CHAPTER 2

ITERATIVE METHODS FOR SOME SYSTEMS OF
NONLINEAR VARIATIONAL INCLUSIONS IN
BANACH SPACES

2.1 INTRODUCTION

One of the most important and interesting problems in the theory of systems of variational inclusions is to developed the numerical methods which provide efficient and implementable iterative algorithms for solving systems of variational inclusions.

The methods based on resolvent operators are generalizations of projection method and have been widely used to study the existence and iterative approximation of solutions of systems of variational inclusions.

In this chapter, we study some properties of maximal $\eta$-monotone, $(m, \eta)$-accretive mappings and their associated resolvent operators. Using these resolvent operators, we study the existence of solutions of some systems of variational inclusions in Banach spaces. Further we develop some iterative algorithms for finding approximate solutions of these systems and discuss their convergence criteria.

The remaining part of this chapter is organized as follows:

In Section 2.2, we consider a system of nonlinear implicit variational inclusions considered by Bai et al. [9]. By using the resolvent operator technique associated with $m$-accretive mappings, we give an iterative algorithm for finding the approximate solutions of this system of variational inclusions and discuss the convergence criteria.

In Section 2.3, we consider a system of generalized nonlinear variational-like inclusions considered by Lan et al. [77]. By using the resolvent operator technique associated with maximal $\eta$-monotone mapping, we give a generalized Mann-type iterative algorithm for finding the approximate solutions of this system of variational inclusions. Further, we discuss the existence of solution of the system and convergence criteria for the iterative algorithm.
In Section 2.4, we consider a system of multi-valued implicit variational-like inclusions considered by Kazmi et al. [66]. By using the resolvent operator technique associated with \((m, \eta)\)-accretive mapping due to Chidume et al. [21] and Nadler technique [89], we give an iterative algorithm for finding the approximate solutions of this system of variational inclusions. Further, we discuss the existence of solution of the system and the convergence criteria of the iterative algorithm.

The chapter is based on work of Bai et al. [9], Lan et al. [77] and Kazmi et al. [66].

Throughout this chapter, unless or otherwise stated, \(E\) is a real Banach space endowed with dual space \(E^*\) and the dual pair \(\langle . , . \rangle\) between \(E\) and \(E^*\). If there is no confusion, we denote the norm of \(E\) and \(E^*\) by \(\| . \|\).

2.2 SYSTEM OF NONLINEAR IMPLICIT VARIATIONAL INCLUSIONS

Let \(T, g : E \to E\) are two single-valued mappings. Suppose that \(A : E \to 2^E\) is an \(m\)-accretive mapping. We consider the following system of nonlinear implicit variational inclusions (in short, SNIVI) [9]:

Find \(x, y, z \in E\) such that

\[
\begin{align*}
\theta &\in \alpha T(y) + g(x) - g(y) + A(g(x)), \\
\theta &\in \beta T(z) + g(y) - g(z) + A(g(y)), \\
\theta &\in \gamma T(x) + g(z) - g(x) + A(g(z)),
\end{align*}
\]

where \(\theta\) is the zero element in \(E\) and \(\alpha, \beta, \gamma > 0\).

Some Special Cases of SNIVI (2.2.1)-(2.2.3):

(i) If \(E = H\) is a Hilbert space and \(A = \partial \varphi\), where \(\varphi : H \to \mathcal{R} \cup \{+\infty\}\) is a proper convex and lower semicontinuous function on \(H\) and \(\partial \varphi\) denotes the subdifferential of function \(\varphi\), then SNIVI (2.2.1)-(2.2.3) is equivalent to finding \(x, y, z \in H\) such that

\[
\langle \alpha T(y) + g(x) - g(y), w - g(x) \rangle \geq \varphi(g(x)) - \varphi(w),
\]

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\[
\langle \beta T(z) + g(y) - g(z), w - g(y) \rangle \geq \varphi(g(y)) - \varphi(w), \quad (2.2.5)
\]
and
\[
\langle \gamma T(x) + g(z) - g(x), w - g(z) \rangle \geq \varphi(g(z)) - \varphi(w), \quad (2.2.6)
\]
for \( \alpha, \beta, \gamma > 0 \) and for all \( w \in \mathcal{H} \).

