CHAPTER 4

Generalized Derivations on Prime Near-Rings

4.1 Introduction

Our main objective in this chapter is to describe some recent work in the area of the commutativity of near-rings with generalized derivations. The results presented in this chapter are based on the work of Bell [15] and Golbasi [42] & [43].

Section 4.2 begins with the definition of generalized derivation in near-ring and the results presented in this section are generalization of the results obtained earlier in the setting of prime rings with derivations. This study is continued in the section 4.3 also. Finally Section 4.4 is devoted to the study of generalized derivation satisfying certain identities in near-rings.

4.2 Generalized derivations and commutativity of prime near-rings

Many analysts have studied generalized derivations in the context of algebras on certain normed spaces. By a generalized derivation on an algebra A one usually means a map of the form $x \mapsto ax + xb$ where $a$ and $b$ are fixed elements in A. We prefer to call such maps generalized inner derivations for the reason they present a generalization of the concepts of inner derivations (i.e., the map of the form $x \mapsto ax - xa$). Now in a ring $R$, let $f$ be a generalized inner derivation given by $f(x) = ax + xb$. Notice that $f(xy) = f(x)y + I_b(y)$, where $I_b(y) = yb - by$ is an inner derivation. Motivated by this observation Bresar [31] introduced the notion of generalized derivation in rings as follows:

**Definition 4.2.1** (Generalized derivation). An additive mapping $f : R \rightarrow R$ is called...
a generalized derivation on a ring \( R \) if there exists a derivation \( d : R \to R \) such that
\[ f(xy) = f(x)y + xd(y) \]
holds for all \( x, y \in N \).

The concept of generalized derivation covers both the concepts of derivation and generalized inner derivation. Motivated by this, Golbasi [42] introduced generalized derivation in near-ring as follows:

**Definition 4.2.2 (Generalized derivation in near-rings).** Let \( N \) be a near-ring, \( d \) a derivation of \( N \). An additive mapping \( f : N \to N \) is said to be a right generalized derivation of \( N \) associated with \( d \) if
\[ f(xy) = f(x)y + xd(y), \quad \text{for all } x, y \in R, \tag{4.2.1} \]
and \( f \) is said to be a left generalized derivation of \( N \) associated with \( d \) if
\[ f(xy) = d(x)y + xf(y), \quad \text{for all } x, y \in R. \tag{4.2.2} \]
\( f \) is said to be a generalized derivation of \( N \) associated with \( d \) if it is both a left as well as a right generalized derivation of \( N \) associated with \( d \).

**Remark 4.2.1.** Throughout this chapter \((f,d)\) will denote generalized derivation \( f \) on a near-ring \( N \) associated with \( d \).

We begin with the following preliminary lemmas regarding generalized derivations in near-rings earlier obtained by Golbasi [42].

**Lemma 4.2.1 (i).** Let \((f,d)\) be a right generalized derivation of a near-ring \( N \). Then
\[ f(xy) = xd(y) + f(x)y \]
for all \( x, y \in N \).

(ii). Let \((f,d)\) be a left generalized derivation of near-ring \( N \). Then
\[ f(xy) = xf(y) + d(x)y \]
for all \( x, y \in N \).

**Proof (i).** For any \( x, y \in N \), we get
\[ f(x(y + y)) = f(x)(y + y) + xd(y + y) = f(x)y + f(x)y + xd(y) + xd(y). \]
On the other hand,
\[ f(xy + xy) = f(x)y + xd(y) + f(x)y + xd(y). \]
Comparing these two expressions, we obtain

\[ f(x)y + xd(y) = xd(y) + f(x)y, \]

and so,

\[ f(xy) = xd(y) + f(x)y, \quad \text{for all } x, y \in N. \]

(ii). For any \( x, y \in N \), we get

\[ f(x(y + y)) = d(x)(y + y) + xf(y + y) = d(x)y + d(x)y + xf(y) + xf(y). \]

On the other hand,

\[ f(xy + xy) = d(x)y + xf(y) + d(x)y + xf(y). \]

Comparing these two expressions, we obtain

\[ d(x)y + xf(y) = xf(y) + d(x)y, \]

and so,

\[ f(xy) = xf(y) + d(x)y, \quad \text{for all } x, y \in N. \]

**Lemma 4.2.2 (i).** Let \((f, d)\) be a generalized derivation of near-ring \( N \). Then

\[ (f(x)y + xd(y))z = f(x)yz + xd(y)z, \quad \text{for all } x, y, z \in N. \]

(ii). Let \((f, d)\) be a generalized derivation of near-ring \( N \). Then

\[ (d(x)y + xf(y))z = d(x)yz + xf(y)z, \quad \text{for all } x, y, z \in N. \]

