CHAPTER 3
RESOLVENT OPERATOR TECHNIQUES FOR SOME
SYSTEMS OF VARIATIONAL INCLUSIONS IN
HILBERT SPACES

3.1 INTRODUCTION

In recent part, the techniques based on different classes of resolvent operators have been developed by many researches to study the existence and iterative approximations of solutions of various classes of systems of variational inclusions.

In this chapter we study some properties of maximal $\eta$-monotone mappings, $H$-monotone mappings and $(H, \eta)$-monotone mappings and their associated resolvent operators. Further using the techniques based on these resolvent operators, we develop iterative algorithms for some systems of variational inclusions in Hilbert spaces. We study the existence of solutions of these systems and discuss the convergence analysis of these iterative algorithms.

The remaining part of this chapter is organized as follows:

In Section 3.2, we consider a system of generalized quasi variational-like inclusions considered by Qiu et al. [106]. Using the resolvent operator technique associated with maximal $\eta$-monotone mappings, we discuss the existence of solution and develop an iterative algorithm of this system of quasi variational-like inclusions. Further, we discuss the convergence criteria for the iterative algorithm.

In Section 3.3, we consider a system of multi-valued variational inclusions considered by Yan et al. [117]. Using the resolvent operator technique associated with maximal $H$-monotone mappings due to Fang and Huang [41,42], we discuss the existence of solution and develop an iterative algorithm of this system of variational inclusions. Further, we discuss the convergence criteria of the iterative algorithm.

In Section 3.4, we consider a system of multi-valued quasi variational-like inclusions considered by Peng et al. [102]. Using the resolvent operator technique associated with $(H, \eta)$-accretive mappings, we discuss the existence of solution and
develop a three-step iterative algorithm of this system of variational inclusions. Fur-
ther, we discuss the convergence criteria for the iterative algorithm.

The chapter is based on work of Qiu et al. [106], Yan et al. [117], and Peng et al. [102].

Throughout this chapter, unless or otherwise stated let $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ are
real Hilbert spaces. If there is no confusion, we denote the inner product and norm
of $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively.

3.2 SYSTEM OF GENERALIZED QUASI-VARIATIONAL-LIKE
INCLUSIONS

For each $i = 1, 2, 3$, let $g_i : H_i \rightarrow H_i$, $\eta_i : H_i \times H_i \rightarrow H_i$ and $F_i, T_i : H_1 \times
H_2 \times H_3 \rightarrow H_i$ are all single-valued mappings ($i = 1, 2, 3$). Let $M_i, A_i, B_i, C_i : H_i \rightarrow
CB(H_i)$, ($i = 1, 2, 3$) be multi-valued mappings. We consider the following system
of generalized quasi-variational-like inclusions (in short, SGQVLI):

Find $(x, y, z) \in H_1 \times H_2 \times H_3$, $u_i \in A_i(x)$, $v_i \in B_i(y)$, $w_i \in C_i(z)$ ($i = 1, 2, 3$)
such that

$$
\begin{align*}
0 & \in g_1(x) - g_2(y) + \rho_1(T_1(x, y, z)) + F_1(u_1, v_1, w_1) + M_1(g_1(x))), \\
0 & \in g_2(y) - g_3(z) + \rho_2(T_2(x, y, z)) + F_2(u_2, v_2, w_2) + M_2(g_2(y))), \\
0 & \in g_3(z) - g_1(x) + \rho_3(T_3(x, y, z)) + F_3(u_3, v_3, w_3) + M_3(g_3(z))),
\end{align*}
$$

(3.2.1)

where $\rho_1 > 0, \rho_2 > 0$ and $\rho_3 > 0$ are constants.

Definition 3.2.1. Let $T : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ and $g : \mathcal{H} \rightarrow \mathcal{H}$ be single-valued
mappings, then $T$ is said to be

(i) $(\xi, \sigma, \zeta)$-Lipschitz continuous if there exist constants $\xi > 0, \eta > 0$ and $\zeta > 0$
such that

$$
\|T(x_1, y_1, z_1) - T(x_2, y_2, z_2)\| \leq \xi \|x_1, x_2\| + \sigma \|y_1, y_2\| + \zeta \|z_1, z_2\|,
\forall x_1, y_1, z_1, x_2, y_2, z_2 \in \mathcal{H};
$$

(ii) monotone with respect to $g$ in the first argument if

$$
\langle T(x_1, \ldots) - T(x_2, \ldots), g(x_1) - g(x_2) \rangle \geq 0, \quad \forall x_1, x_2 \in \mathcal{H};
$$

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(iii) \( \beta \)-strongly monotone with respect to \( g \) in the first argument if there exists a constant \( \beta > 0 \) such that
\[
\langle T(x_1, \ldots) - T(x_2, \ldots), g(x_1) - g(x_2) \rangle \geq \beta \|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in \mathcal{H}.
\]

The following lemma shows that SGQVLI(3.2.1) is equivalent a system of relations.