(ii) If \( E = \mathcal{H} \) is a Hilbert space, \( g = I \) is the identity mapping, and \( \varphi \) is the indicator function of a closed convex subset \( K \) in \( \mathcal{H} \), that is,
\[
\varphi(x) = I_K(x) = \begin{cases} 
0, & x \in K, \\
+\infty, & \text{otherwise},
\end{cases}
\]
then a system of nonlinear variational inclusions (2.2.4)-(2.2.6) is equivalent to finding \( x, y, z \in K \) such that
\[
\langle \alpha T(y) + x - y, w - x \rangle \geq 0, \quad (2.2.7)
\]
\[
\langle \beta T(z) + y - z, w - y \rangle \geq 0, \quad (2.2.8)
\]
and
\[
\langle \gamma T(x) + z - x, w - z \rangle \geq 0, \quad (2.2.9)
\]
for \( \alpha, \beta, \gamma > 0 \) and for all \( w \in K \).

(iii) For \( x = y = z \) and \( \alpha = \beta = \gamma = 1 \), the system of nonlinear variational inequalities (2.2.7)-(2.2.9) reduces to the following standard nonlinear variational inequality (NVI) problem:

Find an element \( x \in K \) such that
\[
\langle T(x), w - x \rangle \geq 0, \quad \forall w \in K.
\]

The following lemma shows an equivalence between SNIVI (2.2.1)-(2.2.3) and a system of relations.

**Lemma 2.2.1[9].** \( x, y, z \in E \) is a solution of SNIVI (2.2.1)-(2.2.3) if and only if \( x, y, z \) satisfy the relations
\[
g(x) = R_A[g(y) - \alpha T(y)], \quad (2.2.10)
\]
\[ g(y) = R_A[g(z) - \beta T(z)], \quad (2.2.11) \]
\[ g(z) = R_A[g(x) - \gamma T(x)], \quad (2.2.12) \]

for \( \alpha, \beta, \gamma > 0. \)

Based on Lemma 2.2.1, we give the following iterative algorithm for finding the approximate solutions of SNIVI (2.2.1)-(2.2.3).

**Iterative Algorithm 2.2.1**[9]. For a given initial point \( x_0 \in E \), compute the sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) by the following iterative scheme:

\[
g(x_{n+1}) = (1 - a_n)g(x_n) + a_nR_A[g(y_n) - \alpha T(y_n)],
\]
\[
g(y_n) = R_A[g(z_n) - \beta T(z_n)],
\]
\[
g(z_n) = R_A[g(x_n) - \gamma T(x_n)],
\]

for \( \alpha, \beta, \gamma > 0, \ 0 \leq a_n \leq 1, \ n = 0, 1, \ldots \) and \( \sum_{n=0}^{\infty} a_n = \infty. \)

If \( E = H \) is a Hilbert space, \( g = I \) is the identity mapping and \( A = \partial \varphi \), where \( \varphi \) is the indicator function of a closed convex subset \( K \) in \( H \), then Iterative Algorithm 2.2.1 reduces to the following iterative algorithm.

**Iterative Algorithm 2.2.2**[9]. For a given initial point \( x_0 \in E \), compute the sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) by the following iterative scheme:

\[
x_{n+1} = (1 - a_n)x_n + a_nP_K[y_n - \alpha T(y_n)],
\]
\[
y_n = P_K[z_n - \beta T(z_n)],
\]
\[
z_n = P_K[x_n - \gamma T(x_n)],
\]

for \( \alpha, \beta, \gamma > 0, \ 0 \leq a_n \leq 1, \ n = 0, 1, \ldots \), and \( \sum_{n=0}^{\infty} a_n = \infty. \) Here, \( P_K \) is the projection of \( H \) onto \( K. \)

**Definition 2.2.1**[9]. Let \( T, g : E \to E \) be single-valued mappings. Then \( T \) is said to be

(i) \( r \)-**strongly accretive** with respect to \( g \) if there exists a constant \( r \in (0, 1) \) such that

\[
\langle T(x) - T(y), j_q(g(x) - g(y)) \rangle \geq r\|g(x) - g(y)\|^q, \quad \forall x, y \in E;
\]
(ii) *s-Lipschitz continuous* with respect to $g$ if there exists a constant $s \geq 1$ such that

$$\|T(x) - T(y)\| \leq s\|g(x) - g(y)\|, \quad \forall x, y \in E.$$ 

The following theorem gives the convergence analysis of Iterative Algorithm 2.2.1.