**Proof (i).** For all \( x, y, z \in N \), we get

\[ f((xy)z) = f(xy)z + xyd(z). \]

On the other hand,

\[ f(xyz) = f(xyz + xd(yz)) = f(xyz + xd(y)z + xyd(z)). \]

From these two expressions of \( f(xyz) \), we obtain that, for all \( x, y, z \in N \),

\[ (f(x)y + xd(y))z = f(x)yz + xd(y)z. \]
(ii). For all $x, y, z \in N$, we get
\[
f((xy)z) = f(xy)z + xyd(z) = (d(x)y + xf(y))z + xyd(z).
\]
On the other hand,
\[
f(x(yz)) = d(x)yz + xf(yz)
\]
\[
= d(x)yz + xf(y)z + xyd(z).
\]
From these two expressions of $f(xyz)$, we obtain that, for all $x, y, z \in N$,
\[
(d(x)y + xf(y))z = d(x)yz + xf(y)z.
\]

Lemma 4.2.3. Let $N$ be a prime near-ring, $(f, d)$ be a non-zero generalized derivation of $N$ and $a \in N$.

(i) if $af(N) = \{0\}$ then $a = 0$.

(ii) if $f(N)a = \{0\}$ then $a = 0$.

Proof (i). For all $x, y \in N$, we get
\[
0 = af(xy) = af(x)y + axd(y).
\]
and so,
\[
aNfd(N) = \{0\}.
\]
Since $N$ is prime near-ring and $d \neq 0$, we obtain $a = 0$.

(ii). For all $x, y \in N$, we get
\[
0 = f(xy)a = d(x)ya + xf(y)a,
\]
and so,
\[
d(N)Na = \{0\}.
\]
Since $N$ is prime near-ring and $d \neq 0$, we obtain $a = 0$.

We are now in position to prove the following theorem:
Theorem 4.2.1 ([42, Theorem 2.6]). Let \((f,d)\) be a generalized derivation on a prime near-ring \(N\). If \(f(N) \subseteq Z(N)\) then \((N,+)\) is abelian. Moreover, if \(N\) is 2-torsion free, then \(N\) is a commutative ring.

Proof. Suppose that \(a \in N\) such that \(f(a) \neq 0\). So, \(f(a) \in Z(N) \setminus \{0\}\) and \(f(a) + f(a) \in Z(N) \setminus \{0\}\). For all \(x,y \in N\), we have

\[(x + y)(f(a) + f(a)) = (f(a) + f(a))(x + y),\]

that is,

\[xf(a) + xf(a) + yf(a) + yf(a) = f(a)x + f(a)y + f(a)x + f(a)y.\]

Since \(f(a) \in Z(N)\), we get

\[f(a)x + f(a)y = f(a)y + f(a)x,\]

and so,

\[f(a)(x,y) = 0 \text{ for all } x,y \in N.\]

Since \(f(a) \in Z(N) \setminus \{0\}\) and \(N\) is a prime near-ring, it follows that \((x,y) = 0\), for all \(x,y \in N\). Thus \((N,+)\) is abelian.

Using the hypothesis, for any \(x,z \in N\),

\[zf(xx) = f(xx)z.\]  

(4.2.3)

By Lemma 4.2.2(ii), we have

\[zf(x)x + zxd(x) = d(x)xz + xf(x)z \text{ for all } x,z \in N.\]

Using \(f(N) \subseteq Z(N)\) and \((N,+)\) is abelian, we obtain that

\[f(x)[z,x] = d(x)xz - zxd(x), \text{ for all } x,z \in N.\]  

(4.2.4)

Substituting \(f(y)\) for \(z\) in (4.2.4), and \(f(N) \subseteq Z(N)\), we get

\[0 = d(x)xf(y) - f(y)xd(x) = f(y)[d(x),x].\]

Using Lemma 4.2.3(ii), we get

\[[d(x),x] = 0 \text{ for all } x \in N.\]  

(4.2.5)
Again, by the hypothesis, for any \( x, y \in N \), we have
\[ f(xy)x = xf(xy). \]

Using Lemma 4.2.2(ii)
\[ d(x)yx + xf(y)x = xd(x)y + xxf(y). \] for all \( x, y \in N \).

Since \( f(N) \subset Z(N) \), we get
\[ d(x)yx = xd(x)y \] for all \( x, y \in N \).

Using equation (4.2.5), we have
\[ d(x)yx = d(x)xy, \] for all \( x, y \in N \).

Replacing \( y \) by \( yz \) in the above relation and using this equation, we get
\[ d(x)N[x, z] = 0 \] for all \( x, z \in N \).