**Lemma 3.2.1**[106]. Let \( \eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i \) be single-valued mappings, \( M_i : \mathcal{H}_i \to 2^{\mathcal{H}_i} \) be maximal \( \eta_i \)-monotone mappings, where \( i = 1, 2, 3 \). Then \( (x, y, z, u_1, v_1, w_1, u_2, v_2, w_2, u_3, v_3, w_3) \) with \( (x, y, z) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3, u_i \in A_i(x), v_i \in B_i(y), w_i \in C_i(z) \) \((i = 1, 2, 3)\) is a solution of SGQVLI(3.2.1) if and only if
\[
\begin{align*}
g_1(x) & = J_{M_1}^{\rho_1}(g_2(y)) - \rho_1 F_1(u_1, v_1, w_1) - \rho_1(T_1(x, y, z)), \\
g_2(y) & = J_{M_2}^{\rho_2}(g_3(z)) - \rho_2 F_2(u_2, v_2, w_2) - \rho_2(T_2(x, y, z)), \\
g_3(z) & = J_{M_3}^{\rho_3}(g_1(x)) - \rho_3 F_3(u_3, v_3, w_3) - \rho_3(T_3(x, y, z)),
\end{align*}
\]
(3.2.2)

where \( J_{M_i}^{\rho_i} = (I + \rho_i M_i)^{-1}, \quad \rho_i > 0, \ i = 1, 2, 3 \).

Based on Lemma 3.2.1, we construct the following iterative algorithm for finding the approximate solution of SGQVLI(3.2.1).

**Iterative Algorithm 3.2.1**[106]. For any given \( x_0 \in \mathcal{H}_1, y_0 \in \mathcal{H}_2, z_0 \in \mathcal{H}_3 \), take \( u_0^i \in A_i(x_0), v_0^i \in B_i(y_0), w_0^i \in C_i(z_0), \) \((i = 1, 2, 3)\). Let
\[
\begin{align*}
p_0 & = J_{M_1}^{\rho_1}(g_2(y_0)) - \rho_1 F_1(u_0^1, v_0^1, w_0^1) - \rho_1(T_1(x_0, y_0, z_0)), \\
y_0 & = J_{M_2}^{\rho_2}(g_3(z_0)) - \rho_2 F_2(u_0^2, v_0^2, w_0^2) - \rho_2(T_2(x_0, y_0, z_0)), \\
z_0 & = J_{M_3}^{\rho_3}(g_1(x_0)) - \rho_3 F_3(u_0^3, v_0^3, w_0^3) - \rho_3(T_3(x_0, y_0, z_0)).
\end{align*}
\]

Hence there exist \( x_1 \in \mathcal{H}_1, y_1 \in \mathcal{H}_2, z_1 \in \mathcal{H}_3 \), such that \( p_0 = g_1(x_1), q_0 = g_2(y_1), r_0 = g_3(z_1) \). By Nadler [89], there exists \( u_1^i \in A_i(x_1), v_1^i \in B_i(y_1) \) and \( w_1^i \in C_i(z_1), \) \((i = 1, 2, 3)\) such that
\[
\begin{align*}
\|u_1^i - u_0^i\| & \leq (1 + 1) \hat{d}(A_i x_i, A_i x_0), \\
\|v_1^i - v_0^i\| & \leq (1 + 1) \hat{d}(B_i y_i, B_i y_0),
\end{align*}
\]
\[ \|w^i_{n+1} - w^0_0\| \leq (1 + 1)\tilde{H}(C_i, z_i, C_i, z_0). \]

Let
\[
\begin{align*}
p_1 &= J_{M_1}^0(g_1(y_1)) - \rho_1 F_1(u^1_1, v^1_1, w^1_1) - \rho_1 (T_1(x_1, y_1, z_1)), \\
q_1 &= J_{M_2}^0(g_2(z_1)) - \rho_2 F_2(u^2_1, v^2_1, w^2_1) - \rho_2 (T_2(x_2, y_1, z_1)), \\
r_1 &= J_{M_3}^0(g_1(x_1)) - \rho_3 F_3(u^3_1, v^3_1, w^3_1) - \rho_3 (T_3(x_1, y_1, z_1)).
\end{align*}
\]

Hence, there exist \(x_2 \in \mathcal{H}_1\), \(y_2 \in \mathcal{H}_2\), \(z_2 \in \mathcal{H}_3\), such that \(p_1 = g_1(x_2)\), \(q_1 = g_2(y_2)\), \(r_1 = g_3(z_2)\). By induction, we can define iterative sequences \(\{x_n\}, \{y_n\}, \{z_n\}\), \(\{u^1_n\}, \{v^1_n\}, \{u^2_n\}, \{v^2_n\}, \{u^3_n\}, \{v^3_n\}\) satisfying
\[
\begin{align*}
g_1(x_{n+1}) &= J_{M_1}^0(g_1(y_n)) - \rho_1 F_1(u^1_n, v^1_n, w^1_n) - \rho_1 (T_1(x_n, y_n, z_n)), \\
g_2(y_{n+1}) &= J_{M_2}^0(g_2(z_n)) - \rho_2 F_2(u^2_n, v^2_n, w^2_n) - \rho_2 (T_2(x_n, y_n, z_n)), \\
g_3(z_{n+1}) &= J_{M_3}^0(g_1(x_n)) - \rho_3 F_3(u^3_n, v^3_n, w^3_n) - \rho_3 (T_3(x_n, y_n, z_n)),
\end{align*}
\]

The following theorem shows that SGQVLI(3.2.1) has a solution and the approximation solution obtained by Iterative Algorithm 3.2.1, converges strongly to the exact solution of SGQVLI(3.2.1).