**Theorem 2.2.1** [9]. Let $E$ be a real $q$-uniformly smooth Banach space and $A : E \to 2^E$ be an $m$-accretive mapping. Let $T : E \to E$ be $r$-strongly accretive and $s$-Lipschitz continuous mapping with respect to $g$ and $g : E \to E$ be invertible. Let $x, y, z \in E$ form a solution of SNIVI (2.2.1)-(2.2.3) and the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated by Iterative Algorithm 2.2.1. Then, we have the following

1. The estimates:
   
   (i) $\|g(x_{n+1}) - g(x)\| \leq (1 - a_n)\|g(x_n) - g(x)\| + a_n\sigma\|g(y_n) - g(y)\|$;
   
   (ii) $\|g(y_n) - g(y)\| \leq \|g(z_n) - g(z)\|$, for $0 < \beta \leq (qr/c_q s^q)^{1/(q - 1)}$;
   
   (iii) $\|g(z_n) - g(z)\| \leq \|g(x_n) - g(x)\|$, for $0 < \gamma \leq (qr/c_q s^q)^{1/(q - 1)}$;
   
   (iv) $\|g(x_{n+1}) - g(x)\| \leq (1 - a_n)\|g(x_n) - g(x)\| + a_n\sigma\|g(x_n) - g(x)\|$

   where $\sigma = (1 - q\alpha r + c_q\alpha^q s^q)^{1/q}$, and $c_q \geq 1$ is a constant.

2. The sequence $\{x_n\}$ converges to $x$ for $0 < \beta, \gamma \leq (qr/c_q s^q)^{1/(q - 1)}$.

Let $E = \mathcal{H}$ is a Hilbert space, we know that $E$ is 2-uniformly smooth and $c_2 = 1$ by [115]. And let $g = I$ (the identity mapping). Then, Theorem 2.2.1 reduces to the following corollary:

**Corollary 2.2.1** [9]. Let $\mathcal{H}$ be a real Hilbert space and $T : K \to \mathcal{H}$ an $r$-strongly accretive and $s$-Lipschitz continuous mapping from a nonempty, closed and convex subset $K$ into $\mathcal{H}$. Let $x, y, z \in K$ from a solution for SNIVI (2.2.7)-(2.2.9) and the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated by Iterative Algorithm 2.2.2 Then, we have the following

1. The estimates
\begin{align*}
(i) \quad \|x_{n+1} - x\| &\leq (1 - a_n)\|x_n - x\| + a_n\sigma\|y_n - y\|
(ii) \quad \|y_n - y\| &\leq \|z_n - z\|, \text{ for } 0 < \beta \leq 2r/s^2;
(iii) \quad \|z_n - z\| &\leq \|x_n - x\|, \text{ for } 0 < \gamma \leq 2r/s^2;
(iv) \quad \|x_{n+1} - x\| &\leq (1 - a_n)\|x_n - x\| + a_n\sigma\|x_n - x\|,
\end{align*}

where \( \sigma = (1 - 2\alpha r + \alpha^2 s^2)^{1/2}. \)

(2) The sequence \( \{x_n\} \) converges to \( x \) for \( 0 < \beta, \gamma \leq 2r/s^2, \) and \( 0 < \alpha < 2r/s^2. \)

### 2.3 SYSTEM OF GENERALIZED NONLINEAR VARIATIONAL-LIKE INCLUSIONS

First, we study some properties of resolvent operator associated with maximal \( \eta \)-monotone mapping in Banach space.