Hence either \( x \in Z(N) \) or \( d(x) = 0 \). Let \( K = \{ x \in N \mid x \in Z(N) \} \) and \( L = \{ x \in N \mid d(x) = 0 \} \). Then \( K \) and \( L \) are two additive subgroups of \( N \) whose union is \( N \). However, a group cannot be the union of two of its proper subgroups, hence either \( N = K \) or \( N = L \). Since \( d \neq 0 \), we are forced to conclude that \( N \) is a commutative ring.

**Remark 4.2.2.** In the original proof presented by Golbasi [42], it has been shown that \((N, \cdot)\) is commutative after (4.2.3). In fact, by the argument given by Golbasi, the commutativity of \((N, \cdot)\) does not follow and the proof contains many errors. However, it has been corrected in the similar setting after relation (4.2.3).

In the year 1992 Daif and Bell [36] established that a semi prime ring \( R \) must be commutative if it admits a derivation \( d \) such that \( d([x, y]) = [x, y] \). Motivated by this result Golbasi [43] studied generalized derivations in near-rings satisfying certain identities and obtained the following results:

**Theorem 4.2.2** ([43, Theorem 3.1]). Let \((f, d)\) be a generalized derivation of a prime near-ring \( N \). If \( f([x, y]) = 0 \) for all \( x, y \in N \), then \( N \) is a commutative ring.
Proof. Assume that \( f([x,y]) = 0 \) for all \( x, y \in N \). Substituting \( xy \) instead of \( y \), obtaining
\[
f([x,xy]) = f(x[x,y]) = d(x)[x,y] + xf([x,y]) = 0.
\]
Since the second term is zero, it is clear that
\[
d(x)xy = d(x)yx \text{ for all } x, y \in N.
\] (4.2.6)

Using the same arguments as used in the proof of Theorem 4.2.1, we get that \( N \) is a commutative ring.

**Theorem 4.2.3** ([43, Theorem 3.2]). Let \( (f,d) \) be a generalized derivation of a prime near-ring \( N \). If \( f([x,y]) = \pm [x,y] \), for all \( x, y \in N \), then \( N \) is a commutative ring.

Proof. Assume that \( f([x,y]) = \pm [x,y] \) for all \( x, y \in N \). Replacing \( y \) by \( xy \) in the hypothesis, we have
\[
f([x,xy]) = \pm (x^2y - xyx) = \pm x[x,y] .
\]
On the other hand,
\[
f([x,xy]) = f(x[x,y])
\]
\[
= d(x)[x,y] + xf([x,y])
\]
\[
= d(x)[x,y] + x(\pm [x,y]).
\]
It follows from the two expressions for \( f([x,xy]) \) that
\[
d(x)xy = d(x)yx \text{ for all } x, y \in N.
\]
Using the same arguments as used in the proof of Theorem 4.2.1, we get that \( N \) is a commutative ring.

**4.3 Generalized derivations as homomorphisms**

In the present section we shall study the generalization of some well known results concerning derivations of prime rings to generalized derivations of prime near-rings. We begin with the following result due to Golbasi [43] which has its independent interests also.
Lemma 4.3.1. Let \((f, d)\) and \((g, h)\) be two generalized derivations of a prime near-ring \(N\). If \(h\) is a non-zero derivation on \(N\) and \(f(x)h(y) = -g(x)d(y)\) for all \(x, y \in N\), then \((N, +)\) is abelian.

\textit{Proof.} Suppose that
\[
f(x)h(y) + g(x)d(y) = 0, \quad \text{for all } x, y \in N.
\]
We substitute \(y + z\) for \(y\), thereby obtaining
\[
f(x)h(y) + f(x)h(z) + g(x)d(y) + g(x)d(z) = 0.
\]
Using the hypothesis, we get
\[
f(x)h(y, z) = 0, \quad \text{for all } x, y, z \in N.
\]
It follows by Lemma 4.2.3(ii) that \(h(y, z) = 0\), for all \(y, z \in N\). For any \(w \in N\), we have
\[
h(wy, wz) = h(w(y, z)) = h(w)(y, z) + wh(y, z) = 0,
\]
and so,
\[
h(w)(y, z) = 0, \quad \text{for all } w, y, z \in N.
\]
By Lemma 1.4.3 (iii) yields that \((N, +)\) is abelian.

Theorem 4.3.1 ([42, Theorem 2.5]). Let \((f, d)\) be a generalized derivation on a prime near-ring \(N\) where \(d\) is a non-zero derivation on \(N\). If \(N\) is 2-torsion free near-ring and \(f^2 = 0\), then \(f = 0\).

