU,$$  \hfill (1.4.15)

where $PU$ and $FU$ are the tangential and normal components of $JU$ respectively. Then $P$ is an endomorphism of the tangent bundle $T(M)$ and $F$ is the normal bundle valued 1-form on $T(M)$.
Similarly, for any vector field $N$ normal to $M$, if we put

$$JN = tN + fN,$$  

(1.4.16)

where $tN$ and $fN$ are the tangential and normal components of $JN$ respectively, then $f$ can be treated as an endomorphism of the normal bundle $T^\perp(M)$ and $t$, a tangent bundle valued 1-form on $T^\perp(M)$ with kernel as $J\mathcal{D}^\perp$ and $\mu$ respectively.

The covariant differentiation of the operators $P$, $F$, $t$ and $f$ are defined respectively as

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V,$$  

(1.4.17)

$$(\bar{\nabla}_U F)V = \nabla_U^1 FV - F\nabla_U V,$$  

(1.4.18)

$$(\bar{\nabla}_U t)N = \nabla_U tN - t\nabla_U^1 N,$$  

(1.4.19)

$$(\bar{\nabla}_U f)N = \nabla_U fN - f\nabla_U^1 N.$$  

(1.4.20)

However, on a submanifold $M$ of an almost contact manifold $(\tilde{M}, \phi, \xi, \eta)$ for any $U \in T(M)$, we also denote the tangential and normal components of $\phi U$ by $PU$ and $FU$ respectively. Similarly, the tangential and normal components of $\phi N$ for $N \in T^\perp(M)$ are denoted by $tN$ and $fN$ respectively i.e., we write

$$\phi U = PU + FU$$  

(1.4.21)

and

$$\phi N = tN + fN.$$  

(1.4.22)

The covariant differentiation of the operators $P$, $F$, $t$ and $f$ are defined in the same manner as in equations (1.4.17) to (1.4.20).

K.A.Khan, V.A.Khan and S.I.Husain [24] considered the tensors $P$ and $Q$ to obtain integrability conditions of the canonical distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on a $CR$-submanifold of an almost Hermitian manifold. They also studied the geometrical properties of the leaves of the distributions using these tensors.

Let $\tilde{M}$ be an almost Hermitian manifold and $M$ be a $CR$-submanifold of $\tilde{M}$. Then for any $U,V$ in $T(M)$, we have

$$(\bar{\nabla}_U J)V = \bar{\nabla}_U JV - J\bar{\nabla}_U V.$$  

Making use of Gauss and Weingarten formulae and equations (1.4.15) and (1.4.16), the above equation takes the form

$$(\bar{\nabla}_U J)V = (\nabla_U P)V - A_F VU - th(U,V) + (\nabla_U F)V + h(U,PV) - fh(U,V).$$  

13
Furthermore, for any $U, V \in T(M)$, let us decompose $(\nabla_U J)V$ into tangential and normal parts as

$$(\nabla_U J)V = P_U V + Q_U V.$$  \hspace{1cm} (1.4.23)

Now comparing the tangential and normal parts of $(\nabla_U J)V$ in the above equation

$$P_U V = (\nabla_U P)V - A_{FV} U - th(U, V),$$  \hspace{1cm} (1.4.24)

$$Q_U V = (\nabla_U F)V + h(U, PV) - fh(U, V)$$  \hspace{1cm} (1.4.25)

Similarly, for $N \in T'^{-1}(M)$ denoting by $P_U N$ and $Q_U N$ respectively, the tangential and normal parts of $(\nabla_U J)$, we get

$$P_U N = (\nabla_U t) N + PA_N U - A_{JN} U,$$  \hspace{1cm} (1.4.26)

$$Q_U N = (\nabla_U f) N + h(tN, U) + fA_N U.$$  \hspace{1cm} (1.4.27)

The following properties of $P$ and $Q$ are used in our subsequent sections of different chapters,

$(p_1)$ (i) $P_{U+W} = P_U W + P_V W,$ \hspace{1cm} (ii) $Q_{U+W} = Q_U W + Q_V W$

$(p_2)$ (i) $P_U (V + W) = P_U V + P_U W,$ \hspace{1cm} (ii) $Q_U (V + W) = Q_U V + Q_U W.$

$(p_3)$ (i) $g(P_U V, W) = -g(V, P_U W),$ \hspace{1cm} (ii) $g(Q_U V, \xi) = -g(V, P_U \xi)$

$(p_4)$ $P_UJV + Q_UJV = -J(P_U V + Q_U V).$

**Definition 1.4.13 [13].** A submanifold $M$ of an almost Hermitian manifold $\tilde{M}$ is said to be a generic submanifold if the maximal holomorphic subspace

$$D_x = T_x(M) \cap JT_x(M)$$

has a constant dimension for each $x \in M$ and it defines a differentiable distribution on $M$. In this case the tangent space $T_x(M)$ of $M$ at each point $x \in M$ is decomposed as

$$T_x(M) = D_x \oplus D_x^\perp.$$ 

Here, $D_x^\perp$ is the orthogonal compliment of the holomorphic subspace $D_x$ and is not necessarily totally real as was in the case of $CR$-submanifold. For this reason generic submanifold is a generalized version of $CR$-submanifold. The distribution $D_x^\perp$ on a generic submanifold is known as purely real distribution.