For that we need the following definitions:

**Definition 2.3.1 [58].** The mapping \( \eta : E \times E \rightarrow E^* \) is said to be

(i) \( \delta \)-strongly monotone if there exists a constant \( \delta > 0 \) such that

\[ \langle x - y, \eta(x, y) \rangle \geq \delta\|x - y\|^2, \quad \forall x, y \in E; \]

(ii) \( \tau \)-Lipschitz continuous if there exists a constant \( \tau > 0 \) such that

\[ \|\eta(x, y)\| \leq \tau\|x - y\|, \quad \forall x, y \in E. \]

**Definition 2.3.2 [58].** Let \( \eta : E \times E \rightarrow E^* \) be a single-valued mapping and \( A : E \rightarrow 2^E \) be a multi-valued mapping. Then \( A \) is said to be

(i) \( \eta \)-monotone if

\[ \langle u - v, \eta(x, y) \rangle \geq 0, \quad \forall x, y \in E, u \in A(x), v \in A(y); \]

(ii) strictly \( \eta \)-monotone if

\[ \langle u - v, \eta(x, y) \rangle \geq 0, \quad \forall x, y \in E, u \in A(x), v \in A(y), \]

and equality holds if and only if \( x = y. \)
(iii) strongly $\eta$-monotone if there exists a constant $r > 0$ such that
\[
\langle u - v, \eta(x, y) \rangle \geq r \|x - y\|^2, \quad \forall x, y \in E, u \in A(x), v \in A(y);
\]

(iv) maximal $\eta$-monotone mapping if $M$ is $\eta$-monotone and $(I + \lambda M)(E) = E$ for all (equivalently, for some) $\lambda > 0$.

**Lemma 2.3.1**[58,59]. Let $\eta : E \times E \to E^*$ be a strictly monotone mapping and $A : E \to 2^E$ be a maximal $\eta$-monotone mapping. Then the following results hold:

(i) $\langle u - v, \eta(x, y) \rangle \geq 0, \quad \forall (y, u) \in \text{Graph}(A)$ implies $(x, u) \in \text{Graph}(A)$, where
\[
\text{Graph}(A) = \{(x, u) \in E \times E : u \in A(x)\};
\]

(ii) for any $\rho > 0$, inverse mapping $(I + \rho A)^{-1}$ is single-valued.

Let $S, A, p : E \to E$ and $N_1, N_2, \eta_1, \eta_2 : E \times E \to E$ be single-valued mappings, $T, B : E \to 2^E$ be multi-valued mappings and $M_1, M_2$ be respectively, maximal $\eta_1$-monotone and maximal $\eta_2$-monotone mappings. For any given $f, g \in E$, we consider the following system of generalized nonlinear variational-like inclusions (in short, SGNVLI) [77]:

Find $x, y \in E$ such that $p(x) \in \text{dom}(M_1)$ and
\[
\begin{cases}
 y - x - \lambda_1 (N_1(S(y), v) - f) \in \lambda_1 M_1(p(x)), & \forall v \in T(y), \\
 x - y - \lambda_2 (N_2(A(x), u) - g) \in \lambda_2 M_2(y), & \forall u \in B(x),
\end{cases}
\]
where $\lambda_1, \lambda_2$ are two positive constants.

If $p \equiv I$, the identity mapping, then SGNVLI(2.3.1) is equivalent to finding $x, y \in E$ such that
\[
\begin{cases}
 y - x - \lambda_1 (N_1(S(y), v) - f) \in \lambda_1 M_1(p(x)), & \forall v \in T(y), \\
 x - y - \lambda_2 (N_2(A(x), u) - g) \in \lambda_2 M_2(y), & \forall u \in B(x),
\end{cases}
\]
for $\lambda_1, \lambda_2 > 0$.

The following lemma gives an equivalence between SGNVLI(2.3.1) and a system of inclusions.
Lemma 2.3.1[77]. For given $x, y$ in $E$, $(x, y)$ is a solution of SGNVLI(2.3.1) if and only if
\[
\begin{cases}
p(x) \in J_{M_1}^1(p(x) + y - x - \lambda_1(N_1(S(y), T(y)) - f)), \\
y \in J_{M_2}^2(x - \lambda_2(N_2(A(x), B(x)) - g)).
\end{cases}
\tag{2.3.3}
\]

Based on Lemma 2.3.4, we have the following iterative algorithms for SGN-VLI(2.3.1) and problem (2.3.2).