\textit{Proof.} For arbitrary \(x, y \in N\) we have
\[
0 = f^2(xy) = f(f(xy)) = f(f(x)y + xd(y)) = f^2(x)y + 2f(x)d(y) + xd^2(y).
\]
By the hypothesis,
\[
2f(x)d(y) + xd^2(y) = 0 \quad \text{for all } x, y \in N. \tag{4.3.1}
\]
Writing $f(x)$ by $x$ in (4.3.1), we get

$$f(x)d^2(y) = 0 \text{ for all } x, y \in N.$$ 

By Lemma 4.2.3(ii), we obtain that $d^2(N) = \{0\}$ or $f = 0$. If $d^2(N) = \{0\}$, then by Lemma 1.4.3(iv), we arrive at a contradiction. So, we find that $f = 0$.

In the year 1989 Bell and Kappe [23] proved that if $d$ is a derivation of a semi-prime ring $R$ which is also either a homomorphism or an anti homomorphism, then $d = 0$. Motivated by this result, Golbasi [43] proved the following theorem for generalized derivation of near-ring as follows:

**Theorem 4.3.2.** Let $(f, d)$ be a non-zero generalized derivation on a prime near-ring $N$. If $f$ acts as a homomorphism on $N$, then $f$ is the identity map on $N$.

**Proof.** Assume that $f$ acts as a homomorphism on $N$. Then we obtain

$$f(xy) = f(x)f(y) = d(x)y + xf(y), \text{ for all } x, y \in N. \quad (4.3.2)$$

Replacing $y$ by $yz$ in (4.3.2), we arrive at

$$f(x)f(yz) = d(x)yz + xf(yz).$$

Since $(f, d)$ be a generalized derivation and $f$ acts as a homomorphism on $N$, we deduce that

$$f(xy)f(z) = d(x)yz + xd(y)z + xyf(z).$$

By Lemma 4.2.2 (ii), we get

$$d(x)zf(z) + xf(y)f(z) = d(x)yz + xd(y)z + xyf(z),$$

and so,

$$d(x)zf(z) + xf(yz) = d(x)yz + xd(y)z + xyf(z).$$

That is,

$$d(x)zf(z) + xd(y)z + xyf(z) = d(x)yz + xd(y)z + xyf(z).$$

Hence, we deduce that

$$d(x)y(f(z) - z) = 0 \text{ for all } x, y, z \in N.$$
Because $N$ is prime and $d \neq 0$, we have $f(z) = z$ for all $z \in N$. Thus, $f$ is the identity map.

**Theorem 4.3.3** ([43, Theorem 3.4]). Let $(f, d)$ be a non-zero generalized derivation on a prime near-ring $N$. If $f$ acts as an anti homomorphism on $N$, then $f$ is the identity map on $N$.

**Proof.** By the hypothesis, we have

$$f(xy) = f(y)f(x) = d(x)y + xf(y) \text{ for all } x, y \in N. \quad (4.3.3)$$

Replacing $y$ by $xy$ in (4.3.3), we obtain

$$f(xy)f(x) = d(x)xy + xf(xy).$$

Since $(f, d)$ is a generalized derivation and $f$ acts as an anti homomorphism on $N$, we deduce that

$$(d(x)y + xf(y))f(x) = d(x)xy + xf(y)f(x).$$

By Lemma 4.2.2 (ii), we get

$$d(x)yf(x) + xf(y)f(x) = d(x)xy + xf(y)f(x),$$

and so

$$d(x)yf(x) = d(x)xy \text{ for all } x, y \in N.$$ 

Replacing $y$ by $yz$ and using this equation, we have

$$d(x)N[f(x), z] = 0 \text{ for all } x, z \in N.$$ 

Hence we obtain that either $d(x) = 0$ or $f(x) \in Z(N)$, for all $x \in N$. By a standard argument, one of these must hold for all $x \in N$. Since $d \neq 0$, the second possibility gives that $N$ is a commutative ring by Theorem 4.2.1, and so we deduce that $f$ is the identity map on $N$ by Theorem 4.3.2.

**4.4 Some results on generalized derivations in near-rings**

The study of generalized derivations have received significant attention in recent years. In [26], Bell and Mason initiated the study of derivations in near-rings. Further,
Bell [15] and Golbasi [42], [43] studied generalized derivations in near-rings. We begin with the following results due to Golbasi [43].

**Theorem 4.4.1.** Let \((f, d)\) be a generalized derivation of a prime near-ring \(N\) such that \(d(Z(N)) \neq \{0\}\), and \(a \in N\). If \([f(x), a] = 0\) for all \(x \in N\), then \(a \in Z(N)\).