**Theorem 3.2.1**[106]. For each \(i = 1, 2, 3\), let \(\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i\) be a \(\delta_i\)-strongly monotone and \(\gamma_i\)-Lipschitz continuous mapping; let \(\varphi_i : \mathcal{H}_i \to \mathcal{H}_i\) be a \(\alpha_i\)-strongly monotone and \(\beta_i\)-Lipschitz continuous mapping; let \(A_i, B_i, C_i : \mathcal{H}_i \to \mathcal{C}(\mathcal{H}_i)\) be \(l_{A_i}, l_{B_i}, l_{C_i}\)-\(\tilde{H}\)-Lipschitz continuous mappings, respectively and let \(M_i : \mathcal{H}_i \to 2^{\mathcal{H}}\) be a maximal \(\eta\)-monotone mapping. Let mapping \(T_1 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_1\) be \((\xi_1, \sigma_1, \zeta_1)\)-Lipschitz continuous and \(k_1\)-strongly monotone with respect to \(g_2\) in the second argument; let the mapping \(T_2 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_2\) be \((\xi_2, \sigma_2, \zeta_2)\)-Lipschitz continuous and \(k_2\)-strongly monotone with respect to \(g_3\) in the third argument;
and let the mapping $T_3 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_3$ be $(\xi_3, \sigma_3, \zeta_3)$-Lipschitz continuous and $k_3$-strongly monotone with respect to $\varphi_1$ in the first argument. Suppose that $F_i : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_i$ is $(\lambda_i, \mu_i, \gamma_i)$-Lipschitz continuous mapping. If $\alpha_i > 1$ and there exist constants $\rho_1 > 0, \rho_2 > 0, \rho_3 > 0$ such that

$$a. \max \{\rho_1 \xi_1 + \lambda_1 \rho_1 l_{A_1} + \rho_2 \xi_2 + \lambda_2 \rho_2 l_{A_2} + \lambda_3 \rho_3 l_{A_3} + \sqrt{\beta_1^2 - 2k_3 \rho_3 + \rho_3^3 \xi_3^2},$$

$$\rho_2 \sigma_2 + \mu_2 \rho_2 l_{B_2} + \rho_3 \sigma_3 + \mu_3 \rho_3 l_{B_3} + \mu_1 \rho_1 l_{B_1} + \sqrt{\beta_2^2 - 2k_1 \rho_1 + \rho_1^2 \sigma_1^2},$$

$$\rho_3 \zeta_3 + \gamma_3 \rho_3 l_{C_3} + \gamma_1 \xi_1 + \gamma_1 \rho_1 l_{C_1} + \gamma_2 \rho_2 l_{C_2} + \sqrt{\beta_3^2 - 2k_2 \rho_2 + \rho_2^3 \zeta_2^2} \} < 1, \quad (3.2.9)$$

where

$$a = \max \left\{ \frac{\tau_1}{\delta_1 \sqrt{2\alpha_1 - 1}}, \frac{\tau_2}{\delta_2 \sqrt{2\alpha_2 - 1}}, \frac{\tau_3}{\delta_3 \sqrt{2\alpha_3 - 1}} \right\}.$$

Then SGQVLI (3.2.1) admits a solution $(x, y, z, u_1, v_1, w_1, u_2, v_2, w_2, u_3, v_3, w_3)$ and sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}, \{u'_n\}, \{w_n\}, \{u_n^1\}, \{w_n^1\}, \{v_n^2\}, \{w_n^2\}, \{v_n^3\}, \{w_n^3\}$ converge to $x, y, z, u_1, v_1, w_1, u_2, v_2, w_2, u_3, v_3, w_3$ respectively, where $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}, \{u'_n\}, \{w_n\}, \{u_n^1\}, \{w_n^1\}, \{v_n^2\}, \{w_n^2\}, \{v_n^3\}, \{w_n^3\}, (i = 1, 2, 3)$ are the sequences generated by Iterative Algorithm 3.2.1.

### 3.3 SYSTEM OF MULTI-VALUED VARIATIONAL INCLUSIONS

First, we study some properties of resolvent operator associated with $H$-monotone mapping, then we study the existence and iterative approximation of solution of a system of multi-valued variational inclusions.

For that we need the following definitions:

**Definition 3.3.1**[41]. Let $H : \mathcal{H} \to \mathcal{H}$ be a single-valued mapping and $M : \mathcal{H} \to 2^\mathcal{H}$ be a multi-valued mapping, then $M$ is said to be $H$-monotone if $M$ is monotone and $(H + \lambda M)\mathcal{H} = \mathcal{H}$ holds for every $\lambda > 0$.

**Remark 3.3.1.** If $H = I$, the identity mapping, then the definition of $I$-monotone mapping is that of maximal monotone mapping. In fact, the class of $H$-monotone mappings has close relation with that of maximal monotone mappings.

**Example 3.3.1**[41]. Let $H : \mathcal{H} \to \mathcal{H}$ be a strictly monotone single-valued mapping and let $M : \mathcal{H} \to 2^\mathcal{H}$ be an $H$-monotone mapping, then $M$ is maximal monotone.
Definition 3.3.2[41]. Let \( H : \mathcal{H} \to \mathcal{H} \) be a single-valued mapping, then \( H \) is said to be coercive if
\[
\lim_{\|x\| \to \infty} \frac{\langle Hx, x \rangle}{\|x\|} = +\infty.
\]

Definition 3.3.3. Let \( A : \mathcal{H} \to \mathcal{H} \) be a single-valued mapping. \( A \) is said to be bounded if \( A(B) \) is bounded for every bounded subset \( B \) of \( \mathcal{H} \). \( A \) is said to be hemi-continuous if for any fixed \( x, y, z \in \mathcal{H} \), the function \( t \to \langle A(x + ty), z \rangle \) is continuous at 0⁺.