Iterative Algorithm 2.3.1[77]. For any given $x_0 \in E$, the generalized Mann iterative sequence with mixed errors $\{x_n\}$ and $\{y_n\}$ in $E$ is defined as follows:
\[
\begin{cases}
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[x_n - p(x_n) \\
+ J_{M_1}^1(p(x_n) + y_n - x_n - \lambda_1(N_1(S(y_n), T(y_n)) - f))] + \alpha_ne_n + s_n, \\
y_n = J_{M_2}^2(x_n - \lambda_2(N_2(A(x_n), B(x_n)) - g)) + f_n, \quad n = 0, 1, 2, ..., 
\end{cases}
\tag{2.3.4}
\]
where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{e_n\}$, $\{f_n\}$, $\{s_n\}$ are the sequences in $E$ satisfying the following conditions:

(i) $e_n = e'_n + e''_n$;

(ii) $\lim_{n \to \infty} ||e'_n|| = \lim_{n \to \infty} ||f_n|| = 0$;

(iii) $\sum_{n=0}^{\infty} ||e''_n|| < \infty$, $\sum_{n=0}^{\infty} ||s_n|| < \infty$.

If $p = I$, then Iterative Algorithm 2.3.1, reduces to the following iterative algorithm for solving problem (2.3.2).

Iterative Algorithm 2.3.2[77]. For any given $x_0 \in E$, define the Mann iterative sequence with mixed errors $x_n$ and $y_n$ in $E$ as follows:
\[
\begin{cases}
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[x_n - p(x_n) \\
+ J_{M_1}^1(p(x_n) + y_n - x_n - \lambda_1(N_1(S(y_n), T(y_n)) - f))] + \alpha_ne_n + s_n, \\
y_n = J_{M_2}^2(x_n - \lambda_2(N_2(A(x_n), B(x_n)) - g)) + f_n, \quad n = 0, 1, 2, ..., 
\end{cases}
\]
where $\{\alpha_n\}$, $\{e_n\}$, $\{f_n\}$ and $\{s_n\}$ are the same as in Iterative Algorithm 2.3.1.

The following theorem gives the existence and iterative approximation of solutions of SGNVLI(2.3.1).
Theorem 2.3.1[77]. Let $E$ be a $q$-uniformly smooth Banach space and the mappings $S, A : E \to E$ be $\mu$-Lipschitz continuous and $\xi$-Lipschitz continuous, respectively. Let $T, B : E \to CB(E)$ be $(\nu, \hat{H})$-Lipschitz continuous and $(\zeta, \hat{H})$-Lipschitz continuous, respectively. Let $p : E \to E$ be $\alpha$-strongly monotone and $\beta$-Lipschitz continuous; let $N_1 : E \times E \to E$ be $\epsilon_1$-Lipschitz continuous and $\sigma_1$-strongly accretive with respect to $S$ in the first argument; let $N_2 : E \times E \to E$ be $\epsilon_2$-Lipschitz continuous and $\sigma_2$-strongly accretive with respect to $A$ in the first argument; let $N_i$ be $\gamma_i$-Lipschitz continuous in the second argument; let $\eta_i : E \times E \to E^*$ be $\tau_i$-Lipschitz continuous and $\delta_i$-strongly monotone and let $M_i : E \to 2^E$ be maximal $\eta_i$-monotone, for $i = 1, 2$.

Suppose that the following condition holds:

$$h = (1 + \frac{\tau_1}{\delta_1})(1 - q\alpha + c_q \beta q)^{\frac{1}{q}} < 1,$$

$$\tau_1 \tau_2 \left[ (1 - q\lambda_1 \sigma_1 + c_q \lambda_1^2 \epsilon_1^2 \mu^2)^{\frac{1}{2}} + \lambda_1 \gamma_1 \nu \right] \times \left[ (1 - q\lambda_2 \sigma_2 + c_q \lambda_2^2 \epsilon_2^2 \xi_2^2)^{\frac{1}{2}} + \lambda_2 \gamma_2 \zeta \right] < \delta_1 \delta_2 (1 - h),$$

where $c_q \geq 1$ and $\lambda_1, \lambda_2 > 0$ are constants. Then SGNVI (2.3.1) has a solution $(x, y)$.

If $p = I$, then Theorem 2.3.1 reduces to following existence theorem of solution system of variational-like inclusions (2.3.2).