*Proof.* Since \(d(Z(N)) \neq \{0\}\), there exists \(c \in Z(N)\) such that \(d(c) \neq 0\). Furthermore, as \(d\) is a derivation, it is clear that \(d(c) \in Z(N)\). Replacing \(x\) by \(cx\) in the hypothesis and using Lemma 4.2.2(ii), we have \(f(cx)a = af(cx)\) i.e.,

\[
d(c)xa + cf(x)a = ad(c)x + acf(x).
\]

Since \(c \in Z(N)\) and \(d(c) \in Z(N)\), we get

\[
d(c)N[x, a] = 0\ 
\text{for all } x \in N.
\]

By the primeness of \(N\) and \(0 \neq d(c) \in Z(N)\), we obtain that \(a \in Z(N)\).

**Theorem 4.4.2.** Let \((f, d)\) be a generalized derivation of a prime near-ring \(N\) and \(a \in N\). If \([f(x), a] = 0\) for all \(x \in N\), then \(d(a) \in Z(N)\).

*Proof.* If \(a = 0\), then there is nothing to prove. Hence we assume that \(a \neq 0\). Replacing \(x\) by \(ax\) in the hypothesis, we have \(f(ax)a = af(ax)\) i.e.,

\[
d(a)xa + af(x)a = ad(a)x + aaf(x).
\]

Using \(f(x)a = af(x)\), we have

\[
d(a)xa = ad(a)x\ 
\text{for all } x \in N.
\]

Taking \(xy\) instead of \(x\) in the last equation and using this, we conclude that

\[
d(a)N[a, y] = 0\ 
\text{for all } y \in N.
\]

Since \(N\) is a prime near-ring, we have either \(d(a) = 0\) or \(a \in Z(N)\). If \(0 \neq a \in Z(N)\), then \((N, +)\) is abelian by Lemma 1.4.3(ii). Thus \(f(xa) = f(ax)\), and hence

\[
f(x)a + xd(a) = d(a)x + af(x),
\]
this implies that,

\[ [d(a), x] = 0 \] for all \( x \in N \).

That is, \( d(a) \in Z(N) \). Hence in either case we have \( d(a) \in Z(N) \).

**Theorem 4.4.3.** Let \((f, d)\) and \((g, h)\) be two generalized derivations of a prime near-ring \( N \). If \( N \) is 2-torsion free and \( f(x)h(y) = -g(x)d(y) \) for all \( x, y \in N \), then \( f = 0 \) or \( g = 0 \).

**Proof.** If \( h = 0 \) or \( d = 0 \), then the proof of the theorem is obvious. So, we may assume that \( h \neq 0 \) and \( d \neq 0 \). Therefore we know that \((N, +)\) is abelian by Lemma 4.3.1.

Now suppose that

\[
\frac{f(x)h(y) + g(x)d(y)}{0} = 0 \] for all \( x, y \in N \).

Replacing \( x \) by \( uv \) in this equation and using the hypothesis, we get

\[
f(uv)h(y) + g(uv)d(y) = uf(v)h(y) + d(u)vh(y) + ug(v)d(y) + h(u)vd(y) = 0,
\]

and so

\[
d(u)vh(y) = -h(u)vd(y) \] for all \( u, v, y \in N \). (4.4.1)

Taking \( yt \) instead of \( y \) in the above relation, we obtain

\[
d(u)vh(y) + d(u)vyh(t) = -h(u)vd(y) + h(u)vyd(t).
\]

That is,

\[
d(u)vyh(t) = -h(u)vyd(t) \] for all \( u, v, y, t \in N \). (4.4.2)

Replacing \( y \) by \( h(y) \) in (4.4.2) and using (4.4.2), we have

\[
h(u)N(d(y)h(t) - h(y)d(t)) = 0 \] for all \( u, y, t \in N \).

Since \( N \) is a prime near-ring and \( h \neq 0 \), we obtain that

\[
d(y)h(t) = h(y)d(t) \] for all \( y, t \in N \). (4.4.3)

Now again taking \( uv \) instead of \( x \) in the initial hypothesis, we get

\[
f(uv)h(y) + ud(v)h(y) + g(u)vd(y) + uh(v)d(y) = 0.
\]
Using (4.4.3) yields that

\[ f(u)v h(y) + 2uh(v)d(y) + g(u)v d(y) = 0 \quad \text{for all } u, v, y \in N. \]

Taking \( h(v) \) instead of \( v \) in this equation, we arrive at

\[ f(u)h(v)h(y) + 2uh^2(v)d(y) + g(u)h(v)d(y) = 0. \]

By the hypothesis and (4.4.3), we have

\[ 0 = -g(u)h(v)d(y) + 2uh^2(v)d(y) + g(u)h(v)d(y), \]

and so,

\[ 2uh^2(v)d(y) = 0 \quad \text{for all } u, v, y \in N. \]

Since \( N \) is a 2-torsion free prime near-ring, we obtain that \( h^2(N)d(N) = \{0\} \). An appeal to Lemma 1.4.3 (ii) and (iv) gives that \( h = 0 \). This contradicts our assumption.