Example 3.3.2[41]. Let \( M : \mathcal{H} \to 2^\mathcal{H} \) be a maximal monotone mapping and \( H : \mathcal{H} \to \mathcal{H} \) be a bounded, coercive, hemi-continuous and monotone mapping. Then \( M \) is \( H \)-monotone.

The following example shows that a maximal monotone mapping need not be \( H \)-monotone for some \( H \).

Example 3.3.3[41]. Let \( \mathcal{H} = \mathbb{R}, \ M = I \) and \( H(x) = x^2 \), for all \( x \in \mathcal{H} \), Then it is easy to see that \( I \) is maximal monotone and the range of \((H + I)\) is \([-1/4, +\infty)\). Therefore, \( I \) is not \( H \)-monotone.

Definition 3.3.4[41]. Let \( H : \mathcal{H} \to \mathcal{H} \) be a strictly monotone mapping and \( M : \mathcal{H} \to 2^\mathcal{H} \) be an \( H \)-monotone mapping. The resolvent operator \( J_{M,\lambda}^H : \mathcal{H} \to \mathcal{H} \) is defined by:
\[
J_{M,\lambda}^H(x) = (H + \lambda M)^{-1}(x), \quad \forall x \in \mathcal{H}.
\]

Remark 3.3.2. When \( H=I \), Definition 3.3.4 reduces to the definition of the resolvent operator of a maximal monotone mapping [9].

Now, we have some properties of \( H \)-monotone mapping and its associated resolvent operator.

Lemma 3.3.1[41]. Let \( H : \mathcal{H} \to \mathcal{H} \) be a strictly monotone mapping and \( M : \mathcal{H} \to 2^\mathcal{H} \) be an \( H \)-monotone mapping. Then the operator \((H + \lambda M)^{-1}\) is single-valued.

Lemma 3.3.2[41]. Let \( H : \mathcal{H} \to \mathcal{H} \) be a strongly monotone mapping with constant \( r \) and \( M : \mathcal{H} \to 2^\mathcal{H} \) be an \( H \)-monotone mapping. Then the resolvent operator
$J_{M,\lambda}^H : \mathcal{H} \to \mathcal{H}$ is Lipschitz continuous with constant $1/r$, i.e.,

$$
\|J_{M,\lambda}^H(x) - J_{M,\lambda}^H(y)\| \leq \frac{1}{r}\|x - y\|, \quad \forall x, y \in \mathcal{H}.
$$

Let $F : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$, $G : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$, $H_1 : \mathcal{H}_1 \to \mathcal{H}_1$, $H_2 : \mathcal{H}_2 \to \mathcal{H}_2$ be nonlinear mappings; let $M : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be an $H_1$-monotone mapping and let $N : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be an $H_2$-monotone mapping. Let $A : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$.

The system of multi-valued variational inclusions (in short, SMVI)[8] is to:

Find $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $u \in A(x)$ and $v \in B(y)$ such that

$$
\begin{cases}
0 \in F(x, v) + M(x), \\
0 \in G(u, y) + N(y).
\end{cases}
$$

(3.3.1)

If $M(x) = \partial \varphi(x)$ and $N(y) = \partial \phi(y)$ for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, where $\varphi, \phi : \mathcal{H}_1 \to R \cup \{+\infty\}$ be proper, convex and lower semi-continuous functionals and $\partial \varphi$ and $\partial \phi$ denotes the subdifferential operators of $\varphi$ and $\phi$, respectively, then SMVI(3.3.1) reduces to the following problem:

Find $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $u \in A(x)$, and $v \in B(y)$ such that

$$
\begin{cases}
< F(x, v), z - x > + \varphi(z) - \varphi(x) \geq 0, \quad \forall z \in \mathcal{H}_1, \\
< G(u, y), w - y > + \phi(w) - \phi(y) \geq 0, \quad \forall w \in \mathcal{H}_2.
\end{cases}
$$

(3.3.2)

If $A$ and $B$ are both identity mapping, then SMVI(3.3.1) reduces to the following problem:

Find $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that

$$
\begin{cases}
0 \in F(x, y) + M(x), \\
0 \in G(x, y) + N(y),
\end{cases}
$$

(3.3.3)

which is called the system of variational inclusions considered by Fang and Huang [42].

**Lemma 3.3.3**[117]. Let $H_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $H_2 : \mathcal{H}_2 \to \mathcal{H}_2$ be strictly monotone mappings; let $M : \mathcal{H}_1 \to \mathcal{H}_1$ be $H_1$-monotone and let $N : \mathcal{H}_2 \to \mathcal{H}_2$ be $H_2$-monotone.
Then \((x, y, u, v)\) is a solution of SMVI(3.3.1) if and only if \((x, y, u, v)\) satisfies the relations

\[
\begin{align*}
x &= J_{\lambda, p}^{H_1}(H_1(x) - \rho F(x, v)), \\
y &= J_{\lambda, \mu}^{H_2}(H_2(y) - \lambda G(u, y)),
\end{align*}
\]  

(3.3.4)

where \(\rho > 0\) and \(\lambda > 0\) be constants.

Based on Lemma 3.3.3, we give the following iterative algorithm for SMVI(3.3.1).