Theorem 2.3.2[77]. Assume that $E, S, A, T, B, N_i, \eta_i$ and $M_i$ for $i = 1, 2$ are the same as in Theorem 2.3.1. If there exist constants $\lambda_1, \lambda_2 > 0$ such that

$$\tau_1 \tau_2 \left[ (1 - q\lambda_1 \sigma_1 + c_q \lambda_1^2 \epsilon_1^2 \mu^2)^{\frac{1}{2}} + \lambda_1 \gamma_1 \nu \right] \times \left[ (1 - q\lambda_2 \sigma_2 + c_q \lambda_2^2 \epsilon_2^2 \xi_2^2)^{\frac{1}{2}} + \lambda_2 \gamma_2 \zeta \right] < \delta_1 \delta_2,$$

where $c_q \geq 1$, then system (2.3.2) has a solution $(x, y)$.

Theorem 2.3.3[77]. Suppose that $E, S, A, T, B, N_i, \eta_i$ and $M_i$ for $i = 1, 2$ are the same as in Theorem 2.3.1. If $\sum_{n=0}^{\infty} \alpha_n = \infty$ and condition (2.3.5) holds, then the generalized Mann iterative sequence $\{x_n\}$ and $\{y_n\}$ defined by Iterative Algorithm 2.3.1, converge strongly to the solution $(x, y)$ of SGNVI(2.3.1).

From Theorem 2.3.1, we can get the following theorem.

Theorem 2.3.4[77]. Assume that $E, S, A, T, B, N_i, \eta_i$ and $M_i$ for $i = 1, 2$ are the same as in Theorem 2.3.2. If $\sum_{n=0}^{\infty} \alpha_n = \infty$ and condition (2.3.6) holds, then the
Mann iterative sequence \( \{x_n\} \) and \( \{y_n\} \) defined by Iterative Algorithm 2.3.2 converge strongly to the solution \((x, y)\) of system (2.3.2).

## 2.4 SYSTEM OF MULTI-VALUED IMPLICIT VARIATIONAL-LIKE INCLUSIONS

First, we give the concept of \((m, \eta)\)-accretive mappings and study some properties of resolvent operator associated with \((m, \eta)\)-accretive mappings.

For that we need the following definitions:

**Definition 2.4.1** [21]. A mapping \( \eta : E \times E \to E \) is said to be

(i) **accretive** if \( \forall x, y \in E \), there exists \( j(x - y) \in J(x - y) \) such that
\[
\langle \eta(x, y), j(x - y) \rangle \geq 0;
\]

(ii) **strictly accretive** if \( \forall x, y \in E \), there exists \( j(x - y) \in J(x - y) \) such that
\[
\langle \eta(x, y), j(x - y) \rangle > 0
\]
and the equality in above holds only when \( x = y \);

(iii) **\( \delta \)-strongly accretive** if \( \forall x, y \in E \), there exists \( j(x - y) \in J(x - y) \) and a constant \( \delta > 0 \) such that
\[
\langle \eta(x, y), j(x - y) \rangle \geq \delta \|x - y\|^2.
\]

**Definition 2.4.2** [21]. Let \( \eta : E \times E \to E \) be a single-valued mapping. Then a multi-valued mapping \( M : E \to 2^E \) is said to be

(i) **\( \eta \)-accretive** if \( \forall x, y \in E \), there exists \( j(x - y) \in J(x - y) \) such that
\[
\langle x - y, j(\eta(x, y)) \rangle \geq 0, \quad \forall x, y \in E \text{ and } \forall u \in Mx, \, v \in My;
\]

(ii) **\( \sigma \)-strongly \( \eta \)-accretive** if \( \forall x, y \in E \), there exists \( j(x - y) \in J(x - y) \) and a constant \( \sigma > 0 \) such that
\[
\langle x - y, j(\eta(x, y)) \rangle \geq \sigma \|x - y\|^2, \quad \forall x, y \in E \text{ and } \forall u \in Mx, \, v \in My;\]
(iii) \((m, \eta)\)-accretive if \(M\) is \(\eta\)-accretive and \((I + \rho M)(E) = E\), for any \(\rho > 0\),

where \(I\) stands for an identity operator on \(E\).

We note that if \(E = \mathcal{H}\), a Hilbert space, then \((m, \eta)\)-accretive mapping is called maximal \(\eta\)-monotone mapping.