**Theorem 4.4.4.** Let \((f, d)\) and \((g, h)\) be two generalized derivations of a prime near-ring \( N \). If \((fg, dh)\) acts as a generalized derivation on \( N \), then \( f = 0 \) or \( g = 0 \).

**Proof.** By calculating \( fg(xy) \) in two different ways, we see that

\[ g(x)d(y) + f(x)h(y) = 0 \quad \text{for all } x, y \in N. \]

This completes the proof by using Theorem 4.4.3.

In year 2008 Bell [15] established near-ring analogue of several results earlier obtained for derivation in near-rings by Bell and Mason [26]. We begin with the following lemma obtained by Bell [15] which allows us limited right distributive property in left near-rings.

**Lemma 4.4.1.** Let \( N \) be an arbitrary near-ring and \((f, d)\) be a generalized derivation of \( N \). Then \((f(a)b + ad(b))c = f(a)bc + ad(b)c\) for all \( a, b, c \in N \).

**Proof.** Clearly

\[ f((ab)c) = f(ab)c + abd(c) = (f(a)b + ad(b))c + abd(c); \]
and using Lemma 2.2.3, we obtain

\[ f(a(bc)) = f(a)bc + ad(bc) = f(a)bc + ad(b)c + abd(c). \]

Comparing these two expressions for \( f(abc) \) gives the desired result.

**Remark 4.4.1.** Lemma 1.4.3(iv) can not be extended to generalized derivation even if \( N \) is assumed to be a ring; as we see that by letting \( N \) be the ring \( M_2(F) \) of \( 2 \times 2 \) matrices over \( F \) and \( f \) be defined by \( f(x) = e_{12}x \).

However, Bell in his paper [15] proved the following results:

**Theorem 4.4.5.** Let \( N \) be a prime near-ring, and \((f,d)\) be a generalized derivation of \( N \). If \( f^2 = 0 \), then \( d^3 = 0 \). Moreover, if \( N \) is 2-torsion-free, then \( d(Z(N)) = \{0\} \).

**Proof.** We have

\[ f^2(xy) = f(f(x)y + xd(y)) = f(x)d(y) + f(x)d(y) + xd^2(y) = 0 \quad \text{for all} \quad x, y \in N. \quad (4.4.4) \]

Applying \( f \) to (4.4.4) gives

\[ f(x)d^2(y) + f(x)d^2(y) + f(x)d^2(y) + xd^2(y) = 0, \quad \text{for all} \quad x, y \in N. \quad (4.4.5) \]

Substituting \( d(y) \) for \( y \) in (4.4.4) gives

\[ f(x)d^2(y) + f(x)d^2(y) + xd^3(y) = 0. \quad (4.4.6) \]

Therefore, by (4.4.5) and (4.4.6),

\[ f(x)d^2(y) = 0 \quad \text{for all} \quad x, y \in N. \quad (4.4.7) \]

It now follows from (4.4.6) that \( xd^3(y) = 0 \) for all \( x, y \in N \); and since \( N \) is prime, we find that \( d^3 = 0 \).

Suppose now that \( N \) is 2-torsion free and that \( d(Z(N)) \neq \{0\} \), and let \( z \in Z(N) \) be such that \( d(z) \neq 0 \). Then if \( x, y \in N \) and \( f(N)x = \{0\} \), then

\[ f(yz)x = f(y)zx + yd(z)x = 0 = yd(z)x, \]

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and since \( N \) is prime and \( d(z) \) is not a zero divisor, then \( x = 0 \). It now follows from (4.4.7) that \( d^2 = 0 \) and hence by Lemma 1.4.3(iv) that \( d = 0 \). But this contradicts our assumption that \( d(Z(N)) \neq \{0\} \), hence \( d(Z(N)) = \{0\} \).

**Theorem 4.4.6.** Let \( N \) be a 2-torsion free prime near-ring with 1. If \( f \) is a generalized derivation on \( N \) such that \( f^2 = 0 \) and \( f(1) \in Z(N) \), then \( f = 0 \).