**Iterative Algorithm 3.3.1**

**Step 1.** Choose \((x_0, y_0) \in H_1 \times H_2\) and choose \(u_0 \in A(x_0)\) and \(v_0 \in B(y_0)\).

**Step 2.** Let

\[
\begin{align*}
x_{n+1} &= J_{\lambda, p}^{H_1}(H_1(x_n) - \rho F(x_n, v_n)), \\
y_{n+1} &= J_{\lambda, \mu}^{H_2}(H_2(y_n) - \lambda G(u_n, y_n)).
\end{align*}
\]  

(3.3.5)

**Step 3.** Choose \(u_{n+1} \in A(x_{n+1})\) and \(v_{n+1} \in B(y_{n+1})\) such that

\[
\begin{align*}
||u_{n+1} - u_n|| &\leq (1 + (1 + n)^{-1})\hat{H}_1(A(a_{n+1}), A(a_n)), \\
||v_{n+1} - v_n|| &\leq (1 + (1 + n)^{-1})\hat{H}_2(B(b_{n+1}), B(b_n)),
\end{align*}
\]  

(3.3.6)

where \(\hat{H}_i(\cdot, \cdot)\) is the Hausdorff pseudo-metric on \(2^{H_i}\) for \(i = 1, 2\).

**Step 4.** If \(x_{n+1}, y_{n+1}, u_{n+1}, v_{n+1}\) satisfy (3.3.5) to sufficient accuracy, stop; otherwise, set \(n := n + 1\) and return to step 2.

The following theorem prove that approximate solution obtained by Iterative Algorithm 3.3.1, converges to a solution of SMVI(3.3.1).

**Theorem 3.3.1.** Let \(H_1 : H_1 \rightarrow H_1\) be a strongly monotone and Lipschitz continuous mapping with constants \(\gamma_1\) and \(\tau_1\), respectively; let \(H_2 : H_2 \rightarrow H_2\) be a strongly monotone and Lipschitz continuous mapping with constant \(\gamma_2\) and \(\tau_2\), respectively; let \(M : H_1 \rightarrow 2^{H_1}\) be \(H_1\)-monotone and let \(N : H_2 \rightarrow 2^{H_2}\) be \(H_2\)-monotone. Suppose that \(A : H_1 \rightarrow C(H_1)\) is \(\hat{H}_1\)-Lipschitz continuous and \(B : H_2 \rightarrow C(H_2)\) is \(\hat{H}_2\)-Lipschitz continuous with constants \(\eta_1\) and \(\eta_2\), respectively. Let \(F : H_1 \times H_2 \rightarrow H_1\) be a nonlinear mapping such that for any given \((x, y) \in H_1 \times H_2, F(\cdot, y)\) is strongly monotone with respect to \(H_1\) and Lipschitz continuous with
constants $r_1$ and $s_1$, respectively, and $F(x,.)$ is Lipschitz continuous with constant $\theta$. Let $G : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$ be a nonlinear mapping such that for any given $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $G(x,.)$ is strongly monotone with respect to $H_2$ and Lipschitz continuous with constants $r_2$ and $s_2$, and $G(., y)$ is Lipschitz continuous with constant $\xi$. If there exist constants $\rho > 0$ and $\lambda > 0$ such that

$$
\begin{align*}
\left\{\begin{array}{l}
\gamma_2 \sqrt{\tau_1^2 - 2 \rho r_1} + \rho \delta s_1^2 + \lambda \xi \eta_1 \gamma_1 < \gamma_1 \gamma_2, \\
\gamma_1 \sqrt{\tau_2^2 - 2 \gamma r_2} + \lambda \xi \eta_1 \gamma_2 < \gamma_1 \gamma_2,
\end{array}\right.
\end{align*}
$$

(3.3.7)

then SMVI(3.3.1) admits a solution $(x, y, u, v)$ and sequences $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ converge to $x$, $y$, $u$ and $v$, respectively, where $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ is the sequences generated by Iterative Algorithm 3.3.1.

### 3.4 SYSTEM OF MULTI-VALUED QUASI VARIATIONAL-LIKE INCLUSIONS

First we extend the concept of $H$-monotone mapping to $(H, \eta)$-monotone mapping and discuss some properties of resolvent operator associated with $(H, \eta)$-monotone mapping.

**Definition 3.4.1** [44]. Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $H : \mathcal{H} \to \mathcal{H}$ be single-valued mappings and $M : \mathcal{H} \to 2^\mathcal{H}$ be a multi-valued mapping, then $M$ is said to be

(i) $\eta$-monotone if

$$
\langle u - v, \eta(x, y) \rangle \geq 0, \quad \forall x, y \in \mathcal{H}, x \in M(x), y \in M(v);
$$

(ii) $(H, \eta)$-monotone if $M$ is $\eta$-monotone and $(H + \lambda M)(\mathcal{H}) = \mathcal{H}, \forall \lambda > 0$.

**Definition 3.4.2** [44]. Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a single-valued mapping, $H : \mathcal{H} \to \mathcal{H}$ be a strongly $\eta$-monotone mapping and $M : \mathcal{H} \to 2^\mathcal{H}$ be an $(H, \eta)$-monotone mapping. Then the resolvent operator $R_{M,\lambda}^{H,\eta} : \mathcal{H} \to \mathcal{H}$ is defined by

$$
R_{M,\lambda}^{H,\eta}(x) = (H + \lambda M)^{-1}(x), \quad \forall x \in \mathcal{H}.
$$

**Remark 3.4.1.** If $\eta(x, y) = x - y$, then the definition of $\eta$-monotonicity is that of monotonicity and the definition of $(H, \eta)$-monotonicity becomes that of $H$-monotonicity.
in [41]. It is easy to see that if \( H = I \) (the identity map on \( H \)), then the definition of \((I, \eta)\)-monotone mapping is that of maximal \( \eta \)-monotone mapping and the definition of \( I \)-monotone mapping is that of maximal monotone mapping, maximal \( \eta \)-monotone mapping, \( H \)-monotone mapping.