**Lemma 2.4.1**[21]. Let \(E\) be a real Banach space. Let \(\eta : E \times E \to E\) be strictly accretive mapping and let \(M : E \to 2^E\) be an \((m, \eta)\)-accretive mapping. Then

(i) \(\langle u - v, \eta(x, y) \rangle \geq 0\), \(\forall (v, y) \in \mathcal{G}(M)\) implies \((u, x) \in \mathcal{G}(M)\), where \(\mathcal{G}(T) = \{(u, x) \in E \times E : u \in M(x)\}\) is the graph of \(M\);

(ii) the mapping \((I + \rho M)^{-1}\) is single-valued for all \(\rho > 0\).

**Remark 2.4.1**[21]. By Lemma 2.4.1 we can define resolvent operator associated with \((m, \eta)\)-accretive mapping \(M\) as follows:

\[
J^\rho_M(x) = (I + \rho M)^{-1}(x), \quad \forall x \in E,
\]

where \(\rho > 0\) is a constant and \(\eta : E \times E \to E\) is strictly accretive mapping.

**Lemma 2.4.2**[21]. Let \(\eta : E \times E \to E\) be \(\delta\)-strongly accretive and \(\tau\)-Lipschitz continuous mapping and let \(M : E \to 2^E\) be an \((m, \eta)\)-accretive mapping. Then the resolvent operator of \(M\), \(J^\rho_M = (I + \rho M)^{-1}\) is \((\tau/\delta)\)-Lipschitz continuous, i.e.,

\[
\|J^\rho_M(x) - J^\rho_M(y)\| \leq \frac{\tau}{\delta} \|x - y\|, \quad \forall x, y \in E,
\]

where \(\rho > 0\) is a constant.

Let \(N, \eta : E \times E \to E\), \(g : E \to E\) be single-valued mappings; let \(A, B : E \to CB(E)\) be multi-valued mappings. Suppose that \(M : E \to 2^E\) is \((m, \eta)\)-accretive mapping. Then we consider the following system of multi-valued variational-like inclusions (in short, SMVLI): Find \(x, y \in E\), \(u \in A(u)\), \(v \in B(u)\), \(u' \in A(v)\), \(v' \in B(v)\) such that

\[
\Theta \in g(x) - g(y) + \rho(N(u', v') + M(g(x))), \quad (2.4.3)
\]
\[
\Theta \in g(y) - g(x) + \gamma(N(u, v) + M(g(y))), \quad (2.4.4)
\]
where $\Theta$ is the zero element in $E$ and $\rho, \gamma > 0$.

If $E = \mathcal{H}$ is a Hilbert space; $N(x, y) = x$, $\forall x \in \mathcal{H}$ and $A$ is single-valued mapping, then SMIVLI (2.4.3)-(2.4.4) reduces to the following system of nonlinear variational-like inclusions studied by Kazmi and Bhat [5]: Find $x, y \in \mathcal{H}$ such that

$$\Theta^* \in g(x) - g(y) + \rho(T(y) + M(g(x))), \quad \rho > 0, \quad (2.4.5)$$

$$\Theta^* \in g(y) - g(x) + \gamma(T(x) + M(g(y))), \quad \gamma > 0, \quad (2.4.6)$$

where $\Theta^*$ is the zero element in $\mathcal{H}$ and $\rho, \gamma > 0$.

The following lemma gives an equivalence between SMIVLI (2.4.1)-(2.4.2) and a system of relations.

**Lemma 2.4.3** [66]. $(x, y, u, v, u', v')$, where $x, y \in E$, $u \in A(u), v \in B(u), u' \in A(v), v' \in B(v)$ is a solution of SMIVLI (2.4.3)-(2.4.4) if and only if $(x, y, u, v, u', v')$ satisfies

$$g(x) = J^{\rho}_* [g(u) - \rho N(u', v')], \quad \rho > 0, \quad (2.4.7)$$

where

$$g(y) = J^{\gamma}_* [g(u) - \gamma N(u, v)], \quad \gamma > 0, \quad (2.4.8)$$

and $J^{\rho}_* = (I + \rho M)^{-1}, J^{\gamma}_* = (I + \gamma)^{-1}$ are the resolvent operators associated with $M$.