**Proof.** Note that \( f(x) = f(1x) = f(1)x + 1d(x), \) so

\[
f(x) = cx + d(x), \quad c \in Z(N).
\]

(4.4.8)

If \( c = 0 \), then \( f = d \) and \( d^2 = 0 \), so \( d = 0 \) by Lemma 1.4.3(iv) and therefore \( f = 0 \).

If \( c \neq 0 \), then \( c \) is not a zero divisor, hence by (4.4.7) \( d^2 = 0 \) and \( d = 0 \). But then \( f(x) = cx \) and \( f^2(x) = c^2x = 0 \) for all \( x \in N \). Since \( c^2 \) is not a zero divisor, we get \( N = \{0\} \), a contradiction. Thus \( c = 0 \).

The following result obtained by Bell [15] is an extension of Theorem 1.4.2.

**Theorem 4.4.7.** Let \( N \) be a 2-torsion free prime near-ring. If \( N \) admits a non-zero generalized derivation \( f \) such that \( f(N) \subseteq Z(N) \), then \( N \) is a commutative ring.

In order to develop the proof of the above theorem, we need the following lemma essentially proved in [15].

**Lemma 4.4.2.** Let \( N \) be a prime near-ring, and let \((f, d)\) be a generalized derivation of \( R \), where \( d \neq 0 \). If \( d(f(N)) = \{0\} \), then \( f(d(N)) = \{0\} \).

**Proof.** Assume that \( d(f(x)) = 0 \) for all \( x \in N \). It follows that \( d(f(xy)) = d(f(x)y) + d(xy) = 0 \) for all \( x, y \in N \), that is,

\[
f(x)d(y) + d(x)d(y) + xd^2(y) = 0 \quad \text{for all } x, y \in N.
\]

(4.4.9)

Applying \( d \) again, we get

\[
f(x)d^2(y) + d^2(x)d(y) + d(x)d^2(y) + d(x)d^2(y) + xd^3(y) = 0 \quad \text{for all } x, y \in N.
\]

(4.4.10)
Taking \( d(y) \) instead of \( y \) in (4.4.9) gives
\[
f(x)d^2(y) + d(x)d^2(y) + xd^2(y) = 0.
\]
Hence (4.4.10) yields
\[
d^2(x)d(y) + d(x)d^2(y) = 0 \text{ for all } x, y \in N. \tag{4.4.11}
\]
Now, substitute \( d(x) \) for \( x \) in (4.4.9), obtaining
\[
f(d(x))d(y) + d^2(x)d(y) + d(x)d^2(y) = 0,
\]
and use (4.4.11) to conclude that \( f(d(x))d(y) = 0 \) for all \( x, y \in N \). Thus, by Lemma 1.4.3 (iii),
\[
f(d(x)) = 0 \text{ for all } x \in N.
\]

**Proof of Theorem 4.4.7.** Since \( f \neq 0 \), there exists \( x \in N \) such that \( 0 \neq f(x) \in Z(N) \). Since \( f(x) + f(x) = f(x + x) \in Z(N), (N,+) \) is abelian by Lemma 1.4.3 (ii). To complete the proof, we show that \( N \) is multiplicatively commutative.

First consider the case \( d = 0 \), so that \( f(xy) = f(x)y \in Z(N) \) for all \( x, y \in N \). Then \( f(x)yw = w f(x)y \), hence \( f(x)(yw - wy) = 0 \) for all \( x, y, w \in N \). Choosing \( x \) such that \( f(x) \neq 0 \) and invoking Lemma 1.4.3 (i), we get \( yw - wy = 0 \) for all \( y, w \in N \).

Now assume that \( d \neq 0 \), and let \( c \in Z(N) \setminus \{0\} \). Then \( f(xc) = f(x)c + xd(c) \in Z(N) \); therefore, \((f(x)c + xd(c))y = y(f(x)c + xd(c))\) for all \( x, y \in N \), and by Lemma 4.4.1, we see that \( f(x)c_y + xd(c)y = yf(x)c + yxd(c) \). Since both \( f(x) \) and \( d(c) \) are in \( Z(N) \), we have \( d(c)(xy - yx) = 0 \) for all \( x, y \in N \), and provided that \( d(Z(N)) \neq \{0\} \), we can conclude that \( N \) is commutative.

Assume that \( d \neq 0 \) and \( d(Z(N)) = \{0\} \). In particular, \( d(f(x)) = 0 \) for all \( x \in N \). Note that for \( c \in N \) such that \( f(c) = 0 \), \( f(cx) = cd(x) \in Z(N) \); hence by Lemma 4.4.2, \( d(x)d(y) \in Z(N) \) and \( d(y)d(x) \in Z(N) \) for each \( x, y \in N \). If one of these is 0, the other is a central element squaring to 0, hence is also 0. The remaining possibility is that \( d(x)d(y) \) and \( d(y)d(x) \) are non-zero central elements, in which case \( d(x) \) is not a zero divisor. Thus \( d(x)d(x)d(y) = d(x)d(y)d(x) \) yields
\[
d(x)(d(x)d(y) - d(y)d(x)) = 0 = d(x)d(y) - d(y)d(x).
\]
Consequently, $N$ is commutative by Theorem 1.4.4.