**Lemma 3.4.1** [44]. Let \( \eta : H \times H \to H \) be a single-valued Lipschitz continuous mapping with constant \( r \); let \( H : H \to H \) be a strongly \( \eta \)-monotone mapping with constant \( \gamma > 0 \) and let \( M : H \to 2^H \) be an \((H, \eta)\)-monotone mapping. Then, the resolvent operator \( R_{M, \lambda}^{H, \eta} : H \to H \) is Lipschitz continuous with constant \( \frac{T}{\gamma} \), i.e.,

\[
\| R_{M, \lambda}^{H, \eta}(x) - R_{M, \lambda}^{H, \eta}(y) \| \leq \frac{T}{\gamma} \| x - y \|, \quad \forall x, y \in H.
\]

**Definition 3.4.3.** Let \( H_1, H_2, H_3 \) be Hilbert spaces, \( g : H_1 \to H_1 \) and \( F_1 : H_1 \times H_2 \times H_3 \to H_1 \) be single-valued mappings.

(i) \( F_1(.,.,.) \) is said to be **Lipschitz continuous** in the first argument if there exists a constant \( \xi > 0 \) such that

\[
\| F_1(x, y, z) - F_1(x', y, z) \| \leq \xi \| x - x' \|, \quad \forall x, x' \in H_1, y \in H_2, z \in H_3;
\]

(ii) \( F_1(.,.,.) \) is said to be **monotone** in the first argument if

\[
\langle F_1(x, y, z) - F_1(x', y, z), x - x' \rangle \geq 0, \quad \forall x, x' \in H_1, y \in H_2, z \in H_3;
\]

(iii) \( F_1(.,.,.) \) is said to be **strongly monotone** in the first argument if there exists a constant \( \alpha > 0 \) such that

\[
\langle F_1(x, y, z) - F_1(x', y, z), x - x' \rangle \geq \alpha \| x - x' \|^2, \quad \forall x, x' \in H_1, y \in H_2, z \in H_3;
\]

(iv) \( F_1(.,.,.) \) is said to be **monotone** with respect to \( g \) in the first argument if

\[
\langle F_1(x, y, z) - F_1(x', y, z), gx - gx' \rangle \geq 0, \quad \forall x, x' \in H_1, y \in H_2, z \in H_3;
\]

(v) \( F_1(.,.,.) \) is said to be **strongly monotone** with respect to \( g \) in the first argument if there exists a constant \( \beta > 0 \) such that

\[
\langle F_1(x, y, z) - F_1(x', y, z), gx - gx' \rangle \geq \beta \| x - x' \|^2, \quad \forall x, x' \in H_1, y \in H_2, z \in H_3.
\]
In a similar way, we can define the Lipschitz continuity and the strong monotonicity (monotonicity) of $F_i(\cdot, \cdot, \cdot)$ with respect to $g$ in the second or third argument.

Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be Hilbert spaces; let $H_1, g_1 : \mathcal{H}_1 \to \mathcal{H}_1$, $H_2, g_2 : \mathcal{H}_2 \to \mathcal{H}_2$, $H_3, g_3 : \mathcal{H}_3 \to \mathcal{H}_3$, $\eta_1 : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathcal{H}_1$, $\eta_2 : \mathcal{H}_2 \times \mathcal{H}_2 \to \mathcal{H}_2$, $\eta_3 : \mathcal{H}_3 \times \mathcal{H}_3 \to \mathcal{H}_3$, $F_1, G_1 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_1$, $F_2, G_2 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_2$, $F_3, G_3 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_3$ are single-valued mappings and let $A, D, Q : \mathcal{H}_1 \to CB(\mathcal{H}_1)$, $B, E, U : \mathcal{H}_2 \to CB(\mathcal{H}_2)$, $C, P, U : \mathcal{H}_3 \to CB(\mathcal{H}_3)$ are all multi-valued mappings. Let $M_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be an $(\eta_1, r)$-monotone mapping with respect to the first argument; let $M_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be an $(\eta_2, r)$-monotone mapping with respect to the first argument and let $M_3 : \mathcal{H}_3 \to 2^{\mathcal{H}_3}$ be an $(\eta_3, r)$-monotone mapping with respect to the first argument. We consider the following system of multi-valued quasi-variational-like inclusions (in short, SMQVLI):

Find $(x, y, z, a, b, c, d, e, p, q, u, v)$ such that $(x, y, z) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$, $a \in A(x)$, $b \in B(y)$, $c \in C(z)$, $d \in D(x)$, $e \in E(y)$, $p \in P(z)$, $q \in Q(x)$, $u \in U(y)$, $v \in V(z)$ and

$$
\begin{align*}
0 & \in F_1(x, y, z) - G_1(a, b, c) + M_1(g_1(x)), \\
0 & \in F_2(x, y, z) - G_2(a, b, c) + M_2(g_2(x)), \\
0 & \in F_3(x, y, z) - G_3(a, b, c) + M_3(g_3(x)).
\end{align*}
$$

(3.4.1)

The following lemma shows that SMQVLI(3.4.1) is equivalent a system of relations.