Based on Lemma 2.4.3, we have the following iterative algorithm for SMIVLI (2.4.1).

**Iterative Algorithm 2.4.1** [66]. For given $x_0 \in E, y_0 \in E, u_0 \in A(u_0), v_0 \in B(u_0), u'_0 \in A(v_0), v'_0 \in B(v_0)$, compute the sequences $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}, \{u'_n\}, \{v'_n\}$ by the following iterative schemes:

$$g(x_{n+1}) = J^{\rho}_* [g(y_n) - \rho N(u'_n, v'_n)], \quad \rho > 0, \quad (2.4.9)$$

where

$$g(y_n) = J^{\gamma}_* [g(x_n) - \gamma N(u_n, v_n)], \quad \gamma > 0, \quad (2.4.10)$$

$u_n \in A(x_n)$, $||u_{n+1} - u_n|| \leq (1 + (1 + n)^{-1}) \hat{H}(A(x_{n+1}), A(x_n));$

$v_n \in B(x_n)$, $||v_{n+1} - v_n|| \leq (1 + (1 + n)^{-1}) \hat{H}(B(x_{n+1}), B(x_n));$
for \( n = 0, 1, 2, 3, \ldots \).

The following theorem gives the existence and iterative approximation of solution of SMIVLI (2.4.1)-(2.4.2).

**Theorem 2.4.1**[66]. Let \( E \) be a real Banach space. Let the multi-valued mapping \( M : E \to 2^E \) be \((\eta, m)\)-accretive; let the multi-valued mappings \( A, B : E \to CB(E) \) be \((\lambda_A, \hat{H})\)-Lipschitz continuous and \((\lambda_B, \hat{H})\)-Lipschitz continuous, respectively; let the mapping \( (g-I) : E \to E \) be \( k\)-strongly accretive, where \( I \) is the identity mapping on \( E \); let the mapping \( g : E \to E \) be \( \lambda_g\)-Lipschitz continuous; let the mapping \( N : E \times E \to E \) be \( \lambda_{N_1}\)-Lipschitz continuous in the first and second arguments, respectively. If for some positive numbers \( \rho \) and \( \gamma \), the following condition is satisfied:

\[
\rho, \gamma < \frac{\sqrt{d(0.5 + k) - \lambda_g^2}}{\lambda_{N_1} \lambda_A + \lambda_{N_2} \lambda_B}; \quad \lambda_g < \sqrt{d(0.5 + k)}, \quad d = \frac{\delta}{\tau}. \tag{2.4.11}
\]

Then the sequences \( \{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}, \{u'_n\}, \{v'_n\} \) generated by Iterative Algorithm 2.4.1, strongly converge to \( x, y, u, v, u', v' \), respectively, in \( E \) and \( x, y, u, v, u', v' \) is a solution of SMIVLI (2.4.1)-(2.4.2).

**Example 2.4.1**[66]. Let \( E \equiv \mathbb{R}^2 \) and let for any \( x = (x_1, x_2) \in \mathbb{R}^2, y = (y_1, y_2) \in \mathbb{R}^2 \), consider \( g(x) \equiv x, A(x) \equiv [x_1, x_1 + 1] \times \{0\}, B(x) \equiv \{x\}, N(x, y) \equiv x + 2y, \eta(x, y) \equiv x - y \). Then we can easily observe that

(i) \( g(x) \) is \( 1\)-strongly monotone and \( 1\)-Lipschitz continuous, that is, \( k = 1, \lambda_g = 1 \);

(ii) \( \eta \) is \( 1\)-strongly monotone and \( 1\)-Lipschitz continuous, that is, \( \delta = 1, \tau = 1 \);

(iii) \( A \) and \( B \) are \( 1\)-\( \hat{H} \)-Lipschitz continuous, that is, \( \lambda_A = 1, \lambda_B = 1 \);

(iv) \( N \) is \( 1\)-Lipschitz continuous in the first argument and \( 2\)-Lipschitz continuous in the second argument, that is, \( \lambda_{N_1} = 1, \lambda_{N_2} = 2 \).

After simple calculation, condition (2.4.11) is satisfied for \( \rho, \gamma \in (0, 0, 4) \) and \( \theta, \theta' \in (0, 1) \).