**Remark 4.4.2.** The following example shows that Theorem 1.4.4 can not be extended to generalized derivations, even if $N$ is assumed to be a ring.

**Example 4.4.1.** Consider the ring $H$ of real quaternions, and define $f : H \rightarrow H$ by $f(x) = ix + xi$. It is easy to check that $f$ is a generalized derivation with associated derivation $d$ given by $d(x) = xi - ix$, and that $f(x)f(y) = f(y)f(x)$ for all $x, y \in H$.

A weak generalization of Theorem 1.4.4, was given in [42] by Golbasi but the proof given by Golbasi was not correct (At one point, both left and right distributivity were assumed). However, Bell [15] rectified the error in the proof given by Golbasi and obtained the following result:

**Theorem 4.4.8([15, Theorem 4.1])** Let $N$ be a 2-torsion free prime near-ring and let $(f, d)$ be a generalized derivation of $N$ such that $f$ satisfies $f(xy) = d(x)y + xf(y)$ for all $x, y \in N$. If $f(x)f(y) = f(y)f(x)$ for all $x, y \in N$, then $N$ is a commutative ring.

**Proof.** The argument used in the proof of Theorem 4.2.1 shows that if both $z$ and $z + z$ commute element wise with $f(N)$, then we have

$$zf(x, y) = 0 \text{ for all } x, y \in N. \quad (4.4.12)$$

Substituting $f(t), t \in N$ for $z$ in (4.4.12), we get $f(t)f(x, y) = 0$. By Lemma 4.2.3(i), we obtain that $f(x, y) = 0$ for all $x, y \in N$. For any $w \in N$, we have

$$0 = f(wx, wy) = f(w(x, y)) = d(w)f(x, y) + wf(x, y)$$

and so, we obtain

$$d(w)(x, y) = 0, \text{ for all } x, y \in N.$$ 

From Lemma 1.4.3 (iii), we get $(x, y) = 0$ for all $x, y \in N$.

Now, assume that $N$ is 2-torsion free. By the assumption $f(x)f(y) = f(y)f(x)$, for all $x, y \in N$, we have

$$f(z)f(f(x)y) = f(f(x)y)f(z) \quad \text{for all } x, y, z \in N.$$
Hence, we get
\[
\begin{align*}
f(z)d(f(x))y + f(z)f(x)f(y) &= d(f(x))yf(z) + f(x)f(y)f(z) \\
f(z)f(f(x))y + f(x)f(z)f(y) &= d(f(x))yf(z) + f(x)f(z)f(y)
\end{align*}
\]
and so,
\[
f(z)d(f(x))y = d(f(x))yf(z), \quad \text{for all } x, y, z \in N. \tag{4.4.13}
\]

If we take \(yw\) instead of \(y\) in (4.4.13), then
\[
d(f(x))yw f(z) = f(z)d(f(x))yw = d(f(x))y f(z)w
\]
and so,
\[
d(f(x))N\{f(z)w - wf(z)\} = 0 \quad \text{for all } x, z \in N.
\]

Since \(N\) is a prime near-ring, we have
\[
d(f(N)) = \{0\} \quad \text{or} \quad f(N) \subset Z(N).
\]

Hence in view of Theorem 4.4.7, we may assume that \(d(f(N)) = \{0\}\) and therefore, by Lemma 4.4.2, \(f(d(N)) = \{0\}\). We calculate \(f(d(x)d(y))\) in two ways. Using the defining property of \(f\), we obtain
\[
f(d(x)d(y)) = f(d(x))d(y) + d(x)d^2(y) = d(x)d^2(y),
\]
and using the given condition, we obtain
\[
f(d(x)d(y)) = d^2(x)d(y) + d(x)f(d(y)) = d^2(x)d(y).
\]

Thus, \(d^2(x)d(y) = d(x)d^2(y)\) for all \(x, y \in N\). But since \(d(f(N)) = \{0\}\), (4.4.11) holds in this case as well, therefore \(d^2(x)d(y) = 0\) for all \(x, y \in N\), hence by Lemma 1.4.3 (iii) \(d^2 = 0\). Thus \(d = 0\), contrary to our original hypothesis, so that the case \(d(f(N)) = \{0\}\) does not in fact occur.