**Lemma 3.4.2[102].** For $i = 1, 2, 3$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$ be a single-valued mapping, $H_i : \mathcal{H}_i \to \mathcal{H}_i$ be a strictly $\eta_i$ monotone mapping and $M_i : \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be an $(\eta_i, r)$-monotone mapping. Then $(x, y, z, a, b, c, d, e, p, q, u, v)$ with $(x, y, z) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$, $a \in A(x)$, $b \in B(y)$, $c \in C(z)$, $d \in D(x)$, $e \in E(y)$, $p \in P(z)$, $q \in Q(x)$, $u \in U(y)$, $v \in V(z)$ is a solution of SMQVLI(3.4.1) if and only if

$$
\begin{align*}
g_1(x) & = \rho_{M_1}^{H_1, \eta_1}(H_1(g_1(x))) - \lambda F_1(x, y, z) + \lambda G_1(a, b, c), \\
g_2(y) & = \rho_{M_2}^{H_2, \eta_2}(H_2(g_2(y))) - \rho F_2(x, y, z) + \rho G_2(d, e, p), \\
g_3(z) & = \rho_{M_3}^{H_3, \eta_3}(H_3(g_3(z))) - \sigma F_3(x, y, z) + \sigma G_3(q, u, v),
\end{align*}
$$

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where $R_{M1}^{H1,n} = (H_1 + \lambda M_1)^{-1}$, $R_{M2,\sigma}^{H2} = (H_2 + \rho M_2)^{-1}$, $R_{M3,\sigma}^{H3} = (H_3 + \sigma M_3)^{-1}$, $\lambda > 0$, $\rho > 0$ and $\sigma > 0$ be constants.

Based on Lemma 3.4.2, we construct the following iterative algorithm for finding the approximate solution of SMQVLI(3.4.1).

**Iterative Algorithm 3.4.1[102].** For any given $x_0 \in \mathcal{H}_1, y_0 \in \mathcal{H}_2$ and $z_0 \in \mathcal{H}_3$, we compute the sequences $x_n, y_n, z_n, a_n, b_n, c_n, d_n, e_n, p_n, q_n, u_n$ and $v_n$ using iterative schemes such that

\begin{align*}
x_{n+1} &= x_n - g_1(x_n) + R_{M1}^{H1,n} (H_1(g_1(x_n))) - \lambda F_1(x_n, y_n, z_n) + \lambda G_1(a_n, b_n, c_n), \quad (3.4.2) \\
y_{n+1} &= y_n - g_1(y_n) + R_{M2,\sigma}^{H2} (H_2(g_2(y_n))) - \rho F_2(x_n, y_n, z_n) + \rho G_2(d_n, e_n, p_n), \quad (3.4.3) \\
z_{n+1} &= z_n - g_3(z_n) + R_{M3,\sigma}^{H3} (H_3(g_3(z_n))) - \sigma F_3(x_n, y_n, z_n) + \sigma G_3(q_n, u_n, v_n), \quad (3.4.4)
\end{align*}

\begin{align*}
a_n &\in A(x_n), \quad \|a_{n+1} - a_n\| \leq (1 + \frac{1}{n+1}) \hat{H}_1(A(x_{n+1}, A(x_n))), \\
b_n &\in B(y_n), \quad \|b_{n+1} - b_n\| \leq (1 + \frac{1}{n+1}) \hat{H}_2(B(x_{n+1}, B(x_n))), \\
c_n &\in C(z_n), \quad \|c_{n+1} - c_n\| \leq (1 + \frac{1}{n+1}) \hat{H}_3(C(z_{n+1}, C(z_n))), \\
d_n &\in D(x_n), \quad \|d_{n+1} - d_n\| \leq (1 + \frac{1}{n+1}) \hat{H}_4(D(x_{n+1}, D(x_n))), \\
e_n &\in E(y_n), \quad \|e_{n+1} - e_n\| \leq (1 + \frac{1}{n+1}) \hat{H}_5(E(y_{n+1}, E(y_n))), \\
p_n &\in P(z_n), \quad \|p_{n+1} - p_n\| \leq (1 + \frac{1}{n+1}) \hat{H}_6(P(z_{n+1}, P(z_n))), \\
q_n &\in Q(x_n), \quad \|q_{n+1} - q_n\| \leq (1 + \frac{1}{n+1}) \hat{H}_7(Q(x_{n+1}, Q(x_n))), \\
u_n &\in U(y_n), \quad \|u_{n+1} - u_n\| \leq (1 + \frac{1}{n+1}) \hat{H}_8(U(y_{n+1}, U(y_n))), \\
v_n &\in V(z_n), \quad \|v_{n+1} - v_n\| \leq (1 + \frac{1}{n+1}) \hat{H}_9(V(z_{n+1}, V(z_n))), \quad (3.4.13)
\end{align*}