Now in view of the Remark 1.4.2, we have a third important class of submanifold of an almost Hermitian manifold (In particular of a Kaehler manifold), called slant
submanifolds.

A slant submanifold is defined as submanifold of $\tilde{M}$ such that for any non zero vector $U \in T_p(M)$, the tangent angle $\theta(U)$ between $JU$ and the tangent space $T_x(M)$ is constant (which is independent of the choice of the point $x \in M$ and choice of the tangent vector $U \in T_x(M)$). It is clear that holomorphic and totally real submanifolds are special classes of slant submanifolds. A slant submanifold is called proper if it is neither holomorphic nor totally real submanifold.

If $M$ is a slant submanifold of an almost Hermitian manifold $\tilde{M}$, we have (cf., [19])

$$P^2 = -\cos^2(\theta) \ I,$$

(1.4.28)

where $I$ is identity map and $\theta$ is the writing angle of $M$ in $\tilde{M}$. Hence, we have

$$g(PU, PV) = \cos^2 \theta g(U, V),$$

(1.4.29)

and

$$g(FU, FV) = \sin^2 \theta g(U, V)$$

(1.4.30)

for all $U, V$ tangent to $M$. A natural generalization of $CR$-submanifolds in terms of slant distribution was given by N. Papaghiuc [45]. These submanifolds are known as semi-slant submanifolds. He defined these submanifolds as follows:

**Definition 1.4.14** [45]. A submanifold $M$ of an almost Hermitian manifold is called a semi-slant submanifold if it is endowed with two orthogonal complimentary distributions $D$ and $D^\theta$ such that $D$ is holomorphic and $D^\theta$ is slant i.e., the angle $\theta(X)$ between $JX$ and $D^\theta_x$ is constant for each $X \in D^\theta_x$.

Hence, $CR$-submanifolds and slant submanifolds are semi-slant submanifolds with $\pi/2$ and $D = \{0\}$ respectively.

For $\pi/2$, the semi-slant submanifold is semi-invariant submanifold. On a semi-slant submanifold $M$, for any $X \in T(M)$, we may write

$$X = P_1X + P_2X + \eta(X)\xi,$$

(1.4.31)

where $P_1X \in D$ and $P_2X \in D^\theta$.

Applying $\phi$, (1.4.31) in view of (1.4.21) yields

$$\phi X_1 = \phi P_1X + TP_2X + NP_2X.$$

(1.4.32)

The differential geometry of semi-invariant or contact $CR$-submanifolds as a generalization of invariant and anti-invariant submanifolds of an almost contact metric
manifold was initiated by A. Bejancu and N. Papaghiuc \[4\] followed by several geometers.

Throughout our dissertation, for a submanifold $M$ of an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$, we assume that the structure vector field $\xi$ is tangential to the submanifold $M$ and therefore the tangent bundle $T(M)$ is decomposed as

$$T(M) = \mathcal{D} \oplus <\xi> \oplus \mathcal{D}^\perp,$$

where $<\xi>$ is the one dimensional distribution on $M$ spanned by structure vector field $\xi$.

**Definition 1.4.15.** A submanifold $M$ of an almost contact metric manifold $\tilde{M}$ is said to be a contact CR-submanifold (or semi-invariant submanifold) if there exists a pair of orthogonal distributions $(\mathcal{D}, \mathcal{D}^\perp)$ satisfying the conditions;

1. $T(M) = \mathcal{D} \oplus \mathcal{D}^\perp \oplus <\xi>,$
2. The distribution $\mathcal{D}$ is invariant by $\phi$ i.e., $\phi \mathcal{D}_x = \mathcal{D}_x$, $x \in M,$
3. The distribution $\mathcal{D}^\perp$ is anti-invariant i.e., $\phi \mathcal{D}^\perp_x \subseteq T^\perp_x(M)$, $x \in M.$

It follows that normal bundle splits as

$$T^\perp(M) = \phi \mathcal{D}^\perp \oplus \mu,$$  \hspace{1cm} (1.4.33)

where $\mu$ is invariant sub-bundle of $T^\perp(M)$. If $\mathcal{D} = \{0\}$ (resp. $\mathcal{D}^\perp = \{0\}$), then $M$ is said to be an anti-invariant (resp. invariant) submanifold. We say that $M$ is proper contact CR-submanifold (or semi-invariant submanifold) if it is neither invariant nor anti-invariant.

**Remark 1.4.4.** Let $M$ be a semi-invariant submanifold of an almost contact metric manifold $\tilde{M}$. Then we have

1. For any $U \in T(M)$, $PU \in \mathcal{D}$ and $FU \in \phi \mathcal{D}^\perp,$
2. For any $N \in T^\perp(M)$, $tN \in \mathcal{D}^\perp$ and $fN \in \mu.$