for all $n = 0, 1, 2, \ldots$

The following theorem prove that approximate solution obtained by Iterative Algorithm 3.4.1, converges to a solution of SMQVLI(3.4.1).
Theorem 3.4.1 [102]. For $i = 1, 2, 3$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \rightarrow \mathcal{H}_i$ be strongly $\eta_i$-monotone and Lipschitz continuous with constant $\gamma_i$ and $\delta_i$, respectively, and let $g_i : \mathcal{H}_i \rightarrow \mathcal{H}_i$ be strongly monotone and Lipschitz continuous with $r_i$ and $s_i$, respectively. Let $A, D, Q : \mathcal{H}_1 \rightarrow CB(\mathcal{H}_1)$, $B, E, U : \mathcal{H}_2 \rightarrow CB(\mathcal{H}_2)$, $C, P, V : \mathcal{H}_3 \rightarrow CB(\mathcal{H}_3)$ be $D$-Lipschitz continuous with constant $l_A > 0$, $l_D > 0$, $l_Q > 0$, $l_B > 0$, $l_E > 0$, $l_U > 0$, $l_C > 0$, $l_P > 0$ and $l_V > 0$, respectively. Let $F_1 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \rightarrow \mathcal{H}_1$ be strongly monotone with respect to $g_1$ in the first argument with constant $\alpha_1 > 0$, Lipschitz continuous in the first argument with constant $\beta_1 > 0$, Lipschitz continuous in the second argument with constant $\xi_1 > 0$, respectively, where $g_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ is defined by $g_1(x) = H_1og_1(x) = H_1(g_1(x)) \quad \forall x \in \mathcal{H}_1$, $\xi_1 > 0$; let $F_2 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \rightarrow \mathcal{H}_2$ be strongly monotone with respect to $g_2$ in the second argument with constant $\alpha_2 > 0$, Lipschitz continuous in the first argument with constant $\xi_2 > 0$, Lipschitz continuous in the third argument with constant $\beta_2 > 0$, Lipschitz continuous in the second argument with constant $\xi_3 > 0$, respectively, where $g_2 : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ is defined by $g_2(x) = H_1og_1(y) = H_2(g_2(y)) \quad \forall y \in \mathcal{H}_2$ and let $F_3 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \rightarrow \mathcal{H}_3$ be strongly monotone with respect to $g_3$ in the third argument with constant $\alpha_3 > 0$, Lipschitz continuous in the third argument with constant $\beta_3 > 0$, Lipschitz continuous in the first argument with constant $\xi_3 > 0$, and Lipschitz continuous in the second argument with constant $\xi_3 > 0$, respectively, where, $g_3 : \mathcal{H}_3 \rightarrow \mathcal{H}_3$ is defined by $g_3(x) = H_3og_3(x) = H_3(g_3(x)) \quad \forall z \in \mathcal{H}_3$. Assume that $G_1 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \rightarrow \mathcal{H}_1$ is Lipschitz continuous in the first, second and third arguments with constants $\mu_1 > 0$, $v_1$ and $w_1$, respectively, $G_2 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \rightarrow \mathcal{H}_2$ is Lipschitz continuous in the first, second and third arguments with constants $\mu_2 > 0$, $v_2$ and $w_2$, respectively, $G_3 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \rightarrow \mathcal{H}_3$ is Lipschitz continuous in the first, second and third arguments with constants $\mu_3 > 0$, $v_3$ and $w_3$, respectively, $M_1 : \mathcal{H}_1 \rightarrow 2_{1}^{\mathcal{H}}$ is an $(H_1, \eta_1)$-monotone mapping, $M_2 : \mathcal{H}_2 \rightarrow 2_{2}^{\mathcal{H}}$ is an $(H_2, \eta_2)$-monotone mapping and $M_3 : \mathcal{H}_3 \rightarrow 2_{3}^{\mathcal{H}}$ is an $(H_3, \eta_3)$-monotone mapping.
Suppose that there exist constants $\lambda > 0$, $\rho > 0$ and $\sigma > 0$ such that

\[
\begin{align*}
\sqrt{1 - 2r_1 + s_3^2 + \frac{d}{\eta_1}} (\sqrt{\delta_1^2 s_1^2 - 2\lambda \alpha_1 + \lambda^2 \beta_1^2 + \lambda \mu_1 l_A} + \rho(\xi_2 + \mu_2 l_B) \frac{a}{\eta_2} + \sigma(\xi_3 + \mu_3 l_Q) \frac{a}{\eta_3} < 1, \\
\sqrt{1 - 2r_2 + s_2^2 + \frac{d}{\eta_2}} (\sqrt{\delta_2^2 s_2^2 - 2\rho \alpha_2 + \rho^2 \beta_2^2 + \rho v_2 l_E} + \lambda(\xi_1 + v_1 l_B) \frac{a}{\eta_1} + \sigma(\xi_3 + v_3 l_V) \frac{a}{\eta_3} < 1, \\
\sqrt{1 - 2r_3 + s_3^2 + \frac{d}{\eta_3}} (\sqrt{\delta_3^2 s_3^2 - 2\sigma \alpha_3 + \alpha^2 \beta_3^2 + \sigma w_3 l_V} + \lambda(\xi_1 + w_1 l_C) \frac{a}{\eta_1} + \rho(\xi_2 + w_2 l_P) \frac{a}{\eta_2} < 1.
\end{align*}
\tag{3.4.13}
\]

Then $\text{SMQVLI}(3.4.1)$ admits a solution $(x, y, z, a, b, c, d, e, p, q, u, v)$ and sequences

\{x_n\}, \{y_n\}, \{z_n\}, \{d_n\}, \{e_n\}, \{p_n\}, \{q_n\}, \{u_n\}, \{v_n\}

converge to $x, y, z, a, b, c, d, e, p, q, u, v$ respectively, where \{x_n\}, \{y_n\}, \{z_n\}, \{d_n\}, \{e_n\}, \{p_n\}, \{q_n\}, \{u_n\}, \{v_n\} be the sequences generated by Iterative Algorithm 3.4.1.